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On the Iterated Duggal Transforms

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Abstract. For a bounded operator $T = U|T|$ (polar decomposition), we consider a transform $\widehat{T} = |T|U$ and discuss the convergence of iterated transform of $\widehat{T}$ under the strong operator topology. We prove that such iteration of quasiaffine hyponormal operator converges to a normal operator under the strong operator topology.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ has the unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{1/2}$ and U is a partial isometry with initial space $(\mathrm{ran}|T|)^-$, the closure of the range of $|T|$, and final space $(\mathrm{ran}T)^-$, the closure of the range of T. The *Aluthge transform* $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, which was first studied in [1] and was applied to the study of the invariant subspace problem via the sequence $\{\widetilde{T}^{(n)}\}_{n\in\mathbb{N}}$ of Aluthge iterates of T, defined by $\widetilde{T}^{(1)} := \widetilde{T}$ and $\widetilde{T}^{(n+1)} := (\widetilde{T}^{(n)})^{\sim}$ for $n \in \mathbb{N}$, in [8]. The problem whether Aluthge iteration of bounded operators on a Hilbert space $\mathcal{H}$ is convergent was introduced in [8]. If $\dim\mathcal{H} < \infty$, the Aluthge iteration converges to a normal operator (cf. [3], [2]). However, in general such iterations do not always converge under the strong operator topology (SOT) (cf. [4]). Moreover, the problem whether the hyponormal operators on $\mathcal{H}$ with $\dim\mathcal{H} = \infty$ has the Aluthge iteration converging to a normal operator in $\mathcal{L}(\mathcal{H})$ under the strong operator topology remains still open. In this paper we consider the transform $\widehat{T} = |T|U$, which is referred as *the Duggal transform* of T in [5], and define $\widehat{T}^{(1)} := \widehat{T}$ and $\widehat{T}^{(n+1)} := \widehat{(\widehat{T}^{(n)})}$, $n \in \mathbb{N}$, similarly.

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In this note we prove that if a quasiaffinity $T \in \mathcal{L}(\mathcal{H})$ is hyponormal, then the iteration of $\{\widehat{T}^{(n)}\}_{n\in\mathbb{N}}$ converges to a normal operator under SOT, whose spectrum is contained in the spectrum of T. In addition we give a sufficient condition with the iteration of $\{\widehat{T}^{(n)}\}_{n\in\mathbb{N}}$ for invariant subspaces for hyponormal operators in $\mathcal{L}(\mathcal{H})$.

2. Duggal iterations

We begin with some elementary properties, but sets forth basic relations between T and $\widehat{T}$ that will be useful in the proofs below and the sequel research.

Basic properties (BP). Let $T = U|T|$ (polar decomposition) be in $\mathcal{L}(\mathcal{H})$. Then the following statements hold.
1) $|T|\,T = \widehat{T}|T|$, $U\widehat{T} = TU$, $|T|^{\frac{1}{2}}\widetilde{T} = \widehat{T}|T|^{\frac{1}{2}}$, and $\widetilde{T}(|T|^{\frac{1}{2}}U) = (|T|^{\frac{1}{2}}U)\widehat{T}$.
2) T is a quasiaffinity if and only if $|T|$ is a quasiaffinity and U is a unitary operator. And $\widehat{T}$ is a quasiaffinity if T is. In this case, T and $\widehat{T}$ are unitarily equivalent. Also, $\widehat{T}$ and $\widetilde{T}$ are quasisimilar.
3) The transform $\widehat{\cdot}: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is SOT-continuous when restricted to the set of quasiaffinities.
4) $\sigma(T) = \sigma(\widehat{T})$, where $\sigma(\cdot)$ is the spectrum.
5) $\widehat{T}$ is invertible if and only if $\widetilde{T}$ is, and in this case, $\widehat{T}$ and $\widetilde{T}$ are similar.

The following theorem is contained in the main theorems of this paper.

Theorem 2.1. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is a quasiaffine hyponormal operator. Then the sequence $\{\widehat{T}^{(n)}\}_{n\in\mathbb{N}}$ converges in* SOT *to a normal operator $\widehat{T}_L$ such that $\sigma(\widehat{T}_L) \subset \sigma(T)$, $\|\widehat{T}_L\| = \|T\|$, or equivalently, $r(\widehat{T}_L) = r(T)$, where $r(\cdot)$ denotes the spectral radius.*

Proof. Suppose $T = U|T|$ (polar decomposition), where U is a unitary operator (via BP 2)). Then $|\widehat{T}| = (U^*|T|^2U)^{\frac{1}{2}} = U^*|T|U$, and so $\widehat{T}$ has the polar decomposition $\widehat{T} = U\,|\widehat{T}|$. Thus $\widehat{T}^{(2)} = |\widehat{T}|\,U = U^*|T|\,U^2$. By the mathematical induction we can see that

$$\widehat{T}^{(n+1)} = U^{*n}|T|\,U^{n+1} \quad \text{for } n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{1}$$

On the other hand, if $T = U\,|T|$ is hyponormal then $|T|^2 = T^*T \geq TT^* = U\,|T|^2U^*$, and hence $|T| \geq U\,|T|\,U^*$, or equivalently,

$$U^*|T|\,U \geq |T|. \tag{2}$$

From (2) we can see that $\{U^{*n}|T|\,U^n\}_{n\in\mathbb{N}}$ is a bounded increasing sequence of Hermitian operators. Therefore by the argument of [6, Prob. 120], $\{U^{*n}|T|\,U^n\}_{n\in\mathbb{N}}$ converges in SOT to a Hermitian operator H. Thus $\{\widehat{T}^{(n)}\}_{n\in\mathbb{N}}$ converges in SOT to HU, namely, $\widehat{T}_L$. In particular, since $U^*HU = H$ it follows that H commutes with U and therefore $\widehat{T}_L$ is normal.

Towards the norm equality, since H is the SOT-limit of $\{U^{*n}|T|\,U^n\}_{n\in\mathbb{N}}$, by (2) we have that

$$U^{*n}|T|\,U^n \le H \quad \text{for } n = 0, 1, 2, \cdots,$$

and hence

$$\|\,|T|\,\| = \sup_{\|x\|=1} \langle |T|x, x\rangle \le \sup_{\|x\|=1} \langle Hx, x\rangle = \|H\|.$$

Therefore

$$\|T\| = \|U|T|\| = \||T|\| \le \|H\| = \|HU\| = \|\widehat{T}_L\|.$$

On the other hand, recall ([7, Cor. 3]) that if $S, S_n \in \mathcal{L}(\mathcal{H})$, for $n \in \mathbb{N}$, are hyponormal operators such that S_n converges in SOT to S, then $\sigma_{ap}(S) \subseteq \liminf \sigma(S_n)$, where $\sigma_{ap}(\cdot)$ denotes the approximate point spectrum. Applying this result with $S_n = \widehat{T}^{(n)}$ and $S = \widehat{T}_L$, by BP 4) we have that

$$\sigma_{ap}(\widehat{T}_L) \subseteq \sigma(T) \quad \text{and so} \quad \sigma(\widehat{T}_L) \subset \sigma\,(T)\,. \tag{3}$$

Thus $r(\widehat{T}_L) \le r(T)$. However since every hyponormal operator is normaloid, i.e., norm equals spectral radius, it follows that $\|\widehat{T}_L\| \le \|T\|$. Therefore we can conclude that $\|\widehat{T}_L\| = \|T\|$, or equivalently, $r(\widehat{T}_L) = r(T)$. □

In Theorem 2.1, the sequence $\{\widehat{T}^{(n)}\}$ does not always converge in SOT without the condition of hyponormality (see Example 2.2 below).

Example 2.2. Let P be a positive quasiaffinity on $\mathcal{H}$ with $2 \le \dim \mathcal{H} \le \infty$ and let $U \in \mathcal{L}(\mathcal{H})$ be unitary such that $|T|U \ne U^*|T|$ and $U^2 = 1$. Consider an operator $T = UP$. Then, by BP 2) T is quasiaffinity. Recall from (1) that $\widehat{T}^{(n+1)} = U^{*n}|T|U^{n+1}$ $(n \in \mathbb{N}_0)$, which implies that

$$\widehat{T}^{(n)} = \begin{cases} U^*|T| & \text{if } n \text{ is odd,} \\ |T|U & \text{if } n \text{ is even.} \end{cases}$$

Hence $\{\widehat{T}^{(n)}\}_{n\in\mathbb{N}}$ does not converge under SOT in $\mathcal{L}(\mathcal{H})$.

In Theorem 2.1 the property $\|\widehat{T}_L\| = \|T\|$ does not hold in general (see Proposition 2.3 below).

Proposition 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a nilpotent operator of order $m \ge 2$. Then $(\widehat{T})^{m-1} = 0$ and $\widehat{T}^{(m-1)} = 0$. Therefore $\widehat{T}_L = 0$.*

Proof. It follows from [9, Prop. 4.6] that $(\widetilde{T})^{m-1} = 0$. Since $(\widehat{T})^{m-1}|T|^{1/2} = |T|^{1/2}(\widetilde{T})^{m-1} = 0$, $(\widehat{T})^{m-1} = 0$ on the space $(\mathrm{ran}|T|)^-$. And also, for $h \in ((\mathrm{ran}|T|)^-)^\perp$, obviously $(\widehat{T})^{m-1}h = 0$. To prove the second part of this proposition, we will claim from the mathematical induction that if $T^k = 0$, then $\widehat{T}^{(k-1)} = 0$. For $k = 2$, if $T^2 = 0$, then $\widetilde{T} = 0$ (because, $|T|U|T| = 0$, which implies from [9, Lem. 4.5] that $|T|^{1/2}U|T|^{1/2} = 0$) and so, by BP 1), we have $\widehat{T}|T|^{1/2} = |T|^{1/2}\widetilde{T} = 0$. Hence it is easy to see that $\widehat{T} = 0$. Assume that the assertion holds for $2 \le k \le m-1$.

Suppose $T^m = 0$. Then, since $(\widehat{T})^{m-1} = 0$, by the induction hypothesis, we have $\widehat{(\widehat{T})}^{(m-2)} = 0$, i.e., $\widehat{T}^{(m-1)} = 0$. Hence the proof is complete. $\square$

Remark 2.4. We don't know whether every hyponormal operator T in $\mathcal{L}(\mathcal{H})$ always has the Duggal iteration convergence $\widehat{T}_L$ under SOT.

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References

[1] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory, **13**(1990), 307-315.

[2] T. Ando and T. Yamazaki, *The iterated Aluthge transforms of a 2-by-2 matrix converge*, Linear Algebra Appl., **375**(2003), 299-309.

[3] J. Antezana, E. Pujals, and D. Stojanoff, *Convergence of the iterated Aluthge transform sequence for diagonalizable matrices,* Advances Math., **216**(2007), 255-278.

[4] M. Cho, I. Jung, and W. Lee, *On Aluthge transforms of p-hyponormal operators*, Integral Equations Operator Theory, **53**(2005), 321-329.

[5] C. Foias, I. Jung, E. Ko, and C. Pearcy, *Complete contractivity of maps associated with the Aluthge and Duggal transforms*, Pacific J. Math., **209**(2003), 249-359.

[6] P. R. Halmos, A Hilbert space problem book, 2nd ed., New York, 1982.

[7] I. S. Hwang and W. Y. Lee, *The spectrum is continuous on the set of p-hyponormal operators*, Math. Z., **235**(2000), 151-157.

[8] I. Jung, E. Ko, and C. Pearcy, *Aluthge transforms of operators*, Integral Equations Operator Theory, **37**(2000), 437-448.

[9] I. Jung, E. Ko, and C. Pearcy, *The iterated Aluthge transform of an operator*, Integral Equations Operator Theory, **45**(2003), 375-387.