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On a Class of Meromorphic Functions Defined by Certain Linear Operators

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Abstract. In the present investigation, we introduce new classes of p-valent meromorphic functions defined by Liu-Srivastava linear operator and the multiplier transform and study their properties by using certain first order differential subordination and superordination.

1. Introduction

Let $\mathscr{H}$ be the class of functions analytic in $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathscr{H}(a,n)$ be the subclass of $\mathscr{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$ and set $\mathscr{H}_1 := \mathscr{H}(1,1)$. Let Σ_p denote the class of all analytic functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (z \in \Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}; p \in \mathbb{N}) \tag{1.1}$$

and set $\Sigma := \Sigma_1$.

For two functions $f(z)$ given by (1.1) and $g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k$, the *Hadamard product* (*or convolution*) of f and g is defined by

$$(f * g)(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{1.2}$$

For $\alpha_j \in \mathbb{C}$ $(j = 1, 2, \cdots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$ $(j = 1, 2, \cdots m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z)$ is defined by the

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infinite series

$$_lF_m(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z):=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n}\frac{z^n}{n!}$$

$$(l\le m+1;l,m\in\mathbb{N}_0:=\{0,1,2,\cdots\})$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n:=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1, & (n=0);\\ a(a+1)(a+2)\dots(a+n-1), & (n\in\mathbb{N}:=\{1,2,3\cdots\}).\end{cases}$$

Corresponding to the function

$$h_p(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z):=z^{-p}\ {}_lF_m(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z),$$

the *Liu-Srivastava operator* [7] $\mathcal{H}_p^{(l,m)}(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m):\Sigma_p\mapsto\Sigma_p$ is defined by the Hadamard product

$$\begin{aligned}\mathcal{H}_p^{(l,m)}(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m)f(z)&:=h_p(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z)*f(z)\\ &=\frac{1}{z^p}+\sum_{n=1-p}^{\infty}\frac{(\alpha_1)_{n+p}\cdots(\alpha_l)_{n+p}}{(\beta_1)_{n+p}\cdots(\beta_m)_{n+p}}\frac{a_nz^n}{(n+p)!}.\end{aligned}$$

To make the notation simple, we write

$$\mathcal{H}_p^{l,m}[\alpha_1]f(z):=\mathcal{H}_p^{(l,m)}(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m)f(z).$$

Special cases of the Liu-Srivastava linear operator includes the meromorphic analogue of the Carlson-Shaffer linear operator

$$\mathcal{L}_p(a,c):=\mathcal{H}_p^{(2,1)}(a,1;c) \tag{1.3}$$

considered by Liu [6] and the operator

$$\mathcal{D}^{n+p-1}:=\mathcal{L}_p(n+p,1) \tag{1.4}$$

investigated by Yang [14]. When $p=1$, the operator was first introduced by Ganigi and Uralegaddi [5] and then generalized by Yang [13] to an operator analogous to the Ruscheweyh derivative operator, and the operator $\mathcal{J}_{c,p}=\mathcal{L}_p(c,c+1)$ was studied, for example, by Uralegaddi and Somanatha [12]. Note that

$$\mathcal{J}_{c,p}f=\frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt\quad(c>0).$$

Motivated by the operator studied by Aouf and Hossen [1] (see also [4], [6], [10]), we define the operator $\mathcal{I}_p(n,\lambda)$ on Σ_p by the following infinite series

$$\mathcal{I}_p(n,\lambda)f(z):=\frac{1}{z^p}+\sum_{k=1-p}^{\infty}\left(\frac{k+\lambda}{\lambda-p}\right)^n a_kz^k\quad(\lambda>p,n\in\mathbb{Z}). \tag{1.5}$$

A function $f(z) \in \Sigma_p$ is said to be in the class $\Omega_p^{l,m}(\alpha_1; A, B)$ if it satisfies the following subordination:

$$\frac{(\mathcal{H}_p^{l,m}[\alpha_1+1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} \prec 1 - \frac{p(A-B)z}{\alpha_1(1+Bz)}$$

$$(z \in \Delta;\ \ \alpha_1 \in \mathbb{C}; -1 < B < A \leq 1;\ p, l, m \in \mathbb{N})$$

This class $\Omega_p^{l,m}(\alpha_1; A, B)$ was introduced by Liu and Srivastava [7] and they have proved the following:

Theorem 1.1([7, Theorem 1, p.23])**.** *Let α_1 be real number. If*

$$\alpha_1 \geq \frac{p(A-B)}{1+B} \qquad (-1 < B < A \leq 1;\ p \in \mathbb{N}),$$

then

$$\Omega_p^{l,m}(\alpha_1+1; A, B) \subset \Omega_p^{l,m}(\alpha_1; A, B). \tag{1.6}$$

Theorem 1.2([7, Theorem 2, p.25])**.** *Let λ be a complex number such that*

$$\Re(\lambda) > \frac{p(A-B)}{1+B} \qquad (-1 < B < A \leq 1;\ p \in \mathbb{N}).$$

If $f \in \Omega_p^{l,m}(\alpha_1; A, B)$, then the function

$$F(z) = \frac{\lambda}{z^{\lambda+p}} \int_0^z t^{\lambda+p-1} f(t)dt$$

also belongs to the class $\Omega_p^{l,m}(\alpha_1; A, B)$.

For two analytic functions f and F, we say that F is *superordinate* to f if f is subordinate to F. Recently Miller and Mocanu [9] considered certain second order differential superordinations. Using the results of Miller and Mocanu [9], Bulboaca [3] has considered certain classes of first order differential superordinations and Bulboaca [2] considered certain superordination-preserving integral operators.

In recent years, many results of various interesting subclasses of the class Σ_p of meromorphically p-valent functions were investigated extensively by (among others) Aouf *et al.* [1], Liu and Srivasava [7], Ravichandran *et al.* [11], Uralegaddi and Somanatha [12] and Yang [14]. In this paper, we generalize the above-stated classes of Liu and Srivasava [7] to a more general classes of meromorphic p-valent functions which we define below using differential subordination and superordination.

Definition 1.1. A function $f(z) \in \Sigma_p$ is said to be in $\Omega_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; \varphi)$ if it satisfies the following subordination:

$$\frac{(\mathcal{H}_p^{l,m}[\alpha_1+1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} \prec \varphi(z) \quad (z \in \Delta), \tag{1.7}$$

and is said to be in $\widetilde{\Omega}_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; \varphi)$ if f satisfies the following superordination:

$$\varphi(z) \prec \frac{(\mathcal{H}_p^{l,m}[\alpha_1+1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} \quad (z \in \Delta), \tag{1.8}$$

where $\varphi(z)$ is analytic in Δ and $\varphi(0) = 1$.

To make the notation simple, we write

$$\Omega_p(\alpha_1; \varphi) := \Omega_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; \varphi)$$

and

$$\widetilde{\Omega}_p(\alpha_1; \varphi) := \widetilde{\Omega}_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; \varphi).$$

Also we define the class $\Omega_p(\alpha_1; \varphi_1, \varphi_2)$ by the following:

$$\Omega_p(\alpha_1; \varphi_1, \varphi_2) := \widetilde{\Omega}_p(\alpha_1; \varphi_1) \cap \Omega_p(\alpha_1; \varphi_2).$$

For

$$\varphi(z) = 1 - \frac{p(A-B)z}{\alpha_1(1+Bz)} \quad (z \in \Delta;\ \ \alpha_1 \in \mathbb{C}; -1 < B < A \leq 1;\ p \in \mathbb{N}),$$

the class $\Omega_p(\alpha_1; \varphi)$ reduces to the class $\Omega_p^{l,m}(\alpha_1; A, B)$, introduced and studied by Liu and Srivastava [7].

Definition 1.2. A function $f(z) \in \Sigma_p$ is said to be in the class $\mathcal{M}_p(n, \lambda; \varphi)$ if it satisfies the following subordination:

$$\frac{(\mathcal{I}_p(n+1, \lambda)f(z))'}{(\mathcal{I}_p(n, \lambda)f(z))'} \prec \varphi(z) \quad (f(z) \in \Sigma_p), \tag{1.9}$$

and is said to be in $\widetilde{\mathcal{M}}_p(n, \lambda; \varphi)$ if f satisfies the following superordination:

$$\varphi(z) \prec \frac{(\mathcal{I}_p(n+1, \lambda)f(z))'}{(\mathcal{I}_p(n, \lambda)f(z))'} \quad (f(z) \in \Sigma_p), \tag{1.10}$$

where $\varphi(z)$ is analytic in Δ and $\varphi(0) = 1$. Also we define the class $\mathcal{M}_p(n, \lambda; \varphi_1, \varphi_2)$ by the following:

$$\mathcal{M}_p(n, \lambda; \varphi_1, \varphi_2) := \widetilde{\mathcal{M}}_p(n, \lambda; \varphi_1) \cap \mathcal{M}_p(n, \lambda; \varphi_2).$$

2. Preliminaries

In order to prove our main results we will need to use the next definition and lemmas.

Definition 2.1([9, Definition 2, p.817]). Denote by $\mathcal{Q}$, the set of all functions $f(z)$ that are analytic and injective on $\overline{\Delta} - E(f)$, where

$$E(f) = \{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Lemma 2.1([8]). *Let $q(z)$ be univalent in the unit disk Δ and ϑ and φ be analytic in a domain $\mathbb{D}$ containing $q(\Delta)$ with $\varphi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) := zq'(z)\varphi(q(z))$, $h(z) := \vartheta(q(z)) + Q(z)$. Suppose that either (i) $h(z)$ is convex, or (ii) $Q(z)$ is starlike univalent in Δ. In addition, assume that*

$$\Re \frac{zh'(z)}{Q(z)} > 0 \quad (z \in \Delta).$$

If $p(z)$ is analytic in Δ, with $p(0) = q(0)$, $p(\Delta) \subset \mathbb{D}$ and

$$\vartheta(p(z)) + zp'(z)\varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)) = h(z), \tag{2.1}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2.2([3]). *Let $q(z)$ be univalent in the unit disk Δ and ϑ and φ be analytic in a domain $\mathbb{D}$ containing $q(\Delta)$. Suppose that*

(1) $\Re\left[\vartheta'(q(z))/\varphi(q(z))\right] > 0$ *for* $z \in \Delta$,

(2) $Q(z) := zq'(z)\varphi(q(z))$ *is starlike univalent in* Δ.

If $p(z) \in \mathscr{H}[q(0), 1] \cap \mathcal{Q}$, with $p(\Delta) \subset \mathbb{D}$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ, then

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.2}$$

implies $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

3. The classes $\Omega_p(\alpha_1; \varphi)$ and $\widetilde{\Omega}_p(\alpha_1; \varphi)$

By making use of Lemma 2.1, we first prove the following result:

Theorem 3.1. *Let $\psi(z)$ be univalent in Δ, $\psi(0) = 1$ and $\psi(z) \neq 0$. Assume that $z\psi'/\psi$ is starlike in Δ and $\Re\{\alpha_1\psi(z)\} > 0$. Let $\chi(z)$ be defined by*

$$\chi(z) := \frac{1}{\alpha_1 + 1}\left[\alpha_1\psi(z) + 1 + \frac{z\psi'(z)}{\psi(z)}\right]. \tag{3.1}$$

If $f(z) \in \Omega_p(\alpha_1 + 1; \chi)$, then $f(z) \in \Omega_p(\alpha_1; \psi)$. If $f(z) \in \widetilde{\Omega}_p(\alpha_1 + 1; \chi)$,

$$0 \neq \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} \in \mathscr{H}_1 \cap \mathcal{Q} \text{ and } \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 2]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'} \text{ is univalent in } \Delta, \tag{3.2}$$

then $f(z) \in \widetilde{\Omega}_p(\alpha_1; \psi)$.

Proof. First of all consider the following identity

$$z(\mathcal{H}_p^{l,m}[\alpha_1]f(z))' = \alpha_1 \mathcal{H}_p^{l,m}[\alpha_1+1]f(z) - (\alpha_1+p)\mathcal{H}_p^{l,m}[\alpha_1]f(z), \tag{3.3}$$

which upon differentiation, yields

$$z(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'' = \alpha_1(\mathcal{H}_p^{l,m}[\alpha_1+1]f(z))' - (\alpha_1+p+1)(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'. \tag{3.4}$$

Now define the function $q(z)$ by

$$q(z) := \frac{(\mathcal{H}_p^{l,m}[\alpha_1+1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'}. \tag{3.5}$$

Then, clearly, $q(z)$ is analytic in Δ.
By logarithmic differentiation of (3.5) with respect to z and using (3.4), we obtain

$$\frac{(\mathcal{H}_p^{l,m}[\alpha_1+2]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1+1]f(z))'} = \frac{1}{\alpha_1+1}\left(\alpha_1 q(z) + 1 + \frac{zq'(z)}{q(z)}\right). \tag{3.6}$$

Since $f(z) \in \Omega_p(\alpha_1+1; \chi)$, we have from (3.6) that

$$\alpha_1 q(z) + \frac{zq'(z)}{q(z)} \prec \alpha_1 \psi(z) + \frac{z\psi'(z)}{\psi(z)}$$

and this can be written as (2.1), by defining

$$\vartheta(w) := \alpha_1 w \text{ and } \varphi(w) := \frac{1}{w}.$$

Note that $\varphi(w) \neq 0$ and $\vartheta(w),\ \varphi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$. Set

$$Q(z) := \frac{z\psi'(z)}{\psi(z)} \tag{3.7}$$

$$h(z) := \vartheta(\psi(z)) + Q(z) = \alpha_1\psi(z) + \frac{z\psi'(z)}{\psi(z)}. \tag{3.8}$$

In light of the hypothesis of our Theorem 3.1, we see that $Q(z)$ is starlike and

$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{\alpha_1\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right\} > 0.$$

By an application of Lemma 2.1, we obtain that $q(z) \prec \psi(z)$ or

$$\frac{(\mathcal{H}_p^{l,m}[\alpha_1+1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} \prec \psi(z),$$

which shows that $f(z) \in \Omega_p(\alpha_1; \psi)$.
The second half of the Theorem 3.1 follows by a similar application of Lemma 2.2. □

Remark 3.1. The subordination result of Theorem 3.1 also holds if we replace the condition $\Re\{\alpha_1\psi(z)\} > 0$ by

$$\Re\left\{\alpha_1\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right\} > 0$$

and hence we obtain as its special case the following results using (1.3) and (1.4) respectively:

(1) Let $\psi(z)$ be univalent in Δ, $\psi(0) = 1$ and $\psi(z) \neq 0$. Assume that $z\psi'/\psi$ is starlike in Δ and

$$\Re\left\{a\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right\} > 0.$$

Let $\chi(z)$ be defined by

$$\chi(z) := \frac{1}{a+1}\left[a\psi(z) + 1 + \frac{z\psi'(z)}{\psi(z)}\right].$$

If $f(z) \in \Sigma_p$ and

$$\frac{[\mathcal{L}_p(a+2,c)f(z)]'}{[\mathcal{L}_p(a+1,c)f(z)]'} \prec \chi(z)$$

then

$$\frac{[\mathcal{L}_p(a+1,c)f(z)]'}{[\mathcal{L}_p(a,c)f(z)]'} \prec \psi(z)$$

and $\psi(z)$ is the best dominant.

(2) Let $\psi(z)$ be univalent in Δ, $\psi(0) = 1$ and $\psi(z) \neq 0$. Assume that $z\psi'/\psi$ is starlike in Δ and

$$\Re\left\{(n+p)\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right\} > 0.$$

Let $\chi(z)$ be defined by

$$\chi(z) := \frac{1}{n+p+1}\left[(n+p)\psi(z) + 1 + \frac{z\psi'(z)}{\psi(z)}\right].$$

If $f(z) \in \Sigma_p$ and

$$\frac{[\mathcal{D}^{n+p+1}f(z)]'}{[\mathcal{D}^{n+p}f(z)]'} \prec \chi(z)$$

then

$$\frac{[\mathcal{D}^{n+p}f(z)]'}{[\mathcal{D}^{n+p-1}f(z)]'} \prec \psi(z)$$

and $\psi(z)$ is the best dominant.

Using Theorem 3.1, we obtain the following "sandwich result":

Corollary 3.2. *Let $\psi_i(z) \neq 0$ $(i = 1, 2)$ be univalent in Δ. Further assume that $z\psi_i'(z)/\psi_i(z)$ $(i = 1, 2)$ is starlike univalent in Δ and $\Re\{\alpha_1\psi_i(z)\} > 0$. If $f(z) \in \Omega_p(\alpha_1 + 1; \chi_1, \chi_2)$ satisfies (3.2), then $f(z) \in \Omega_p(\alpha_1; \psi_1, \psi_2)$, where*

$$\chi_i(z) := \frac{1}{\alpha_1 + 1}\left[\alpha_1\psi_i(z) + 1 + \frac{z\psi_i'(z)}{\psi_i(z)}\right] \quad (i = 1, 2).$$

Theorem 3.3. *Let ψ be univalent in Δ, $\psi(0) = 1$, and λ be a complex number. Assume that $z\psi'/(\lambda - p - \alpha_1 + \alpha_1\psi)$ is starlike in Δ and $\Re\{\lambda - p - \alpha_1 + \alpha_1\psi(z)\} > 0$. Define the functions $\mathcal{F}$ and h by*

$$\mathcal{F}(z) := \frac{\lambda - p}{z^\lambda}\int_0^z t^{\lambda-1} f(t)dt, \tag{3.9}$$

$$h(z) := \psi(z) + \frac{z\psi'(z)}{\lambda - p - \alpha_1 + \alpha_1\psi(z)}.$$

If $f(z) \in \Omega_p(\alpha_1; h)$, then $\mathcal{F} \in \Omega_p(\alpha_1; \psi)$. If $f(z) \in \widetilde{\Omega}_p(\alpha_1; h)$,

$$0 \neq \frac{\mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z)}{\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z)} \in \mathscr{H}_1 \cap \mathcal{Q} \text{ and } \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} \text{ is univalent in } \Delta, \tag{3.10}$$

then the function $\mathcal{F} \in \widetilde{\Omega}_p(\alpha_1; \psi)$.

Proof. From the definition of $\mathcal{F}(z)$ and (3.4), we obtain that

$$\begin{aligned}(\lambda - p)\mathcal{H}_p^{l,m}[\alpha_1]f(z) &= \lambda\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z) + z(\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z))' \\ &= \alpha_1\mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z) + (\lambda - p - \alpha_1)\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z).\end{aligned} \tag{3.11}$$

Define the function $q(z)$ by

$$q(z) := \frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z))'}. \tag{3.12}$$

Then, clearly, $q(z)$ is analytic in Δ. Using (3.11) and (3.12), we have

$$(\lambda - p)\frac{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z))'} = \lambda - p - \alpha_1 + \alpha_1 q(z). \tag{3.13}$$

Upon logarithmic differentiation of (3.13) and using (3.4), and (3.12), we get

$$\frac{(\mathcal{H}_p^{l,m}[\alpha_1 + 1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'} = q(z) + \frac{zq'(z)}{\lambda - p - \alpha_1 + \alpha_1 q(z)}. \tag{3.14}$$

Since $f(z) \in \Omega_p(\alpha_1; h)$, we have, from (3.14),

$$q(z) + \frac{zq'(z)}{\lambda - p - \alpha_1 + \alpha_1 q(z)} \prec \psi(z) + \frac{z\psi'(z)}{\lambda - p - \alpha_1 + \alpha_1 \psi(z)}$$

and this can be written as (2.1), by defining

$$\vartheta(w) := w \text{ and } \varphi(w) := \frac{1}{\lambda - p - \alpha_1 + \alpha_1 w}.$$

The first half of Theorem 3.3 now follows by an application of Lemma 2.1 and the second half follows by a similar application of Lemma 2.2. □

Remark 3.2. The subordination result of Theorem 3.3 also holds if we replace the condition $\Re\{\lambda - p - \alpha_1 + \alpha_1\psi(z)\} > 0$ by

$$\Re\left\{\lambda - p - \alpha_1 + \alpha_1\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{\alpha_1 z\psi'(z)}{\lambda - p - \alpha_1 + \alpha_1\psi(z)}\right\} > 0$$

and hence we obtain as its special case the following results using (1.3) and (1.4) respectively:

(1) Let ψ be univalent in Δ, $\psi(0) = 1$, and λ be a complex number. Assume that $z\psi'/(\lambda - p - a + a\psi)$ is starlike in Δ and

$$\Re\left\{\lambda - p - a + a\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{az\psi'(z)}{\lambda - p - a + a\psi(z)}\right\} > 0.$$

Let F be defined as in (3.9) and $h(z)$ defined by

$$h(z) := \psi(z) + \frac{z\psi'(z)}{\lambda - p - a + a\psi(z)}.$$

If $f(z) \in \Sigma_p$ and

$$\frac{[\mathcal{L}_p(a+1,c)f(z)]'}{[\mathcal{L}_p(a,c)f(z)]'} \prec h(z)$$

then

$$\frac{[\mathcal{L}_p(a+1,c)F(z)]'}{[\mathcal{L}_p(a,c)F(z)]'} \prec \psi(z)$$

and $\psi(z)$ is the best dominant.

(2) Let ψ be univalent in Δ, $\psi(0) = 1$, and λ be a complex number. Assume that $z\psi'/(\lambda - n + (n+p)\psi)$ is starlike in Δ and

$$\Re\left\{\lambda - n + (n+p)\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{(n+p)z\psi'(z)}{\lambda - n + (n+p)\psi(z)}\right\} > 0.$$

Let F be defined as in (3.9) and $h(z)$ defined by

$$h(z) := \psi(z) + \frac{z\psi'(z)}{\lambda - n + (n+p)\psi(z)}.$$

If $f(z) \in \Sigma_p$ and

$$\frac{[\mathcal{D}^{n+p} f(z)]'}{[\mathcal{D}^{n+p-1} f(z)]'} \prec h(z)$$

then

$$\frac{[\mathcal{D}^{n+p} F(z)]'}{[\mathcal{D}^{n+p-1} F(z)]'} \prec \psi(z)$$

and $\psi(z)$ is the best dominant.

Using Theorem 3.3, we have the following result:

Corollary 3.4. *Let ψ_i be univalent in Δ $(i = 1, 2)$ and λ be a complex number. Assume that $\dfrac{z\psi_i'}{\lambda - p - \alpha_1 + \alpha_1\psi_i}$ is starlike in Δ and $\Re\{\lambda - p - \alpha_1 + \alpha_1\psi_i(z)\} > 0$ for $i = 1,\ 2$. If $f(z) \in \Omega_p(\alpha_1; h_1, h_2)$ satisfies* (3.10)*, then the function $\mathcal{F}$ defined by* (3.9) *belongs to $\Omega_p(\alpha_1; \psi_1, \psi_2)$ where*

$$h_i(z) := \psi_i(z) + \frac{z\psi_i'(z)}{\lambda - p - \alpha_1 + \alpha_1\psi_i(z)} \quad (i = 1, 2).$$

Theorem 3.5. *Let $f(z) \in \Sigma_p$ and $\alpha_1 \neq -1$. Define $\mathcal{F}$ by*

$$\mathcal{F}(z) := \frac{\alpha_1}{z^{\alpha_1+p}} \int_0^z t^{\alpha_1+p-1} f(t)dt. \tag{3.15}$$

Then $f(z) \in \Omega_p(\alpha_1; \varphi)$ if and only if $\mathcal{F} \in \Omega\left(\alpha_1 + 1; \dfrac{1+\alpha_1\varphi}{1+\alpha_1}\right)$. Also $f(z) \in \widetilde{\Omega}_p(\alpha_1; \varphi)$ if and only if $\mathcal{F} \in \widetilde{\Omega}\left(\alpha_1 + 1; \dfrac{1+\alpha_1\varphi}{1+\alpha_1}\right)$.

Proof. From (3.15), we have

$$\alpha_1 f(z) = (\alpha_1 + p)\mathcal{F}(z) + z\mathcal{F}'(z). \tag{3.16}$$

By convoluting (3.16) with $h_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z)$ and using the fact that $z(f * g)'(z) = f(z) * zg'(z)$, we obtain

$$\alpha_1 \mathcal{H}_p^{l,m}[\alpha_1] f(z) = (\alpha_1 + p)\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z) + z(\mathcal{H}_p^{l,m}[\alpha_1]\mathcal{F}(z))'$$

and by using (3.4), we get

$$\mathcal{H}_p^{l,m}[\alpha_1] f(z) = \mathcal{H}_p^{l,m}[\alpha_1 + 1]\mathcal{F}(z) \tag{3.17}$$

and

$$\begin{aligned}\alpha_1 \mathcal{H}_p^{l,m}[\alpha_1+1]f(z) &= z(\mathcal{H}_p^{l,m}[\alpha_1]f(z))' + (\alpha_1+p)\mathcal{H}_p^{l,m}[\alpha_1]f(z)\\ &= z(\mathcal{H}_p^{l,m}[\alpha_1+1]\mathcal{F}(z))' + (\alpha_1+p)\mathcal{H}_p^{l,m}[\alpha_1+1]\mathcal{F}(z)\\ &= (\alpha_1+1)\mathcal{H}_p^{l,m}[\alpha_1+2]\mathcal{F}(z) - (\alpha_1+p+1)\mathcal{H}_p^{l,m}[\alpha_1+1]\mathcal{F}(z)\\ &\qquad + (\alpha_1+p)\mathcal{H}_p^{l,m}[\alpha_1+1]\mathcal{F}(z)\\ &= (\alpha_1+1)\mathcal{H}_p^{l,m}[\alpha_1+2]\mathcal{F}(z) - \mathcal{H}_p^{l,m}[\alpha_1+1]\mathcal{F}(z). \qquad (3.18)\end{aligned}$$

Therefore, from (3.17) and (3.18), we have

$$\frac{(\mathcal{H}_p^{l,m}[\alpha_1+2]\mathcal{F}(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1+1]\mathcal{F}(z))'} = \frac{1}{\alpha_1+1}\left[1+\alpha_1\frac{(\mathcal{H}_p^{l,m}[\alpha_1+1]f(z))'}{(\mathcal{H}_p^{l,m}[\alpha_1]f(z))'}\right],$$

and the desired results follow at once. □

Using (1.3), we have the following result:

Example 3.1. Let $f(z) \in \mathcal{A}_p$ and $a \neq -1$. Define F as in (3.15). Then

$$\frac{[\mathcal{L}_p(a+1,c)f(z)]'}{[\mathcal{L}_p(a,c)f(z)]'} \prec \varphi(z) \quad \text{if and only if} \quad \frac{[\mathcal{L}_p(a+2,c)F(z)]'}{[\mathcal{L}_p(a+1,c)F(z)]'} \prec \frac{1+a\varphi}{1+a}.$$

Also

$$\varphi(z) \prec \frac{[\mathcal{L}_p(a+1,c)f(z)]'}{[\mathcal{L}_p(a,c)f(z)]'} \quad \text{if and only if} \quad \frac{1+a\varphi}{1+a} \prec \frac{[\mathcal{L}_p(a+2,c)F(z)]'}{[\mathcal{L}_p(a+1,c)F(z)]'}.$$

Using Theorem 3.5, we have

Corollary 3.6. *Let $f(z) \in \Sigma_p$ and $\alpha_1 \neq -1$. Then $f(z) \in \Omega_p(\alpha_1;\varphi_1,\varphi_2)$ if and only if $\mathcal{F}$ given by* (3.15) *is in* $\Omega\left(\alpha_1+1; \frac{1+\alpha_1\varphi_1}{1+\alpha_1}, \frac{1+\alpha_1\varphi_2}{1+\alpha_1}\right)$.

4. The classes $\mathcal{M}_p(n,\lambda;\varphi)$ and $\widetilde{\mathcal{M}}_p(n,\lambda;\varphi)$

By making use of Lemma 2.1, we prove the following result:

Theorem 4.1. *Let $\psi(z)$ be univalent in Δ, $\psi(0) = 1$, $\Re\psi(z) > 0$ and $z\psi'/\psi$ be starlike in Δ. Let $\chi(z)$ be defined by*

$$\chi(z) := \frac{1}{\lambda-p}\left[(\lambda-p)\psi(z) + \frac{z\psi'(z)}{\psi(z)}\right].$$

If $f(z) \in \mathcal{M}_p(n+1,\lambda;\chi)$, then $f(z) \in \mathcal{M}_p(n,\lambda;\psi)$. If $f(z) \in \widetilde{\mathcal{M}}_p(n+1,\lambda;\chi)$,

$$0 \neq \frac{(\mathcal{I}_p(n+1,\lambda)f(z))'}{(\mathcal{I}_p(n,\lambda)f(z))'} \in \mathscr{H}_1 \cap \mathcal{Q} \text{ and } \frac{(\mathcal{I}_p(n+2,\lambda)f(z))'}{(\mathcal{I}_p(n+1,\lambda)f(z))'} \text{ is univalent in } \Delta, \qquad (4.1)$$

then $f(z) \in \widetilde{\mathcal{M}}_p(n, \lambda; \psi)$.

Proof. First of all consider the following identity

$$(\lambda - p)\mathcal{I}_p(n+1, \lambda)f(z) = z[\mathcal{I}_p(n, \lambda)f(z)]' + \lambda\mathcal{I}_p(n, \lambda)f(z), \tag{4.2}$$

which yields up on differentiation

$$z[\mathcal{I}_p(n, \lambda)f(z)]'' = (\lambda - p)[\mathcal{I}_p(n+1, \lambda)f(z)]' - (\lambda + 1)[\mathcal{I}_p(n, \lambda)f(z)]'. \tag{4.3}$$

Now define the function $q(z)$ by

$$q(z) := \frac{(\mathcal{I}_p(n+1, \lambda)f(z))'}{(\mathcal{I}_p(n, \lambda)f(z))'}. \tag{4.4}$$

Then, clearly, $q(z)$ is analytic in Δ. By logarithmic differentiation of (4.4) and using (4.3), we obtain

$$\frac{(\mathcal{I}_p(n+2, \lambda)f(z))'}{(\mathcal{I}_p(n+1, \lambda)f(z))'} = \frac{1}{\lambda - p}\left((\lambda - p)q(z) + \frac{zq'(z)}{q(z)}\right). \tag{4.5}$$

Since $f(z) \in \mathcal{M}_p(n+1, \lambda; \chi)$ and in view of (4.5), we have

$$(\lambda - p)q(z) + \frac{zq'(z)}{q(z)} \prec (\lambda - p)\psi(z) + \frac{z\psi'(z)}{\psi(z)}.$$

The first result follows by an application of Lemma 2.1. Similarly the second result follows from Lemma 2.2. □

Remark 4.1. The subordination result of Theorem 4.1 also holds if we replace the condition $\Re\psi(z) > 0$ by

$$\Re\left\{(\lambda - p)\psi(z) + 1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right\} > 0.$$

Using Theorem 4.1, we obtain the following "sandwich result":

Corollary 4.2. *Let $\psi_i(z)$ be univalent in Δ, $\Re\psi_i(z) > 0$ and $z\psi_i'(z)/\psi_i(z)$ be starlike univalent in Δ for $i = 1, 2$. Define*

$$\chi_i(z) := \frac{1}{\lambda - p}\left[(\lambda - p)\psi_i(z) + \frac{z\psi_i'(z)}{\psi_i(z)}\right] \quad (i = 1, 2).$$

If $f(z) \in \mathcal{M}_p(n+1, \lambda; \chi_1, \chi_2)$ satisfies (4.1), then $f(z) \in \mathcal{M}_p(n, \lambda; \psi_1, \psi_2)$.

Theorem 4.3. *Let ψ be univalent in Δ, $\psi(0) = 1$, δ be a complex number, $z\psi'/(\delta -$*

$\lambda+(\lambda-p)\psi)$ *be starlike in* Δ *and* $\Re\{\delta-\lambda+(\lambda-p)\psi(z)\}>0$. *Define the function* $\mathcal{F}$ *by*

$$\mathcal{F}(z) := \frac{\delta-p}{z^{\delta}}\int_0^z t^{\delta-1}f(t)dt, \tag{4.6}$$

$$h(z) := \psi(z)+\frac{z\psi'(z)}{\delta-\lambda+(\lambda-p)\psi(z)}.$$

If $f(z)\in\mathcal{M}_p(n,\lambda;h)$, *then* $\mathcal{F}\in\mathcal{M}_p(n,\lambda;\psi)$. *If* $f(z)\in\widetilde{\mathcal{M}}_p(n,\lambda;h)$,

$$0\neq\frac{(\mathcal{I}_p(n+1,\lambda)\mathcal{F}(z))'}{(\mathcal{I}_p(n,\lambda)\mathcal{F}(z))'}(z)\in\mathscr{H}_1\cap\mathcal{Q} \text{ and } \frac{(\mathcal{I}_p(n+1,\lambda)f(z))'}{(\mathcal{I}_p(n,\lambda)f(z))'} \text{ is univalent in } \Delta, \tag{4.7}$$

then $\mathcal{F}\in\widetilde{\mathcal{M}}_p(n,\lambda;\psi)$.

Proof. From the definition of $\mathcal{F}(z)$ and

$$z(\mathcal{I}_p(n,\lambda)\mathcal{F}(z))' = (\lambda-p)\mathcal{I}_p(n+1,\lambda)\mathcal{F}(z)-\lambda\mathcal{I}_p(n,\lambda)\mathcal{F}(z), \tag{4.8}$$

we have

$$\begin{aligned}(\delta-p)\mathcal{I}_p(n,\lambda)f(z) &= \delta\mathcal{I}_p(n,\lambda)\mathcal{F}(z)+z(\mathcal{I}_p(n,\lambda)\mathcal{F}(z))'\\ &= (\lambda-p)\mathcal{I}_p(n+1,\lambda)\mathcal{F}(z)+(\delta-\lambda)\mathcal{I}_p(n,\lambda)\mathcal{F}(z).\end{aligned} \tag{4.9}$$

Define the function $q(z)$ by

$$q(z) := \frac{(\mathcal{I}_p(n+1,\lambda)\mathcal{F}(z))'}{(\mathcal{I}_p(n,\lambda)\mathcal{F}(z))'}. \tag{4.10}$$

Then, clearly, $q(z)$ is analytic in Δ. Using (4.9) and (4.10), we have

$$(\delta-p)\frac{(\mathcal{I}_p(n,\lambda)f(z))'}{(\mathcal{I}_p(n,\lambda)\mathcal{F}(z))'} = \delta-\lambda+(\lambda-p)q(z). \tag{4.11}$$

Upon logarithmic differentiation of (4.11) and using (4.3), (4.8) and (4.10), we get

$$\frac{(\mathcal{I}_p(n+1,\lambda)f(z))'}{(\mathcal{I}_p(n,\lambda)f(z))'} = q(z)+\frac{zq'(z)}{\delta-\lambda+(\lambda-p)q(z)}. \tag{4.12}$$

For $f(z)\in\mathcal{M}_p(n,\lambda,h)$, we have from (4.11),

$$q(z)+\frac{zq'(z)}{\delta-\lambda+(\lambda-p)q(z)} \prec \psi(z)+\frac{z\psi'(z)}{\delta-\lambda+(\lambda-p)\psi(z)}.$$

The first part of our result now follows by an application of Lemma 2.1. Similarly the second part follows from Lemma 2.2. □

Remark 4.2. The subordination result of Theorem 4.3 also holds if we replace the condition $\Re\{\delta-\lambda+(\lambda-p)\psi(z)\}>0$ by

$$\Re\left\{\delta-\lambda+(\lambda-p)\psi(z)+1+\frac{z\psi''(z)}{\psi'(z)}-\frac{(\lambda-p)z\psi'(z)}{\delta-\lambda+(\lambda-p)\psi(z)}\right\}>0.$$

Using Theorem 4.3, we have the following result:

Corollary 4.4. *Let ψ_i be univalent in Δ and δ be a complex number. Assume that $z\psi_i'/(\delta-\lambda+(\lambda-p)\psi_i)$ is starlike in Δ and $\Re\{\delta-\lambda+(\lambda-p)\psi_i(z)\}>0$ for $i=1,\ 2$. Define the functions h_i by*

$$h_i(z):=\psi_i(z)+\frac{z\psi_i'(z)}{\delta-\lambda+(\lambda-p)\psi_i(z)}\quad(i=1,2).$$

If $f(z)\in\mathcal{M}_p(n,\lambda;h_1,h_2)$, then $\mathcal{F}$ defined by (4.6) belongs to $\mathcal{M}_p(n,\lambda;\psi_1,\psi_2)$.

Theorem 4.5. *Let $f(z)\in\Sigma_p$. Then $f(z)\in\mathcal{M}_p(n,\lambda;\varphi)$ if and only if*

$$\mathcal{F}(z):=\frac{\lambda-p}{z^\lambda}\int_0^z t^{\lambda-1}f(t)dt\in\mathcal{M}_p(n+1,\lambda;\varphi). \tag{4.13}$$

Also $f(z)\in\widetilde{\mathcal{M}}_p(n,\lambda;\varphi)$ if and only if $\mathcal{F}\in\widetilde{\mathcal{M}}_p(n+1,\lambda;\varphi)$.

Proof. From (4.13), we have

$$(\lambda-p)f(z)=\lambda\mathcal{F}(z)+z\mathcal{F}'(z). \tag{4.14}$$

By convoluting (4.14) with

$$\phi_p(k,\lambda;z):=\frac{1}{z^p}+\sum_{k=1-p}^{\infty}\left(\frac{k+\lambda}{\lambda-p}\right)^r z^k$$

and using the fact that $z(f*g)'(z)=f(z)*zg'(z)$, we obtain

$$(\lambda-p)\mathcal{I}_p(n,\lambda)f(z)=\lambda\mathcal{I}_p(n,\lambda)\mathcal{F}(z)+z(\mathcal{I}_p(n,\lambda)\mathcal{F}(z))'$$

and by using (4.3), we get

$$\mathcal{I}_p(n,\lambda)f(z)=\mathcal{I}_p(n+1,\lambda)\mathcal{F}(z) \tag{4.15}$$

and

$$\begin{aligned}(\lambda-p)\mathcal{I}_p(n+1,\lambda)f(z)&=z(\mathcal{I}_p(n,\lambda)f(z))'+\lambda\mathcal{I}_p(n,\lambda)f(z)\\&=z(\mathcal{I}_p(n+1,\lambda)\mathcal{F}(z))'+\lambda\mathcal{I}_p(n+1,\lambda)\mathcal{F}(z)\\&=(\lambda-p)\mathcal{I}_p(n+2,\lambda)\mathcal{F}(z).\end{aligned} \tag{4.16}$$

Therefore, from (4.15) and (4.16), we have

$$\frac{(\mathcal{I}_p(n+2,\lambda)\mathcal{F}(z))'}{(\mathcal{I}_p(n+1,\lambda)\mathcal{F}(z))'} = \frac{(\mathcal{I}_p(n+1,\lambda)f(z))'}{(\mathcal{I}_p(n,\lambda)f(z))'},$$

and the desired result follows at once. □

Using Theorem 4.5, we have

Corollary 4.6. *Let* $f(z) \in \Sigma_p$. *Then* $f(z) \in \mathcal{M}_p(n,\lambda;\varphi_1,\varphi_2)$ *if and only if* $\mathcal{F} \in \mathcal{M}_p(n+1,\lambda;\varphi_1,\varphi_2)$.

References

[1] M. K. Aouf and H. M. Hossen, *New criteria for meromorphic p-valent starlike functions*, Tsukuba J. Math., **17(2)**(1993), 481-486.

[2] T. Bulboacă, *A class of superordination-preserving integral operators*, Indag. Math. (N. S.), **13(3)**(2002), 301-311.

[3] T. Bulboacă, *Classes of first-order differential superordinations*, Demonstratio Math., **35(2)**(2002), 287-292.

[4] N. E. Cho and S. H. Lee, *Certain subclasses of meromorphic functions defined by subordination. II*, Kyungpook Math. J., **36(2)**(1996), 283-291.

[5] M. R. Ganigi and B. A. Uralegaddi, *New criteria for meromorphic univalent functions*, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.), **33(81)(1)**(1989), 9-13.

[6] J. -L. Liu, *A linear operator and its applications on meromorphic p-valent functions*, Bull. Inst. Math. Acad. Sinica, **31(1)**(2003), 23-32.

[7] J. -L. Liu and H. M. Srivastava, *Classes of meromorphically multivalent functions associated with the generalized hypergeometric function*, Math. Comput. Modelling, **39(1)**(2004), 21-34.

[8] S. S. Miller and P. T. Mocanu, Differential subordinations, Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker Inc., New York, 2000.

[9] S. S. Miller and P. T. Mocanu, *Subordinants of differential superordinations*, Complex Var. Theory Appl., **48(10)**(2003), 815-826.

[10] J. Patel and P. Sahoo, *On certain subclasses of meromorphically p-valent functions*, Bull. Calcutta Math. Soc., **93(6)**(2001), 455-464.

[11] V. Ravichandran, S. Sivaprasad Kumar, and K. G. Subramanian, *Convolution conditions for spirallikeness and convex spirallikeness of certain meromorphic p-valent functions*, JIPAM. J. Inequal. Pure Appl. Math., **5(1)**(2004), Article 11, 7 pp. (electronic).

[12] B. A. Uralegaddi and C. Somanatha, *New criteria for meromorphic starlike univalent functions*, Bull. Austral. Math. Soc., **43(1)**(1991), 137-140.

[13] D. Yang, *On new subclasses of meromorphic p-valent functions*, J. Math. Res. Exposition, **15(1)**(1995), 7-13.

[14] D. Yang, *On a class of meromorphic starlike multivalent functions*, Bull. Inst. Math. Acad. Sinica, **24(2)**(1996), 151-157.