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Approximating Common Fixed Points of One-step Iterative Scheme with Error for Asymptotically Quasi-nonexpansive Type Nonself-Mappings

Narongrit Puturong
Mathematics Program, Faculty of Science, Udon Thani Rajabhat University, Udon Thani 41000, Thailand
e-mail : naputurong@hotmail.com

Abstract. In this paper, a new one-step iterative scheme with error for approximating common fixed points of asymptotically quasi-nonexpansive type nonself-mappings in Banach space is defined. The results obtained in this paper extend and improve the recent ones, announced by H. Y. Zhou, Y. J. Cho, and S. M. Kang [Zhou et al.,(2007), namely, A new iterative algorithm for approximating common fixed points for asymptotically nonexpansive mappings, published to Fixed Point Theory and Applications 2007 : 1-9], and many others.

1. Introduction

Let X be a real Banach space and let C be a nonempty subset of X. Further, for a mapping $T : C \to C$, let $\phi \neq F(T)$ be the set of all fixed points of T. A mapping $T : C \to C$ is said to be asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\}$ of real number with $k_n \geq 1$ and $\lim_n k_n = 1$ such that for all $x \in C,\ q \in F(T)$

$$\|T^n x - q\| \leq k_n \|x - q\|, \text{ for all } n \geq 1. \tag{1}$$

T is called asymptotically quasi-nonexpansive type [5] provided T is uniformly continuous and

$$\limsup_n \{\sup_{x \in \mathrm{C}} (\|T^n x - q\| - \|x - q\|)\} \leq 0, \text{ for all } q \in \mathrm{F(T)}. \tag{2}$$

The mapping T is called uniformly L-Lipschitzian if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \text{ for all } x, y \in \mathrm{C} \text{ and all } n \geq 1. \tag{3}$$

$T : C \to X$ is completely continuous [6] if for all bounded sequence $\{x_n\} \subset C$ there exists a convergent subsequence of $\{Tx_n\}$.

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Recall that a Banach space X is called uniformly convex [4] if for every $0 < \varepsilon \leq 2$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\left\|\frac{x+y}{2}\right\| \leq 1-\delta$ for every $x, y \in S_X$ and $\|x-y\| \geq \varepsilon,\ S_X = \{x \in X : \|x\| = 1\}$.

A Banach space X is said to be smooth[8] if $\lim\limits_{t \to o} \frac{\|x+ty\| - \|x\|}{t}$ exists for all $x, y \in S_X,\ \ S_X = \{x \in X : \|x\| = 1\}$.

In 2004, S. S. Chang, J. K. Kim, and S. M. Kang [1] gave necessary and sufficient condition for the Ishikawa iterative sequence with mixed errors of asymptotically quasi-nonexpansive type mapping in Banach space to converge to a fixed point in Banach spaces. In 2006, J. Quan, S. S. Chang, and X. J. Long [3] gave necessary and sufficient condition for finite-step iterative sequences with mean errors for a family of asymptotically quasi-nonexpansive and type mapping in Banach spaces to converge to a common fixed point.

Recently, H. Y. Zhou, Y. J. Cho, and S. M. Kang [8] gave a new iterative scheme(5) for approximating common fixed point of two asymptotically nonexpansive nonself-mappings with respect to P and proving some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.

The purpose of this paper is to study approximating common fixed point of two asymptotically quasi-nonexpansive type nonself-mappings with respect to P and to a strong convergence theorem, and to define one-step iterative scheme with error which modified of H. Y. Zhou, Y. J. Cho, and S. M. Kang [8] iteration as follows.

A subset C of X is called retract of X if there exists a continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to C$ is called retraction if $P^2 = P$. Note that if a mapping P is a retraction, then $Pz = z$ for all z in the range of P.

Let D, E be subsets of a Banach space X. Then, a mapping $P : D \to E$ is said to be sunny if $P(Px + t(x - Px)) = Px$, whenever $Px + t(x - Px) \in D$ for all $x \in D$ and $t \geq 0$.

Let C be a subset of a Banach space X. For all $x \in C$ is defined a set $I_C(x)$ by $I_C(x) = \{x + \lambda(y - x) : \lambda > 0,\ y \in C\}$.

A nonself-mapping $T : C \to X$ is said to be inward if $Tx \in I_C(x)$ for all $x \in C$, and T is said to be weakly inward if $Tx \in \overline{I_C(x)}$ for all $x \in C$.

Let X be a real normed linear space and let C be a nonempty closed convex subset of X. Let $P : X \to C$ be the nonexpansive retraction of X onto C and let $T_1 : C \to X$ and $T_2 : C \to X$ be two asymptotically quasi-nonexpansive type nonself-mappings.

Algorithm 1. For a given $x_1 \in C$, computed the sequence $\{x_n\}$ by the iterative scheme

$$x_{n+1} = \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n + \nu_n u_n,\ \forall n \geq 1, \tag{4}$$

where $\{\alpha_n\},\ \{\beta_n\},\ \{\gamma_n\}$ and $\{\nu_n\}$ are real sequences in $[0, 1]$ and satisfying $\alpha_n +$

$\beta_n + \gamma_n + \nu_n = 1$ and $\{u_n\}$ is bounded sequence in C. The iterative scheme (4) is called the one-step iterative scheme with error. If $\nu_n \equiv 0$, then (4) reduced to the iterative scheme, defined by H. Y. Zhou, Y. J. Cho, and S. M. Kang [8], as follows:

Algorithm 2. For a given $x_1 \in C$, computed the sequence $\{x_n\}$ by the iterative scheme

$$x_{n+1} = \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n, \ \forall n \geq 1, \tag{5}$$

where $\{\alpha_n\}, \ \{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $[0, 1]$ and satisfying $\alpha_n + \beta_n + \gamma_n = 1$.

Now, we recall the well known concept, and the following essential lemmas to prove our main results.

Definition 1.1([8, Definition 1.5(2)])**.** Let C be a nonempty subset of real normed linear space X. Let $P : X \to C$ nonexpansive retraction of X onto C. $T : C \to X$ is said to be uniformly L-Lipschitzian nonself-mapping with respect to P if there exists a constant $L > 0$ such that

$$\|(PT)^n x - (PT)^n y\| \leq L\| \ x - y\|, \ \forall x, y \in C, \ n \geq 1. \tag{6}$$

Lemma 1.1([7, Lemma 1])**.** *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $a_{n+1} \leq a_n + b_n$, for all $n \geq 1$.*

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_n a_n$ exists.

Lemma 1.2([2, Lemma 3])**.** *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| < r, \ r > 0\}$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \xi z + \nu w\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \nu\|w\|^2 - \lambda\mu g(\|x - y\|)$$

for all $x, \ y, \ z, \ w \in B_r$ and $\lambda, \mu, \xi, \nu \in [0, 1]$ with $\lambda + \mu + \xi + \nu = 1$.

Lemma 1.3([8, Lemma 2.2])**.** *Let X be a real smooth Banach space, let C be a nonempty closed subset of X with P as a sunny nonexpansive retraction, and let $T : C \to X$ be a mapping satisfying weakly inward condition. Then $F(PT) = F(T)$.*

2. Main results

In this section, we prove convergence theorem of one-step iterative scheme with error for two asymptotically quasi-nonexpansive type nonself-mappings. In order to prove our main results,the following definition and lemmas are needed.

Definition 2.1. Let C be a nonempty subset of real normed linear space X. Let $P : X \to C$ be a nonexpansive retraction of X onto $C, \emptyset \neq F(T)$ be the set of all fixed points of T. $T : C \to X$ is called asymptotically quasi-nonexpansive type nonself-mapping with respect to P if T is uniformly continuous and

$$\lim sup_n\{\sup_{x\in C}(\|(PT)^n x - q\| - \|x - q\|)\} \leq 0, \text{ for all } q \in F(T). \tag{7}$$

Remark 2.1. If T is self-mapping, then P is become the identity mapping, so that (6) and (7) are reduced to (3) and (2), respectively.

Lemma 2.1. *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and $T_1,\ T_2 : C \to X$ be asymptotically quasi-nonexpansive type nonself-mappings with respect to P nonexpansive retraction of X onto C. Put*

$$\begin{aligned} G_n &= \max\{0,\ \sup_{x\in C}(\|(PT_1)^n x - q\| - \|x - q\|)\} \\ \textit{and} \quad K_n &= \max\{0,\ \sup_{x\in C}(\|(PT_2)^n x - q\| - \|x - q\|)\},\ \forall n \geq 1, \end{aligned}$$

so that $\sum_{n=1}^{\infty} G_n < \infty$ and $\sum_{n=1}^{\infty} K_n < \infty$, respectively.
Suppose that $\{x_n\}$ is the sequence defined by (4) with $\sum_{n=1}^{\infty} \nu_n < \infty$.

If $F(T_1) \cap F(T_2) \neq \emptyset$, then $\lim_n \|x_n - q\|$ exists for any $q \in F(T_1) \cap F(T_2)$.

Proof. For any $q \in F(T_1) \cap F(T_2)$, using the fact that P is nonexpansive retraction and (4), then we have

$$\begin{aligned} \|x_{n+1} - q\| &= \big\|[\alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n + \nu_n u_n] - q\big\| \\ &\leq \alpha_n \|x_n - q\| + \beta_n \big\|(PT_1)^n x_n - q\| + \ \gamma_n \big\|(PT_2)^n x_n - q\| + \nu_n \|u_n - q\| \\ &= \alpha_n \|x_n - q\| + \beta_n \big(\|(PT_1)^n x_n - q\| - \|x_n - q\|\big) + \beta_n \|x_n - q\| \\ &\quad + \ \gamma_n \big(\|(PT_2)^n x_n - q\| - \|x_n - q\|\big) + \gamma_n \|x_n - q\| + \nu_n \|u_n - q\| \\ &\leq (\alpha_n + \beta_n + \gamma_n)\|x_n - q\| + \beta_n \sup_{x\in C} \big(\|(PT_1)^n x - q\| - \|x - q\|\big) \\ &\quad + \ \gamma_n \sup_{x\in C} \big(\|(PT_2)^n x - q\| - \|x - q\|\big) + \nu_n \|u_n - q\| \end{aligned}$$

$$\begin{aligned} \|x_{n+1} - q\| &\leq \|x_n - q\| + \beta_n G_n + \gamma_n K_n + \nu_n \|u_n - q\| \\ &= \|x_n - q\| + d_n, \end{aligned}$$

where $d_n = \beta_n G_n + \gamma_n K_n + \nu_n \|u_n - q\|$.
Since $\sum_{n=1}^{\infty} G_n < \infty$, $\sum_{n=1}^{\infty} K_n < \infty$ and $\sum_{n=1}^{\infty} \nu_n < \infty$, we see that $\sum_{n=1}^{\infty} d_n < \infty$.
If follows from Lemma 1.1 that $\lim_n \|x_n - q\|$ exists. The proof is completed. □

Lemma 2.2. *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and $T_1,\ T_2 : C \to X$ be asymptotically quasi-nonexpansive type nonself-mappings with respect to P nonexpansive retraction of X onto C and T_1, T_2*

be uniformly L-Lipschitzian nonself-mappings with respect to P. Put

$$\begin{aligned} G_n &= \max\{0,\ \sup_{x\in C}(\|(PT_1)^n x - q\| - \|x-q\|)\} \\ \text{and} \quad K_n &= \max\{0,\ \sup_{x\in C}(\|(PT_2)^n x - q\| - \|x-q\|)\},\ \forall n \geq 1, \end{aligned}$$

so that $\sum_{n=1}^{\infty} G_n < \infty$ *and* $\sum_{n=1}^{\infty} K_n < \infty$*, respectively.*

Suppose that $\{x_n\}$ *is the sequence defined by (4) with* $\sum_{n=1}^{\infty} \nu_n < \infty$*, and the additional assumption that* $0 < \lim_n \inf \alpha_n,\ 0 < \lim_n \inf \beta_n$ *and* $0 < \lim_n \inf \gamma_n$*.*

(8) *If* $F(T_1) \cap F(T_2) \neq \emptyset$*, then* $lim_n \|x_n - (PT_1)x_n\| = lim_n \|x_n - (PT_2)x_n\| = 0$.

Proof. By lemma 2.1, $\lim_n \|x_n - q\|$ exists for any $q \in F(T_1) \cap F(T_2)$. If follows that $\{x_n - q\}$, $\{(PT_1)^n x_n - q\}$ and $\{(PT_2)^n x_n - q\}$ are bounded. Also, $\{u_n - q\}$ is bounded by the assumption. We may assume that such sequences belong to B_r, where $B_r = \{x \in X : \|x\| < r,\ r > 0\}$.

From (4) by the property of P, and Lemma 1.2, we have

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &\quad = \big\|[\alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n + \nu_n u_n] - q\big\|^2 \\ &\quad = \big\|\alpha_n (x_n - q) + \beta_n\big((PT_1)^n x_n - q\big) + \gamma_n\big((PT_2)^n x_n - q\big) + \nu_n (u_n - q)\big\|^2 \end{aligned}$$

so

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &\quad \leq \alpha_n \|x_n - q\|^2 + \beta_n \|(PT_1)^n x_n - q\|^2 + \gamma_n \|(PT_2)^n x_n - q\|^2 + \nu_n \|u_n - q\|^2 \\ &\qquad - \alpha_n \beta_n g\big(\|x_n - (PT_1)^n x_n\|\big) \\ &\quad = \alpha_n \|x_n - q\|^2 + \beta_n\big(\|(PT_1)^n x_n - q\|^2 - \|x_n - q\|^2\big) + \beta_n \|x_n - q\|^2 \\ &\qquad + \gamma_n\big(\|(PT_2)^n x_n - q\|^2 - \|x_n - q\|^2\big) + \gamma_n \|x_n - q\|^2 \\ &\qquad + \nu_n \|u_n - q\|^2 - \alpha_n \beta_n g\big(\|x_n - (PT_1)^n x_n\|\big), \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &\quad \leq \alpha_n \|x_n - q\|^2 + \beta_n \|(PT_1)^n x_n - q\|^2 + \gamma_n \|(PT_2)^n x_n - q\|^2 + \nu_n \|u_n - q\|^2 \\ &\qquad - \alpha_n \gamma_n g\big(\|x_n - (PT_2)^n x_n\|\big) \\ &\quad = \alpha_n \|x_n - q\|^2 + \beta_n\big(\|(PT_1)^n x_n - q\|^2 - \|x_n - q\|^2\big) + \beta_n \|x_n - q\|^2 \\ &\qquad + \gamma_n\big(\|(PT_2)^n x_n - q\|^2 - \|x_n - q\|^2\big) + \gamma_n \|x_n - q\|^2 \\ &\qquad + \nu_n \|u_n - q\|^2 - \alpha_n \gamma_n g\big(\|x_n - (PT_2)^n x_n\|\big). \end{aligned} \tag{10}$$

From (9);

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &\quad \leq \alpha_n \|x_n - q\|^2 + \beta_n\big(\|(PT_1)^n x_n - q\| + \|x_n - q\|\big)\big(\|(PT_1)^n x_n - q\| - \|x_n - q\|\big) \\ &\qquad + \beta_n \|x_n - q\|^2 + \gamma_n\big(\|(PT_2)^n x_n - q\| + \|x_n - q\|\big)\big(\|(PT_2)^n x_n - q\| - \|x_n - q\|\big) \\ &\qquad + \gamma_n \|x_n - q\|^2 + \nu_n \|u_n - q\|^2 - \alpha_n \beta_n g\big(\|x_n - (PT_1)^n x_n\|\big) \end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n\|x_n-q\|^2+\beta_n\big(\|(PT_1)^nx_n-q\|+\|x_n-q\|\big)\sup_{x\in C}\big(\|(PT_1)^nx-q\|-\|x-q\|\big)\\
&\quad+\ \beta_n\|x_n-q\|^2+\gamma_n\big(\|(PT_2)^nx_n-q\|+\|x_n-q\|\big)\sup_{x\in C}\big(\|(PT_2)^nx-q\|-\|x-q\|\big)\\
&\quad+\ \gamma_n\|x_n-q\|^2+\nu_n\|u_n-q\|^2-\alpha_n\beta_n g\big(\|x_n-(PT_1)^nx_n\|\big)\\
&\le (\alpha_n+\beta_n+\gamma_n)\|x_n-q\|^2+\beta_n\big(\|(PT_1)^nx_n-q\|+\|x_n-q\|\big)G_n\\
&\quad+\gamma_n\big(\|(PT_2)^nx_n-q\|+\|x_n-q\|\big)K_n+\nu_n\|u_n-q\|^2-\alpha_n\beta_n g\big(\|x_n-(PT_1)^nx_n\|\big)\\
&\le \|x_n-q\|^2+\beta_n\big(\|(PT_1)^nx_n-q\|+\|x_n-q\|\big)G_n\\
&\quad+\gamma_n\big(\|(PT_2)^nx_n-q\|+\|x_n-q\|\big)K_n+\nu_n\|u_n-q\|^2-\alpha_n\beta_n g\big(\|x_n-(PT_1)^nx_n\|\big),\\
&\|x_{n+1}-q\|^2\\
&\quad\le \|x_n-q\|^2+w_nG_n+v_nK_n+\nu_n\|u_n-q\|^2-\alpha_n\beta_n g\big(\|x_n-(PT_1)^nx_n\|\big),
\end{aligned}
$$

where $w_n=\beta_n\big(\|(PT_1)^nx_n-q\|+\|x_n-q\|\big)$ and $v_n=\gamma_n\big(\|(PT_2)^nx_n-q\|+\|x_n-q\|\big)$. Then

$$
\begin{aligned}
&\alpha_n\beta_n g\big(\|x_n-(PT_1)^nx_n\|\big)\\
&\qquad\le \|x_n-q\|^2-\|x_{n+1}-q\|^2+w_nG_n+v_nK_n+\nu_n\|u_n-q\|^2. \qquad (11)
\end{aligned}
$$

Since $\sum\limits_{n=1}^{\infty}G_n<\infty$, $\sum\limits_{n=1}^{\infty}K_n<\infty$, $\sum\limits_{n=1}^{\infty}\nu_n<\infty$, and $\lim_n\|x_n-q\|$ exists, it follow (11) that

$$\limsup_n \alpha_n\beta_n g\big(\|x_n-(PT_1)^nx_n\|\big)=0.$$

From g is continuous strictly increasing with $g(0)=0$ and $0<\liminf_n\alpha_n$, $0<\liminf_n\beta_n$, we have $\lim_n\|x_n-(PT_1)^nx_n\|=0$.
By using a similar method, together with inequality (10), it can be shown that $\lim_n\|x_n-(PT_2)^nx_n\|=0$.

Next, we show that $\lim_n\|x_n-(PT_1)x_n\|=0$ and $\lim_n\|x_n-(PT_2)x_n\|=0$. Using (4), we have

$$\|x_{n+1}-x_n\|\le\beta_n\|(PT_1)^nx_n-x_n\|+\gamma_n\|(PT_2)^nx_n-x_n\|+\nu_n\|u_n-x_n\|.$$

Since $\sum\limits_{n=1}^{\infty}\nu_n<\infty$, and $\|(PT_1)^nx_n-x_n\|\to0$, $\|(PT_2)^nx_n-x_n\|\to0$ as $n\to\infty$. Then $\|x_{n+1}-x_n\|\to0$ as $n\to\infty$.
We consider

$$
\begin{aligned}
&\|x_n-(PT_1)x_n\|\\
&\quad\le\|x_n-x_{n+1}\|+\|x_{n+1}-(PT_1)^{n+1}x_{n+1}\|\\
&\qquad+\|(PT_1)^{n+1}x_{n+1}-(PT_1)^{n+1}x_n\|+\|(PT_1)^{n+1}x_n-(PT_1)x_n\|\\
&\quad\le\|x_n-x_{n+1}\|+\|x_{n+1}-(PT_1)^{n+1}x_{n+1}\|\\
&\qquad+\ L\|x_{n+1}-x_n\|+L\|(PT_1)^nx_n-x_n\|,\ \text{for some } L\in\mathbb{R}^+ \qquad (12)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|x_n - (PT_2)x_n\| \\
&\quad \le \|x_n - x_{n+1}\| + \|x_{n+1} - (PT_2)^{n+1}x_{n+1}\| \\
&\qquad + \ \|(PT_2)^{n+1}x_{n+1} - (PT_2)^{n+1}x_n\| + \ \|(PT_2)^{n+1}x_n - (PT_2)x_n\| \\
&\quad \le \|x_n - x_{n+1}\| + \|x_{n+1} - (PT_2)^{n+1}x_{n+1}\| \\
&\qquad + \ L\|x_{n+1} - x_n\| + \ L\|(PT_2)^n x_n - x_n\|, \text{ for some } L \in \mathbb{R}^+ \qquad (13)
\end{aligned}
$$

Since $\|x_n - (PT_1)^n x_n\| \to 0$, $\|x_n - (PT_2)^n x_n\| \to 0$, $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and uniformly continuity T_1, T_2, together with inequalities (12) and (13) it can be shown that $\|x_n - (PT_1)x_n\| \to 0$, $\|x_n - (PT_2)x_n\| \to 0$, as $n \to \infty$, respectively. □

Remark 2.2. Expressions (12) and (13) : assumption T_1 and T_2 are uniformly L-Lipschitzian nonself-mapping with respect to P as same as assume that the mappings PT_1 and PT_2 are uniformly L-Lipschitzian.

Theorem 2.3. *Let C be a nonempty closed convex subset of a real smooth and uniformly convex Banach space X. Let $T_1,\ T_2 : C \to X$ be two weakly inward and asymptotically quasi-nonexpansive type nonself-mappings with respect to P as a sunny nonexpansive retraction of X onto C and T_1, T_2 be uniformly L-Lipschitzian nonself-mappings with respect to P. Put*

$$
\begin{aligned}
G_n &= \max\{0,\ \sup_{x\in C}(\|(PT_1)^n x - q\| - \|x - q\|)\} \\
\text{and} \quad K_n &= \max\{0,\ \sup_{x\in C}(\|(PT_2)^n x - q\| - \|x - q\|)\},\ \forall n \ge 1,
\end{aligned}
$$

so that $\sum\limits_{n=1}^{\infty} G_n < \infty$ and $\sum\limits_{n=1}^{\infty} K_n < \infty$, respectively.
Suppose that $\{x_n\}$ is the sequence defined by (4) with $\sum\limits_{n=1}^{\infty} \nu_n < \infty$, and the additional assumption that $0 < \lim_n \inf \alpha_n,\ 0 < \lim_n \inf \beta_n$ and $0 < \lim_n \inf \gamma_n$.
If one of T_1 and T_2 is completely continuous and $F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ is converged strongly to a common fixed point of T_1 and T_2.

Proof. From Lemma 2.1, we know that $\lim_n \|x_n - q\|$ exists for any $q \in F(T_1) \cap F(T_2)$, then $\{x_n\}$ is bounded. By Lemma 2.2, we have

$$
\lim_n \|x_n - (PT_1)x_n\| = 0 \text{ and } \lim_n \|x_n - (PT_2)x_n\| = 0. \qquad (14)
$$

Suppose that T_1 is completely continuous, and noting that $\{x_n\}$ is bounded. We conclude that there exists subsequence $\{PT_1 x_{n_j}\}$ of $\{PT_1 x_n\}$ such that $\{PT_1 x_{n_j}\}$ converges. Therefore, from (14), $\{x_{n_j}\}$ is converged. Let $x_{n_j} \to r$ as $j \to \infty$. By the continuity of P, T_1, T_2 and (14), we have $r = PT_1 r = PT_2 r$. Since $F(PT_1) = F(T_1)$ and $F(PT_2) = F(T_2)$ by Lemma 1.3, we have $r = T_1 r = T_2 r$. Thus, $\{x_n\}$ is converged strongly to a common fixed point r of T_1 and T_2. The proof is completed. □

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