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Scaling Law of the Hausdorff Measure

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Abstract. We exhibit an example that the scaling property of Hausdorff measure on $\mathbb{R}^\infty$ does not hold.

1. Introduction

The scaling properties of length, area and volume are well known. On magnification by a factor λ, the length of a curve is multiplied by λ, the area of a plane region is multiplied by λ^2 and the volume of a 3-dimensional object is multiplied by λ^3. It is well known that the scaling factor of s-dimensional Hausdorff measure is λ^s [3]. Such scaling properties are fundamental to the theory of fractals . Especially, the authors in [2], [4] have proved that the scaling property of the Hausdorff measure H^h on $\mathbb{R}^m$ holds for a continuous increasing function h defined on $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$ with $h(0) = 0$ such that h is strictly concave down on a right neighborhood of 0 if $\lim_{t\to 0} \dfrac{h(\lambda t)}{h(t)} = \lambda^\alpha, 0 \le \alpha \le 1, \lambda > 0$. i.e., $H^h(\lambda E) = \lambda^\alpha H^h(E)$ for $E \in \mathbb{R}^m$. Now, we question whether the scaling property of the Hausdorff measure H^h on $\mathbb{R}^m$ can be extended to on $\mathbb{R}^\infty$ or not. In other word, we wonder if the property $H^h(\lambda E) = \lambda^\alpha H^h(E)$ holds for $E \subset \mathbb{R}^\infty$.

The purpose of this paper is to give a negative answer on above question. For this, we consider $\mathbb{R}^\infty = \prod_{i=1}^\infty \mathbb{R}$ with the topology induced by a metric $\rho(x, y) = \sum_{i=1}^\infty \dfrac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}$ for $x = (x_i), y = (y_i) \in \mathbb{R}^\infty$. We can define the *Hausdorff measure* $H^h(E)$ on $\mathbb{R}^\infty$(cf. [1]) by

$$H^h(E) = \lim_{\delta\to 0} H^h_\delta(E),$$

where

$$H^h_\delta(E) = \inf\{\sum_{i=1}^\infty h(\rho(U_i)) : E \subset \cup_{i=1}^\infty, \rho(U_i) \le \delta\}$$

and $\rho(U_i)$ denotes the diameter of U_i with respect to the metric ρ. We also call

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$\{U_i\}$ *a* δ*-cover* of E if $\{U_i\}$ is a cover of E and $\rho(U_i) \leq \delta$ for all i.

2. Main Results

Throughout this paper, we let $\alpha \geq 0$ and h be a continuous increasing function h defined on $\mathbb{R}^+$ with $h(0) = 0$ such that h is strictly concave down on a right neighborhood of 0.

Proposition 2.1. *If* $\lim_{t\to 0} \dfrac{h(\lambda t)}{h(t)} = \lambda^\alpha$ *and* $\lambda > 1$*, then* $H^h(\lambda E) \leq \lambda^\alpha H^h(E)$ *for* $E \subset \mathbb{R}^\infty$.

Proof. If the value $H^h(E)$ is infinite, the inequality is obviously true. Thus, W. L. O. G., we may assume that the value $H^h(E)$ is finite. In order to show this proposition, firstly, we note that

$$\begin{aligned}\rho(\lambda x, \lambda y) &= \sum_{i=1}^{\infty} \frac{\lambda |x_i - y_i|}{2^i(1 + \lambda |x_i - y_i|)} \\ &< \sum_{i=1}^{\infty} \frac{\lambda |x_i - y_i|}{2^i(1 + |x_i - y_i|)} (\text{ since } \lambda > 1) \\ &= \lambda \rho(x, y).\end{aligned}$$

If $\{U_i\}$ is a δ-cover of E, then $\{\lambda U_i\}$ is also a $\lambda\delta$ - cover of λE. Since $\lim_{t\to 0} \dfrac{h(\lambda t)}{h(t)} = \lambda^\alpha$, we can obtain that $h(\lambda t) < \lambda^\alpha h(t) + \epsilon h(t)$ for sufficiently small $\epsilon > 0$ and $0 < t < \epsilon$. So we get that $\sum_{i=1}^\infty h(\lambda \rho(U_i)) \leq \sum_{i=1}^\infty (\lambda^\alpha h(\rho(U_i)) + \epsilon h(\rho(U_i)))$. From the definition of the $H^h_{\lambda\delta}(\lambda E)$, we obtain $H^h_{\lambda\delta}(\lambda E) \leq \sum_{i=1}^\infty h(\lambda \rho(U_i))$ and hence $H^h_{\lambda\delta}(\lambda E) \leq \sum_{i=1}^\infty (\lambda^\alpha h(\rho(U_i)) + \epsilon h(\rho(U_i)))$ for all δ-covers $\{U_i\}$ of E. Thus we have $H^h_{\lambda\delta}(\lambda E) \leq \lambda^\alpha H^h_\delta(E) + \epsilon H^h_\delta(E)$. By letting $\delta, \epsilon \to 0$, we obatin $H^h(\lambda E) \leq \lambda^\alpha H^h(E)$. □

Proposition 2.2. *If* $0 < \lambda < 1$*, then* $H^h(\lambda E) \leq H^h(E)$ *for* $E \subset \mathbb{R}^\infty$.

Proof. Let $\{U_i\}$ be a δ-cover of E. Then $\{\lambda U_i\}$ is a δ-cover of λE. Since h is increasing, $H^h_\delta(\lambda E) \leq H^h_\delta(E)$. Therefore we have $H^h(\lambda E) \leq H^h(E)$. □

Corollary 2.3. *Let* E *be in* $\mathbb{R}^\infty$ *and* $\lim_{t\to 0} \dfrac{h(\lambda t)}{h(t)} = \lambda^\alpha$. *Then*

(1) $H^h(E) \leq H^h(\lambda E) \leq \lambda^\alpha H^h(E)$ *if* $\lambda > 1$.
(2) $\lambda^\alpha H^h(E) \leq H^h(\lambda E) \leq \lambda^\alpha H^h(E)$ *if* $0 < \lambda < 1$.

Remark 2.4. Corollary 2.3 is a slight improvement of Proposition 3.3(2) in [1].

We will show that the strict inequality in the Proposition 2.2 might occur via an example. Essential parts and ideas of the following example can be found in Examples 4.2 and 4.3 of [1].

Example 2.5. Let $I = \{0,1\}$ and $I^\infty = \prod_{i=1}^\infty I$. Define $h(x) = x$. That is, $\alpha = 1$. In this case, H^1 denotes H^h. We can easily obtain from [1] that $H^1(I^\infty) = \frac{1}{2}$. Then $\frac{1}{2}H^1(I^\infty) = \frac{1}{4}$. Consider $\frac{1}{2}I^\infty = \prod_{i=1}^\infty \{0, \frac{1}{2}\}$ and define a map $f : \frac{1}{2}I^\infty \to [0,1]$ by $f(x) = \sum_{i=1}^\infty \dfrac{2x_i}{2^i}$, where $x = (x_i) \in \frac{1}{2}I^\infty$. Then

$$\begin{aligned}
|f(x) - f(y)| &\le 2\sum_{i=1}^\infty \frac{|x_i - y_i|}{2^i} = 2\sum_{i=1}^\infty \frac{|x_i - y_i|(1 + |x_i - y_i|)}{2^i(1 + |x_i - y_i|)} \\
&\le 2\sum_{i=1}^\infty \frac{\frac{3}{2}|x_i - y_i|}{2^i(1 + |x_i - y_i|)} (\text{since } x_i, y_i = 0 \text{ or } \frac{1}{2}) \\
&= 3\rho(x,y)
\end{aligned}$$

It follows from the Theorem 2.1 in [1] that $H^1([0,1]) \le 3H^1(\frac{1}{2}I^\infty)$.

Since $H^1([0,1]) = 1$, $(\frac{1}{2})^1 H^1(I^\infty) = \frac{1}{4} < \frac{1}{3} \le H^1(\frac{1}{2}I^\infty)$. That is, the Hausdorff measure on $\mathbb{R}^\infty$ does not hold the scaling property.

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