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On the Hilbert Type Integral Inequalities with Some Parameters and Its Reverse

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Abstract. This paper deals with some new generalizations of the Hardy-Hilbert type integral inequalities with some parameters. We also consider the equivalent inequalities and the reverse forms.

1. Introduction

If $f(x),\ g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$ then

$$(1.1)\qquad \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \quad < \quad \pi\left(\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right)^{\frac{1}{2}},$$

where the constant factor π is the best possible (see [1]). Inequality (1.1) had been extended by Hardy-Riesz as:

If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $f(x),\ g(x) \geq 0$, such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then we have the following Hardy-Hilbert's integral inequality:

$$(1.2)\qquad \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \quad < \quad \frac{\pi}{\sin(\frac{\pi}{p})}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}},$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible constant (see [2]). This inequality play an important role in mathematical analysis and its applications (see [3]). In [4] and [5], Yang gave some new generalizations of (1.2) by introducing a parameter $\lambda > 0$, and Yang et al. [6] gave an extension of the above results by introducing the index of conjugate parameter (r,s) $(r > 1, \frac{1}{r}+\frac{1}{s} = 1)$ as follows:

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If $f(x), g(x) \geq 0$ and $0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx < \infty$, $0 < \int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x)dx < \infty$, then

(1.3)
$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B(\tfrac{\lambda}{r}, \tfrac{\lambda}{s}) \left(\int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x)dx \right)^{\frac{1}{q}},$$

where the constant factor $B(\frac{\lambda}{r}, \frac{\lambda}{s})$ is the best possible constant. In particular, for $\lambda = 1$, $r = p$, inequality (1.3) reduces to (1.2); for $\lambda = 4$, $r = s = 2$, inequality (1.3) reduces to:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^4} dxdy < \frac{1}{6} \left(\int_0^\infty \frac{1}{x^{p+1}} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty \frac{1}{x^{q+1}} g^q(x)dx \right)^{\frac{1}{q}}. \tag{1.4}$$

Recently, Xie et al.[8] gave a new Hilbert type integral inequality with some parameters and its reverse as follows:

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a, b > 0$, $a \neq b$, $f(x), g(x) \geq 0$, such that $0 < \int_0^\infty \frac{1}{x^{p+1}} f^p(x)dx < \infty$ and $0 < \int_0^\infty \frac{1}{x^{q+1}} g^q(x)dx < \infty$, then

(1.5)
$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+ay)^2 (x+by)^2} dxdy < K \left(\int_0^\infty \frac{1}{x^{p+1}} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty \frac{1}{x^{q+1}} g^q(x)dx \right)^{\frac{1}{q}},$$

where the constant factor $K = \frac{a+b}{(b-a)^2} \left[\frac{1}{b-a} \ln(\frac{b}{a}) - \frac{2}{a+b} \right]$ is the best possible constant. If $0 < p < 1$, then

(1.6)
$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+ay)^2 (x+by)^2} dxdy > K \left(\int_0^\infty \frac{1}{x^{p+1}} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty \frac{1}{x^{q+1}} g^q(x)dx \right)^{\frac{1}{q}},$$

where K the constant factor is the best possible.

In this paper, by introducing some parameters and estimating the weight function, we prove Hilbert type integral inequality with a best constant factor similar to (1.4) and (1.5). The equivalent inequalities and the reverse forms are considered.

2. Main Results

In order to obtain our results, we need the following lemmas.

Lemma 2.1. *If $a, b > 0$, $a \neq b$, $\alpha > 0$, the weight function $\bar{\omega}(x)$ and $\omega(y)$ defined by*

$$\bar{\omega}(x) = \int_0^\infty \frac{x^{2\alpha} y^{2\alpha-1}}{(x^\alpha + ay^\alpha)^2 (x^\alpha + by^\alpha)^2} dy, \; x \in (0, \infty), \tag{2.1}$$

$$\omega(y) = \int_0^\infty \frac{x^{2\alpha-1}y^{2\alpha}}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}dx,\ y\in(0,\infty), \tag{2.2}$$

then we have

$$\bar{\omega}(x) = \omega(y) = K := \frac{1}{\alpha}\frac{a+b}{(b-a)^2}\left[\frac{1}{b-a}\ln\left(\frac{b}{a}\right)-\frac{2}{a+b}\right]. \tag{2.3}$$

Proof. For fixed x, setting $u=\frac{y^\alpha}{x^\alpha}$ in (2.1), we obtain

$$\begin{aligned}\bar{\omega}(x) &= \frac{1}{\alpha}\int_0^\infty \frac{u}{(1+au)^2(1+bu)^2}du\\ &= \frac{1}{\alpha}\frac{a+b}{(b-a)^2}\left[\frac{1}{b-a}\ln\left(\frac{b}{a}\right)-\frac{2}{a+b}\right].\end{aligned}$$

Hence we obtain $\bar{\omega}(x)=K$. In the same way, we obtain $\omega(y)=K$. □

Xie proved the following lemma in [8, Lemma 2.2].

Lemma 2.2. *If $a,b>0$, $a\neq b$ and $\alpha>0$ for $0<\varepsilon<p$, we have*

$$\int_0^\infty \frac{u^{1-\frac{\varepsilon}{p}}}{(1+au)^2(1+bu)^2}du = K+o(1),\quad \varepsilon\to 0^+. \tag{2.4}$$

Lemma 2.3. *If $p>1$ (or $0<p<1$), $\frac{1}{p}+\frac{1}{q}=1$, $a,b>0$, $a\neq b$, $\alpha>0$ and $0<\varepsilon<p$, setting*

$$I := \int_1^\infty\left[\int_1^\infty \frac{y^{\alpha(2-\frac{\varepsilon}{p})-1}dy}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}\right]x^{\alpha(2-\frac{\varepsilon}{q})-1}dx$$

then we have

$$\frac{1}{\alpha\varepsilon}(K+o(1))-O(1) \le I \le \frac{1}{\alpha\varepsilon}(K+o(1)),\quad \varepsilon\to 0^+. \tag{2.5}$$

Proof. For fixed x, $(1+au)^2(1+bu)^2>(a+b)u$, setting $y^\alpha=x^\alpha u$, then we obtain the following inequality by (2.4).

$$\begin{aligned}I &= \int_1^\infty x^{-\alpha\varepsilon-1}\left[\int_{x^{-\alpha}}^\infty \frac{u^{1-\frac{\varepsilon}{p}}du}{(1+au)^2(1+bu)^2}\right]dx\\ &= \int_1^\infty x^{-\alpha\varepsilon-1}\left[\int_0^\infty \frac{u^{1-\frac{\varepsilon}{p}}du}{(1+au)^2(1+bu)^2}\right]dx\\ &\quad -\int_1^\infty x^{-\alpha\varepsilon-1}\left[\int_0^{x^{-\alpha}} \frac{u^{1-\frac{\varepsilon}{p}}du}{(1+au)^2(1+bu)^2}\right]dx\\ &\ge \frac{1}{\alpha\varepsilon}(K+o(1))-\frac{1}{a+b}\int_1^\infty x^{-1}\left(\int_0^{x^{-\alpha}} u^{-\frac{\varepsilon}{p}}du\right)dx\\ &= \frac{1}{\alpha\varepsilon}(K+o(1))-\frac{1}{\alpha(a+b)}\frac{1}{\left(1-\frac{\varepsilon}{p}\right)^2}\\ &= \frac{1}{\alpha\varepsilon}(K+o(1))-O(1).\end{aligned}$$

By the same way, we have

$$I \leq \int_1^\infty \left[\int_0^\infty \frac{y^{\alpha(2-\frac{\varepsilon}{p})-1} dy}{(x^\alpha + a y^\alpha)^2 (x^\alpha + b y^\alpha)^2} \right] x^{\alpha(2-\frac{\varepsilon}{q})-1} dx = \frac{1}{\alpha\varepsilon} (K + o(1)).$$

□

Theorem 2.1. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a, b > 0$, $a \neq b$, $\alpha > 0$ *and* $f(x)$, $g(x) \geq 0$, *such that* $0 < \int_0^\infty \frac{1}{x^{p(2\alpha-1)+1}} f^p(x) dx < \infty$ *and* $0 < \int_0^\infty \frac{1}{x^{q(2\alpha-1)+1}} g^q(x) dx < \infty$, *then*

(2.6)
$$\int_0^\infty \int_0^\infty \frac{f(x) g(y)}{(x^\alpha + a y^\alpha)^2 (x^\alpha + b y^\alpha)^2} dx dy < K \left(\int_0^\infty \frac{1}{x^{p(2\alpha-1)+1}} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty \frac{1}{x^{q(2\alpha-1)+1}} g^q(x) dx \right)^{\frac{1}{q}},$$

where the constant factor K *is the best possible and* K *is defined by* (2.3).

Proof. By Hölder's inequality, with weight (see [7]) and (2.1)-(2.3), we have

(2.7)
$$\begin{aligned}
J &:= \int_0^\infty \int_0^\infty \frac{f(x) g(y)}{(x^\alpha + a y^\alpha)^2 (x^\alpha + b y^\alpha)^2} dx dy \\
&= \int_0^\infty \int_0^\infty \frac{1}{(x^\alpha + a y^\alpha)^2 (x^\alpha + b y^\alpha)^2} \left[\frac{y^{\frac{2\alpha-1}{p}}}{x^{\frac{2\alpha-1}{q}}} f(x) \right] \left[\frac{x^{\frac{2\alpha-1}{q}}}{y^{\frac{2\alpha-1}{p}}} g(y) \right] dx dy \\
&\leq \left\{ \int_0^\infty \int_0^\infty \frac{1}{(x^\alpha + a y^\alpha)^2 (x^\alpha + b y^\alpha)^2} \left(\frac{y^{2\alpha-1}}{x^{(p-1)(2\alpha-1)}} \right) f^p(x) dy dx \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \int_0^\infty \int_0^\infty \frac{1}{(x^\alpha + a y^\alpha)^2 (x^\alpha + b y^\alpha)^2} \left(\frac{x^{2\alpha-1}}{y^{(q-1)(2\alpha-1)}} \right) g^q(y) dx dy \right\}^{\frac{1}{q}} \\
&= \left\{ \int_0^\infty \bar{\omega}(x) \frac{1}{x^{p(2\alpha-1)+1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \omega(y) \frac{1}{y^{q(2\alpha-1)+1}} g^q(y) dy \right\}^{\frac{1}{q}} \\
&= K \left\{ \int_0^\infty \frac{1}{x^{p(2\alpha-1)+1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \frac{1}{x^{q(2\alpha-1)+1}} g^q(x) dx \right\}^{\frac{1}{q}}.
\end{aligned}$$

If (2.7) takes the form of equality, then the exists constants M and N, such that they are not all zero, and (see [7])

$$M\left(\frac{y}{x^{p(2\alpha-1)-1}}\right) f^p(x) = N\left(\frac{x}{y^{q(2\alpha-1)-1}}\right) g^q(y)$$

a.e. in $(0, \infty) \times (0, \infty)$. Hence, there exists a constant C, such that

$$M x^{-p(2\alpha-1)} f^p(x) = N y^{-q(2\alpha-1)} g^q(y) = C$$

a.e. in $(0, \infty)$. We claim that $M = 0$. In fact, if $M \neq 0$, then $x^{-p(2\alpha-1)-1} f^p(x) = \frac{C}{Mx}$ a.e. in $(0, \infty)$, which contradicts the fact that $0 < \int_0^\infty x^{-p(2\alpha-1)-1} f^p(x) dx < \infty$.

In the same way, we claim that $N = 0$. This is a contradiction. Hence by (2.7), we have (2.6).

If the constant factor K in (2.6) is not the best possible, then there exists a positive constant H (with $H < K$), such that (2.6) is still valid if we replace K by H. For $0 < \varepsilon < p$ small enough, setting f_ε and g_ε as: $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for $x \in (0,1)$; $f_\varepsilon(x) = x^{\alpha(2-\frac{\varepsilon}{p})-1}$; $g_\varepsilon(x) = x^{\alpha(2-\frac{\varepsilon}{q})-1}$, for $x \in [1,\infty)$, then we have

$$H\left\{\int_0^\infty \frac{1}{x^{p(2\alpha-1)+1}} f_\varepsilon^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty \frac{1}{x^{q(2\alpha-1)+1}} g_\varepsilon^q(x)dx\right\}^{\frac{1}{q}}$$
$$= H\left\{\int_1^\infty x^{-\alpha\varepsilon-1}dx\right\}^{\frac{1}{p}} \left\{\int_1^\infty x^{-\alpha\varepsilon-1}dx\right\}^{\frac{1}{q}} = H\frac{1}{\alpha\varepsilon}.$$

By (2.5), we have

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}dxdy &= \int_1^\infty \left[\int_1^\infty \frac{y^{\alpha(2-\frac{\varepsilon}{p})-1}dy}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}\right] x^{\alpha(2-\frac{\varepsilon}{q})-1}dx \\ &\geq \frac{1}{\alpha\varepsilon}(K+o(1)) - O(1).\end{aligned}$$

Hence, we find

$$\frac{1}{\alpha\varepsilon}(K+o(1)) - O(1) < \frac{H}{\alpha\varepsilon} \quad \text{or} \quad (K+o(1)) - \alpha\varepsilon O(1) < H.$$

For $\varepsilon \to 0^+$, it follows that $K \leq H$. This contradicts the fact that $H < K$. Hence the constant factor K in (2.6) is the best possible. □

Theorem 2.2. *If* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a, b > 0$, $a \neq b$, $\alpha > 0$ *and* $f(x)$, $g(x) \geq 0$, *such that* $0 < \int_0^\infty \frac{1}{x^{p(2\alpha-1)+1}} f^p(x)dx < \infty$ *and* $0 < \int_0^\infty \frac{1}{x^{q(2\alpha-1)+1}} g^q(x)dx < \infty$, *then*

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}dxdy \\ &> K\left(\int_0^\infty \frac{1}{x^{p(2\alpha-1)+1}} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty \frac{1}{x^{q(2\alpha-1)+1}} g^q(x)dx\right)^{\frac{1}{q}},\end{aligned} \tag{2.8}$$

where the constant factor K *is the best possible and* K *is defined by* (2.3).

Proof. By the reverse Hölder's inequality with weight (see [7]) and the same way of giving (2.7), we obtain (2.8).

If the constant factor K in (2.8) is not the best possible, then there exists a positive constant H (with $H > K$), such that (2.8) is still valid if we replace K by H. For $0 < \varepsilon < p$ small enough, setting f_ε and g_ε as: $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for

$x \in (0,1)$; $f_\varepsilon(x) = x^{\alpha(2-\frac{\varepsilon}{p})-1}$; $g_\varepsilon(x) = x^{\alpha(2-\frac{\varepsilon}{q})-1}$, for $x \in [1,\infty)$, then we have

$$H\left\{\int_0^\infty \frac{1}{x^{p(2\alpha-1)+1}} f_\varepsilon^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty \frac{1}{x^{q(2\alpha-1)+1}} g_\varepsilon^q(x)dx\right\}^{\frac{1}{q}}$$

$$= H\left\{\int_1^\infty x^{-\alpha\varepsilon-1}dx\right\}^{\frac{1}{p}} \left\{\int_1^\infty x^{-\alpha\varepsilon-1}dx\right\}^{\frac{1}{q}} = H\frac{1}{\alpha\varepsilon}.$$

By (2.5), we have

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}dxdy &= \int_1^\infty \left[\int_0^\infty \frac{y^{\alpha(2-\frac{\varepsilon}{p})-1}dy}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}\right] x^{\alpha(2-\frac{\varepsilon}{q})-1}dx \\ &\le \frac{1}{\alpha\varepsilon}(K+o(1)).\end{aligned}$$

Hence, we find

$$\frac{1}{\alpha\varepsilon}(K+o(1)) > \frac{H}{\alpha\varepsilon} \quad \text{or} \quad (K+o(1)) > H.$$

For $\varepsilon \to 0^+$, it follows that $K \ge H$. This contradicts the fact that $H > K$. Hence the constant factor K in (2.8) is the best possible. □

Theorem 2.3. *Under the same assumption of Theorem 2.1 we have*

$$\int_0^\infty y^{2\alpha p-1}\left(\int_0^\infty \frac{f(x)}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}dx\right)^p dy \;<\; K^p\int_0^\infty \frac{f^p(x)}{x^{p(2\alpha-1)+1}}dx, \tag{2.9}$$

where the constant factor K^p is the best possible. Inequalities (2.9) *and* (2.6) *are equivalent.*

Proof. Setting $g(y) = y^{2\alpha p-1}\left(\int_0^\infty \frac{f(x)}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}dx\right)^{p-1}$, by (2.6), we have

$$\begin{aligned}\int_0^\infty y^{-q(2\alpha-1)-1}g^q(y)dy &= \int_0^\infty y^{2\alpha p-1}\left(\int_0^\infty \frac{f(x)}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}dx\right)^p dy \\ &= \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}dxdy \\ &\le K\left(\int_0^\infty \frac{f^p(x)}{x^{p(2\alpha-1)+1}}dx\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{g^q(y)}{y^{q(2\alpha-1)+1}}dy\right)^{\frac{1}{q}}.\end{aligned} \tag{2.10}$$

$$0 < \int_0^\infty y^{-q(2\alpha-1)-1}g^q(y)dy \le K^p\int_0^\infty \frac{f^p(x)}{x^{p(2\alpha-1)+1}}dx < \infty. \tag{2.11}$$

Hence by (2.6), (2.10) and (2.11) preserve the form of strict inequalities, and we have (2.9). By Hölder's inequality, we have

(2.12)

$$\int_0^\infty y^{2\alpha p-1}\left(\int_0^\infty \frac{f(x)}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}dx\right)^p dy$$

$$= \int_0^\infty y^{(2\alpha-1)+\frac{1}{q}}\left(\int_0^\infty \frac{f(x)}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}dx\right) y^{-(2\alpha-1)-\frac{1}{q}}g(y)dy$$

$$= \left\{\int_0^\infty y^{p(2\alpha-1)+\frac{p}{q}}\left(\int_0^\infty \frac{f(x)}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}dx\right)^p dy\right\}^{\frac{1}{p}} \left\{\int_0^\infty y^{-q(2\alpha-1)-1}g^q(y)dy\right\}^{\frac{1}{q}}$$

$$= \left\{\int_0^\infty y^{2\alpha p-1}\left(\int_0^\infty \frac{f(x)}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}dx\right)^p dy\right\}^{\frac{1}{p}} \left\{\int_0^\infty y^{-q(2\alpha-1)-1}g^q(y)dy\right\}^{\frac{1}{q}}.$$

Then by (2.9), we have (2.6). Hence inequalities (2.6) and (2.9) are equivalent.

If the constant factor in (2.9) is not the best possible, then by (2.12), we can get a contradiction that the constant factor in (2.6) is not the best possible. □

Theorem 2.4. *Under the same assumption of Theorem 2.2 we have*

$$\int_0^\infty y^{2\alpha p-1}\left(\int_0^\infty \frac{f(x)}{(x^\alpha+ay^\alpha)^2(x^\alpha+by^\alpha)^2}dx\right)^p dy \quad > \quad K^p\int_0^\infty \frac{f^p(x)}{x^{p(2\alpha-1)+1}}dx, \tag{2.13}$$

where the constant factor K^p *is the best possible. Inequalities* (2.13) *and* (2.8) *are equivalent.*

Proof. The proof of Theorem 2.3 is the similar. □

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