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Uniqueness and Value-sharing of Entire Functions

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Abstract. In this paper, we study the uniqueness problems on entire functions sharing one value. We improve and generalize some previous results of Zhang and Lin [11]. On the one hand, we consider the case for some more general differential polynomials $[f^nP(f)]^{(k)}$ where $P(w)$ is a polynomial; on the other hand, we relax the nature of sharing value from CM to IM.

1. Introduction and main results

Let $f(z)$ be a nonconstant meromorphic function in the whole complex plane. We shall use the following standard notations of value distribution theory: $T(r,f), m(r,f), N(r,f), \overline{N}(r,f), S(r,f) \cdots$. We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$, possibly outside a set of finite linear measure that is not necessarily the same at each occurrence. Let a be a complex number, and k be a positive integer. For a constant a, we denote by $N_{k)}\left(r, \frac{1}{f-a}\right)$ the counting function of the zeros $f(z)-a$ with multiplicity$\leq k$, and by $\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)$ the corresponding one ignoring multiplicity. Let $N_{(k}\left(r, \frac{1}{f-a}\right)$ be the counting function of the zeros $f(z)-a$ with multiplicity$\geq k$, and $\overline{N}_{(k}\left(r, \frac{1}{f-a}\right)$ the corresponding one ignoring multiplicity. Moreover, we define

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right),$$

$$\Theta(a,f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r,f)} .$$

Let $g(z)$ be another meromorphic function. If $f(z)-a$ and $g(z)-a$ assume the same zeros with the same multiplicities, then we call that $f(z)$ and $g(z)$ share the value a CM, where a is a complex number. We say f and g share the value a IM, if $f-a$ and $g-a$ assume the same zeros for which multiplicity is not counted.

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In 1997, corresponding to one famous question of Hayman [3], Fang and Hua [2], Yang and Hua [9] obtained the following unicity theorem:

Theorem A. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \geqslant 6$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share 1 CM, then either $f(z) = c_1e^{cz}, g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

In 2002, Fang [1] considered k-th derivative instead of 1-th derivative, and obtained the following theorems.

Theorem B. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, let n, k be two positive integers with $n > 2k+4$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 CM, then either $f(z) = c_1e^{cz}, g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

Theorem C. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, let n, k be two positive integers with $n > 2k+8$. If $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

Recently, Zhang and Lin [11] studied some cases for some special differential polynomials $[f^n(z)(\mu f^m(z)+\lambda)]^{(k)}$ or $[f^n(z)(f(z)-1)^m]^{(k)}$ and got the following Theorems.

For the sake of simplicity, we denote $m^* := \chi_\mu m$, where

$$\chi_\mu = \begin{cases} 0, & \mu = 0, \\ 1, & \mu \neq 0. \end{cases}$$

Theorem D. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, let n, m, k be three positive integers with $n > 2k+m^*+4$, and λ, μ be constants such that $|\lambda|+|\mu| \neq 0$. If $[f^n(z)(\mu f^m(z)+\lambda)]^{(k)}$ and $[g^n(z)(\mu g^m(z)+\lambda)]^{(k)}$ share 1 CM, then*
(i) *when $\lambda\mu \neq 0$, $f(z) \equiv g(z)$;*
(ii) *when $\lambda\mu = 0$, either $f(z) \equiv tg(z)$, where t is a constant satisfying $t^{n+m^*} = 1$, or $f(z) = c_1e^{cz}, g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying*

$$(-1)^k\lambda^2(c_1c_2)^{n+m^*}(n+m^*)^{2k} = 1 \text{ or } (-1)^k\mu^2(c_1c_2)^{n+m^*}(n+m^*)^{2k} = 1.$$

Theorem E. *Let $f(z)$ and $g(z)$ be three nonconstant entire functions, let n, k, m be two positive integers with $n > 2k+m^*+4$. If $[f^n(z)(f(z)-1)^m]^{(k)}$ and $[g^n(z)(g(z)-1)^m]^{(k)}$ share 1 CM, then either $f(z) \equiv g(z)$ or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(f,g) = w_1^n(w_1-1)^m - w_2^n(w_2-1)^m$.*

From the above results and Proposition in Section 2, naturally we ask whether there exists a corresponding unicity theorem to Theorem D and Theorem E for $[f^nP(f)]^{(k)}$ where $P(w)$ is a polynomial. In this paper, we give a positive answer to above question by proving the following theorem.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions. Let $P(f) =$*

$a_mf^m + a_{m-1}f^{m-1} + \cdots + a_0$ ($a_m \neq 0$), and a_i is the first nonzero coefficient from the right, and n, k, m be three positive integers with $n > 2k + m + 4$. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share 1 CM, then
(1) If $0 \leq i < m$, then either $f(z) \equiv g(z)$ or f, g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = w_1^nP(w_1) - w_2^nP(w_2)$.
(2) If $i = m$, then either $f(z) \equiv tg(z)$, where t is a constant satisfying $t^{n+m} = 1$ or $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k a_m^2 (c_1c_2)^{n+m}[(n + m)c]^{2k} = 1.$$

Recently, Meng [6] replaced the CM sharing value by an IM sharing value in Theorem B and proved the following result.

Theorem F. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, n, k two positive integers with $n \geqslant 5k + 8$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 IM. Then either $f(z) = c_1e^{cz}, g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

In this paper, we relax the nature of sharing value from CM to IM in Theorem 1.1 and prove the following theorem.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two nonconstant transcendental entire functions. Let $P(f) = a_mf^m + a_{m-1}f^{m-1} + \cdots + a_0$ ($a_m \neq 0$), and a_i is the first nonzero coefficient from the right, and n, k, m be three positive integers with $n + m > (5k + 7)(m + 1)$, If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share 1 IM, then*
(1) If $0 \leq i < m$, then f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = w_1^nP(w_1) - w_2^nP(w_2)$.
(2) If $i = m$, then $f(z) \equiv tg(z)$, where t is a constant satisfying $t^{n+m} = 1$ or $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1,c_2 and c are three constants satisfying

$$(-1)^k a_m^2 (c_1c_2)^{n+m}[(n + m)c]^{2k} = 1.$$

Remark. Theorem 1.2 does not hold for the case that $f(z)$ and $g(z)$ are polynomials. For example, let $f(z) = z - 1$, $g(z) = (z - 1)^2$, $P(w) = w - 1$ and $k = 1, m = 1, n = 24$, then $[f^nP(f)]^{(k)}$and $[g^nP(g)]^{(k)}$ share 1 IM, but $R(f, g) \not\equiv 0$.

2. Some Lemmas

Proposition 2.3. *Let $f(z)$ be a transcendental entire function, and n, k, m be three positive integers with $n \geq k + 2$, Then $[f^nP(f)]^{(k)} = 1$ ($a_m \neq 0$) has infinitely many solutions.*

In order to prove the above proposition, we require the following lemmas.

Lemma 2.1([8]). *Let $f(z)$ be a nonconstant meromorphic function, and let*

$a_m(z)\,(a_m \not\equiv 0)$, $a_{m-1}(z), \cdots, a_0(z)$ be meromorphic functions such that $T(r, a_i) = S(r, f)$, $i = 0, 1, 2, \cdots, n$. Then

$$T(r, a_m f^m + a_{m-1} f^{m-1} + \cdots + a_0) = mT(r, f) + S(r, f).$$

Lemma 2.2([4], [10])**.** *Let $f(z)$ be a transcendental entire function, let k be a positive integer, and let c be a nonzero finite complex number. Then*

$$\begin{aligned} T(r,f) &\leq N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leq N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f), \end{aligned}$$

where $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Proof of Proposition. By Lemma 1 and Lemma 2, we have

$$\begin{aligned} &(n+m)T(r,f) \\ &\quad = T(r, f^n P(f)) + S(r,f) \\ &\quad \leq N_{k+1}\left(r, \frac{1}{f^n P(f)}\right) + \overline{N}\left(r, \frac{1}{[f^n P(f)]^{(k)} - 1}\right) + S(r,f) \\ &\quad \leq N_{k+1}\left(r, \frac{1}{f^n}\right) + N_{k+1}\left(r, \frac{1}{P(f)}\right) + \overline{N}\left(r, \frac{1}{[f^n P(f)]^{(k)} - 1}\right) + S(r,f) \\ &\quad \leq (k+1+m)T(r,f) + \overline{N}\left(r, \frac{1}{[f^n P(f)]^{(k)} - 1}\right) + S(r,f). \end{aligned}$$

Thus we get

$$(n-k-1)T(r,f) \leq \overline{N}\left(r, \frac{1}{[f^n P(f)]^{(k)} - 1}\right) + S(r,f). \tag{2.1}$$

So we deduce by (2.1) and $n \geq k+2$ that $[f^n P(f)]^{(k)} = 1$ has infinitely many solutions. □

Lemma 2.3([4], [8])**.** *Let $f(z)$ be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be meromorphic functions such that $T(r, a_i) = S(r, f), i = 1, 2$ and $a_1 \not\equiv a_2$. Then*

$$T(r,f) \leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r,f).$$

Lemma 2.4([5]). *Let $F(z)$ and $G(z)$ be two transcendental entire functions such that* $\Theta(0,F) > \frac{5k+6}{5k+7}$, $\Theta(0,G) > \frac{5k+6}{5k+7}$. *If $F(z)^{(k)}$ and $G(z)^{(k)}$ share the value 1 IM, then either $F(z)^{(k)}G(z)^{(k)} \equiv 1$ or $F(z) \equiv G(z)$.*

3. Proof of the main results

Proof of Theorem 1.2. (i) Firstly we consider the case: $a_0 \neq 0$. Let

$$F = f^n P(f), \qquad G = g^n P(g). \tag{3.1}$$

Thus we obtain that F and G share 1 IM. Moreover, by Lemma 2.2, we have

$$T(r,F) = (n+m)T(r,f) + S(r,f), \tag{3.2}$$

$$T(r,G) = (n+m)T(r,g) + S(r,f). \tag{3.3}$$

In the view of the assumption $n+m > (5k+7)(m+1)$, we get

$$\Theta(0,F) \geq 1 - \frac{m+1}{m+n} > \frac{5k+6}{5k+7}\ , \tag{3.4}$$

$$\Theta(0,G) \geq 1 - \frac{m+1}{m+n} > \frac{5k+6}{5k+7}\ , \tag{3.5}$$

since

$$\begin{aligned}\overline{N}\left(r, \frac{1}{F}\right) &\leq \overline{N}\left(r, \frac{1}{f^n}\right) + \overline{N}\left(r, \frac{1}{P(f)}\right)\\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + \sum_{l=1}^{m} \overline{N}\left(r, \frac{1}{f-\lambda_l}\right)\\ &\leq (m+1)T(r,f),\end{aligned}$$

where λ_l satisfies $P(\lambda_l) = 0$. Hence by (3.4), (3.5) and Lemma 2.4 we deduce that

$$F(z)^{(k)}G(z)^{(k)} \equiv 1 \ \textit{or}\ F(z) \equiv G(z).$$

Next we consider the following two cases:
Case1. $F(z)^{(k)}G(z)^{(k)} \equiv 1$, that is

$$\left[f^n P(f)\right]^{(k)} \left[g^n P(g)\right]^{(k)} \equiv 1. \tag{3.6}$$

By $f(z)$ and $g(z)$ are two nonconstant entire functions and $n+m > (5k+7)(m+1)$, we deduce

$$f(z) \neq 0, \quad g(z) \neq 0.$$

Let $f(z)=e^{\alpha(z)}$, where α is a nonconstant entire function, so $T(r,\alpha')=S(r,f)$. Thus, by induction we get

$$\left[a_m f^{m+n}\right]^{(k)}=p_m(\alpha',\alpha'',\cdots,\alpha^{(k)})e^{(m+n)\alpha(z)},$$

$$\vdots$$

$$[a_0 f^n]^{(k)}=p_0(\alpha',\alpha'',\cdots,\alpha^{(k)})e^{n\alpha(z)},$$

where $p_i(\alpha',\alpha'',\cdots,\alpha^{(k)})$ $(i=0,1,\cdots,m)$ are differential polynomials. Obviously,

$$p_m(\alpha',\alpha'',\cdots,\alpha^{(k)})\not\equiv 0 \quad and \quad p_0(\alpha',\alpha'',\cdots,\alpha^{(k)})\not\equiv 0.$$

Considering g is an entire function, we obtain from (3.6) that $[f^nP(f)]^{(k)}\neq 0$, that is

$$p_m(\alpha',\alpha'',\cdots,\alpha^{(k)})e^{m\alpha(z)}+\cdots+p_0(\alpha',\alpha'',\cdots,\alpha^{(k)})\neq 0.$$

Since $T(r,\alpha^{(j)})\leq T(r,\alpha')+S(r,f)=S(r,f)$, for $j=1,2,\cdots,k$. We deduce that

$$T(r,p_m)=S(r,f),\cdots,T(r,p_0)=S(r,f).$$

Note that $f=e^{\alpha}$. Thus, we have

$$\begin{aligned}
&mT(r,f)\\
&= T(r,p_me^{m\alpha}+\cdots+p_1e^{\alpha})+S(r,f)\\
&\leq \overline{N}\left(r,\frac{1}{p_me^{m\alpha}+\cdots+p_1e^{\alpha}}\right)+\overline{N}\left(r,\frac{1}{p_me^{m\alpha}+\cdots+p_1e^{\alpha}+p_0}\right)+S(r,f)\\
&\leq \overline{N}\left(r,\frac{1}{p_me^{(m-1)\alpha}+\cdots+p_1}\right)+S(r,f)\\
&\leq (m-1)T(r,f)+S(r,f),
\end{aligned}$$

which is a contradiction.

Case2. $F(z)\equiv G(z)$, that is $f^nP(f)\equiv g^nP(g)$, Then f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $R(w_1,w_2)=w_1^nP(w_1)-w_2^nP(w_2)$.

(ii)If $a_j=0\,(0\leq j<i)$ and $a_i\neq 0$, then

$$F=f^{n+i}\left(a_mf^{m-i}+\cdots+a_i\right),\quad G=g^{n+i}\left(a_mg^{m-i}+\cdots+a_i\right).$$

By the assumption $n+m>(5k+7)\,(m+1)$, we get

$$\Theta(0,F)\geq 1-\frac{m-i+1}{m+n}>\frac{5k+6}{5k+7}\,, \tag{3.7}$$

$$\Theta(0,G)\geq 1-\frac{m-i+1}{m+n}>\frac{5k+6}{5k+7}\,, \tag{3.8}$$

since

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F}\right) &\le \overline{N}\left(r,\frac{1}{f^{n+i}}\right)+\overline{N}\left(r,\frac{1}{a_mf^{m-i}+\cdots+a_i}\right)\\ &\le \overline{N}\left(r,\frac{1}{f}\right)+\sum_{l=0}^{m-i}\overline{N}\left(r,\frac{1}{f-\lambda_l}\right)\\ &\le (m-i+1)\,T(r,f),\end{aligned}$$

where λ_l satisfies $a_m\lambda_l^{m-i}+\cdots+a_i=0$. From the above result, we deduce that f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where $R(w_{1,}w_2)=w_1^{n+i}\left(a_mw_1^{m-i}+\cdots+a_i\right)-w_2^{n+i}\left(a_mw_2^{m-i}+\cdots+a_i\right)=w_1^nP(w_1)-w_2^nP(w_2)$.
(iii) If $a_j=0\,(0\le j<m)$ and $a_m\ne 0$, then

$$F=a_mf^{n+m},\quad G=a_mg^{n+m}.$$

Since $\overline{N}\left(r,\frac{1}{F}\right)=\overline{N}\left(r,\frac{1}{f}\right)<T(r,f)$ and $n+m>(5k+7)(m+1)$, we obtain that

$$\Theta(0,F)\ge 1-\frac{1}{m+n}>\frac{5k+6}{5k+7}\ , \tag{3.9}$$

$$\Theta(0,G)\ge 1-\frac{1}{m+n}>\frac{5k+6}{5k+7}\ . \tag{3.10}$$

Therefore by Lemma 2.4 we deduce that

$$F(z)^{(k)}G(z)^{(k)}\equiv 1\ \ \textit{or}\ \ F(z)\equiv G(z).$$

Next we consider two cases.
Case1. $F(z)^{(k)}G(z)^{(k)}\equiv 1$, that is $[a_mf^{n+m}]^{(k)}\,[a_mg^{n+m}]^{(k)}\equiv 1$. Since $f(z)$ and $g(z)$ are two nonconstant entire functions, we see that

$$\left[a_mf^{n+m}\right]^{(k)}\ne 0,\quad \left[a_mg^{n+m}\right]^{(k)}\ne 0.$$

Proceeding as in the proof of Theorem D, we obtain the desired result, that is $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k\,a_m^2\,(c_1c_2)^{n+m}\,[(n+m)c]^{2k}=1.$$

Case2. $F(z)\equiv G(z)$, that is $f^{n+m}=g^{n+m}$. Hence, we get that $f(z)\equiv tg(z)$, where t is a constant satisfying $t^{n+m}=1$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.1. Proceeding as in the proof of Theorem D, Theorem E and Theorem 1.2, we can get the conclusion of Theorem 1.1. □

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