KYUNGPOOK Math. J. 49(2009), 595-603

Riccati Equation and Positivity of Operator Matrices

Jun Ichi Fujii
Department of Information Sciences, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan
e-mail : fujii@@cc.osaka-kyoiku.ac.jp

Masatoshi Fujii*
Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan
e-mail : mfujii@@cc.osaka-kyoiku.ac.jp

Ritsuo Nakamoto
Faculty of Engineering, Ibaraki University, Hitachi, Ibaraki 316-8511, Japan
e-mail : nakamoto@@base.ibaraki.ac.jp

Abstract. We show that for an algebraic Riccati equation $X^*B^{-1}X - T^*X - X^*T = C$, its solutions are given by $X = W + BT$ for some solution W of $X^*B^{-1}X = C + T^*BT$. To generalize this, we give an equivalent condition for $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$ for given positive operators B and A, by which it can be regarded as Riccati inequality $X^*B^{-1}X \leq A$. As an application, the harmonic mean $B\ !\ C$ is explicitly written even if B and C are noninvertible.

1. Introduction

The following equation is said to be the algebraic Riccati equation:

$$X^*B^{-1}X - T^*X - X^*T = C \tag{1}$$

for positive definite matrices B, C and arbitrary T. The simple case $T = 0$ in (1)

$$X^*B^{-1}X = C \tag{2}$$

is called the Riccati equation by several authors. It is known that the geometric mean $B \sharp C$ is the unique positive definite solution of (2), see [6] and [4]. Recall Ando's definition of it in terms of operator matrix [2]; for positive operators B, C

---

* Corresponding author.
Received April 5, 2009; accepted May 4, 2009.
2000 Mathematics Subject Classification: 47A63, 47A64, 47B15.
Key words and phrases: Riccati equation, operator matrix, geometric mean and harmonic mean.

on a Hilbert space,

$$B \sharp C = \max \left\{ X \geq 0; \begin{pmatrix} B & X \\ X & C \end{pmatrix} \geq 0 \right\}. \tag{3}$$

If B is invertible, it is expressed by

$$B \sharp C = B^{\frac{1}{2}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}}. \tag{4}$$

Very recently, Izumino and Nakamura [5] gave some interesting consideration to weakly positive operators introduced by Wigner [12]. An operator T is weakly positive if $T = SCS^{-1}$ for some $S,\ C > 0$, where $X > 0$ means it is positive and invertible. It is equivalent to be of form $T = AB$ for some $A,\ B > 0$. (Take $A = S^2$ and $B = S^{-1}CS^{-1}$.) They pointed out that the square root $T^{\frac{1}{2}}$ of a weakly positive operator $T = SCS^{-1} = AB$ can be defined by $T^{\frac{1}{2}} = SC^{\frac{1}{2}}S^{-1} = A^{\frac{1}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}A^{-\frac{1}{2}}$, and

$$A^{-1} \sharp B = A^{-1}(AB)^{\frac{1}{2}}.$$

As an easy consequence, $A^{-1} \sharp B$ is a (unique) positive solution of Riccati equation $XAX = B$ for given $A,\ B > 0$.

In this note, we discuss a relation between solutions of (1) and (2), and give their solutions. We show that every solution of (1) is given by $X = W + BT$ for some solution W of $X^*B^{-1}X = C + T^*BT$. To generalize this, we next investigate an operator W satisfying $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$ for given positive operators B and A, by which W can be regarded as a solution of Riccati inequality $X^*B^{-1}X \leq A$ if B is invertible. Precisely, we show that $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$ if and only if $W = B^{\frac{1}{2}}X$ for some operator X and $A \geq X^*X$. As an application, the harmonic mean $B \,!\, C$ is explicitly written even if B and C are noninvertible.

2. Riccati equation

In this section, we discuss a relation between solutions of Riccati equations (1) and (2), by which solutions of (1) can be given. The following lemma says that (2) is substantial in a mathematical sense.

Lemma 1. *Let B be positive invertible, C positive and T arbitrary operators on a Hilbert space. Then W is a solution of Riccati equation*

$$W^*B^{-1}W = C + T^*BT$$

if and only if $X = W + BT$ is a solution of an algebraic Riccati equation

$$X^*B^{-1}X - T^*X - X^*T = C.$$

Proof. Put $X = W + BT$. Since

$$X^*B^{-1}X - T^*X - X^*T = W^*B^{-1}W - T^*BT$$

we have the conclusion immediately. □

Next we determine solutions of Riccati equation (2):

Lemma 2. *Let B be positive invertible and A positive. Then W is a solution of Riccati equation*

$$W^*B^{-1}W = A \tag{5}$$

if and only if W is of form $W = B^{\frac{1}{2}}UA^{\frac{1}{2}}$ for some partial isometry U whose initial space contains ran $A^{\frac{1}{2}}$.

Proof. If W is a solution, then $\|B^{-\frac{1}{2}}Wx\| = \|A^{\frac{1}{2}}x\|$ for all vectors x. It ensures the existence of a partial isometry U such that $B^{-\frac{1}{2}}W = UA^{\frac{1}{2}}$, i.e., $W = B^{\frac{1}{2}}UA^{\frac{1}{2}}$. □

Consequently, we have solutions of an algebraic Riccati equation (1).

Theorem 3. *The solutions of an algebraic Riccati equation* (1)

$$X^*B^{-1}X - T^*X - X^*T = C.$$

is given by $X = B^{\frac{1}{2}}U(C + T^*BT)^{\frac{1}{2}} + BT$ for some partial isometry U whose initial space contains ran $(C + T^*BT)^{\frac{1}{2}}$.

In addition, the following result due to Trapp [11] is obtained by Lemma 1.

Corollary 4. *Under the assumption that BT is selfadjoint, the selfadjoint solution of an algebraic Riccati equation* (1) *is of form*

$$X = (T^*BT + C) \sharp B + BT.$$

Proof. The uniqueness of solution follows from the fact that $A \sharp B$ is the unique positive solution of $XB^{-1}X = A$. □

3. Riccati inequality

In this section, we will generalize Riccati equation, say Riccati inequality. Actually it is realized as the positivity of an operator matrix $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$ for given positive operators B and A. Roughly speaking, it is regarded as an operator inequality $W^*B^{-1}W \leq A$. As a matter of fact, it is correct if B is invertible.

Lemma 5. *Let A be a positive operator. Then*

$$\begin{pmatrix} 1 & X \\ X^* & A \end{pmatrix} \geq 0 \quad \textit{if and only if} \quad A \geq X^*X.$$

Proof. Since

$$\begin{pmatrix} 1 & 0 \\ 0 & A - X^*X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -X^* & 1 \end{pmatrix} \begin{pmatrix} 1 & X \\ X^* & A \end{pmatrix} \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix},$$

it follows that

$$\begin{pmatrix} 1 & X \\ X^* & A \end{pmatrix} \geq 0 \quad \text{if and only if} \quad A \geq X^*X.$$

□

Lemma 6. *Let A and B be positive operators. Then*

$$\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \quad \textit{implies} \quad \operatorname{ran} W \subseteq \operatorname{ran} B^{\frac{1}{2}}$$

and so $X = B^{-\frac{1}{2}}W$ is well-defined as a mapping.

Proof. Let $S = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix}$ be the square root of $R = \begin{pmatrix} B & W \\ W^* & A \end{pmatrix}$. Then

$$R = S^2 = \begin{pmatrix} a^2 + bb^* & ab + bd \\ b^*a + db^* & b^*b + d^2 \end{pmatrix},$$

that is,

$$B = a^2 + bb^* \text{ and } W = ab + bd.$$

Since ran $B^{\frac{1}{2}}$ contains both ran a and ran b by Douglas' majorization theorem [3], it contains ran $a+$ ran b. Moreover ran W is contained in ran $a+$ ran b by $W = ab + bd$. □

Theorem 7. *Let A and B be positive operators on K and H respectively, and W be an operator from K to H. Then $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$ if and only if $W = B^{\frac{1}{2}}X$ for some operator X from K to H and $A \geq X^*X$.*

Proof. Suppose that $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$. Since ran $W \subseteq$ ran $B^{\frac{1}{2}}$ by Lemma 6, Douglas' majorization theorem [3] says that $W = B^{\frac{1}{2}}X$ for some operator X. Moreover we restrict X by $P_BX = X$, where P_B is the range projection of B. Noting that $y \in$ ran B if and only if $y = B^{\frac{1}{2}}x$ for some $x \in$ ran $B^{\frac{1}{2}}$, the assumption implies that

$$\left(\begin{pmatrix} P_B & X \\ X^* & A \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right) = \left(\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}, \begin{pmatrix} x \\ z \end{pmatrix} \right) \geq 0$$

for all $y \in$ ran B and $z \in K$. This means that $\begin{pmatrix} P_B & X \\ X^* & A \end{pmatrix} \geq 0$, and so

$$\begin{pmatrix} P_B & 0 \\ 0 & A - X^*X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -X^* & A \end{pmatrix} \begin{pmatrix} P_B & X \\ X^* & A \end{pmatrix} \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix} \geq 0,$$

that is, $A \geq X^*X$, as required. The converse is easily checked. □

The above theorem was essentially shown in Šmul'jan [10, Theorem 1.7]. The following factorization theorem due to Ando [2] is led by Theorem 7.

Theorem 8. *Let A and B be positive operators. Then* $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$ *if and only if* $W = B^{\frac{1}{2}}VA^{\frac{1}{2}}$ *for some contraction* V.

Proof. Suppose that $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$. Then it follows from Theorem 7 that $W = B^{\frac{1}{2}}X$ for some bounded X satisfying $A \geq X^*X$. Hence we can find a contraction V with $X = VA^{\frac{1}{2}}$ by [3], so that $W = B^{\frac{1}{2}}VA^{\frac{1}{2}}$ is shown.

The converse is proved by Lemma 5 as follows:

$$\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} = \begin{pmatrix} B & B^{\frac{1}{2}}VA^{\frac{1}{2}} \\ A^{\frac{1}{2}}V^*B^{\frac{1}{2}} & A \end{pmatrix} = \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & V \\ V^* & 1 \end{pmatrix} \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} \geq 0.$$

□

4. The geometric mean and harmonic mean

Finally we consider the geometric mean and the harmonic one, as an application of the preceding section. The former is defined by

$$B \sharp C = \max \left\{ X \geq 0; \begin{pmatrix} B & X \\ X & C \end{pmatrix} \geq 0 \right\}.$$

If B is invertible, then Theorem 7 says that $\begin{pmatrix} B & X \\ X & C \end{pmatrix} \geq 0$ if and only if $C \geq X^*B^{-1}X$.

By the way, we can directly obtain the desired inequality $C \geq X^*B^{-1}X$ by the following identity:

$$\begin{pmatrix} 1 & 0 \\ -X^*B^{-1} & 1 \end{pmatrix} \begin{pmatrix} B & X \\ X^* & C \end{pmatrix} \begin{pmatrix} 1 & -B^{-1}X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & C - X^*B^{-1}X \end{pmatrix}.$$

Anyway the maximum is given by

$$B \sharp C = B^{\frac{1}{2}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}}.$$

Next we review a work of Pedersen and Takesaki [8]. They proved that if B and C are positive operators and B is nonsingular, then there exists a positive solution X of $XBX = C$ if and only if $(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} \leq kB$ holds for some $k > 0$.

From the viewpoint of Riccati inequality, we add another equivalent condition to Pedersen-Takesaki theorem:

Theorem 9. *Let B and C be positive operators and B be nonsingular. Then the following statements are mutually equivalent:*

(1) *Riccati equation $XBX = C$ has a positive solution.*

(2) $(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} \leq kB$ *for some* $k > 0$.

(3) *There exists the minimum of* $\{X \geq 0; C \leq XBX\}$.

(3') *There exists the minimum of* $\left\{X \geq 0; \begin{pmatrix} 1 & C^{\frac{1}{2}} \\ C^{\frac{1}{2}} & XBX \end{pmatrix} \geq 0\right\}$.

Proof. We first note that (3) and (3') are equivalent by Lemma 5.

Now we suppose (1), i.e., $X_0BX_0 = C$ for some $X_0 \geq 0$. If $X \geq 0$ satisfies $C \leq XBX$, then

$$(B^{\frac{1}{2}}X_0B^{\frac{1}{2}})^2 = B^{\frac{1}{2}}CB^{\frac{1}{2}} \leq (B^{\frac{1}{2}}XB^{\frac{1}{2}})^2$$

and so

$$B^{\frac{1}{2}}X_0B^{\frac{1}{2}} \leq B^{\frac{1}{2}}XB^{\frac{1}{2}}.$$

Since B is nonsingular, we have $X_0 \leq X$, namely (3) is proved.

Next we suppose (3). Since $C \leq XBX$ for some X, we have

$$B^{\frac{1}{2}}CB^{\frac{1}{2}} \leq (B^{\frac{1}{2}}XB^{\frac{1}{2}})^2$$

and so

$$(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} \leq B^{\frac{1}{2}}XB^{\frac{1}{2}} \leq \|X\|B,$$

which shows (2).

The implication (2) $\to$ (1) has been shown by Pedersen-Takesaki [8] and Nakamoto [7], but we sketch it for convenience. By Douglas' majorization theorem [3], we have

$$(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{4}} = ZB^{\frac{1}{2}}$$

for some Z, so that

$$(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} = B^{\frac{1}{2}}Z^*ZB^{\frac{1}{2}} \quad \text{and} \quad B^{\frac{1}{2}}CB^{\frac{1}{2}} = B^{\frac{1}{2}}(Z^*ZBZ^*Z)B^{\frac{1}{2}}.$$

Since B is nonsingular, Z^*Z is a solution of $XBX = C$. □

In adition, we consider an operator matrix $M_{B,C}(X) = \begin{pmatrix} 1 & B^{\frac{1}{2}}X \\ XB^{\frac{1}{2}} & C \end{pmatrix}$ for $B, C, X \geq 0$. We know that $M_{B,C}(X) \geq 0$ if and only if $C \geq XBX$ by Lemma 5. We remark that there exists the maximum of $\{X \geq 0; M_{B,C}(X) \geq 0\}$ if (1) in Theorem 9 holds. As a matter of fact, if $X_0BX_0 = C$ for some $X_0 \geq 0$, then it follows from Lemma 5 that

$$X_0BX_0 = C \geq XBX \quad \text{and so} \quad B^{\frac{1}{2}}X_0B^{\frac{1}{2}} \geq B^{\frac{1}{2}}XB^{\frac{1}{2}}$$

for $X \geq 0$ with $M_{B,C}(X) \geq 0$. Finally the nonsingularity of B implies $X_0 \geq X$, as desired.

On the other hand, the harmonic mean is defined by

$$B \,!\, C = \max\left\{X \geq 0; \begin{pmatrix} 2B & 0 \\ 0 & 2C \end{pmatrix} \geq \begin{pmatrix} X & X \\ X & X \end{pmatrix}\right\}.$$

To obtain the exact expression of the harmonic mean, we need the following lemma which is more explicit than Theorem 7.

Lemma 10. *If* $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$*, then* $X = B^{-\frac{1}{2}}W$ *is bounded and* $A \geq X^*X$*.*

Proof. For a fixed vector x, we put $x_1 = B^{-\frac{1}{2}}Wx$. Since $B^{\frac{1}{2}}x_1 = Wx$, we may assume $x_1 \in (\ker B^{\frac{1}{2}})^{\perp}$. So it follows that

$$\begin{aligned} \|B^{-\frac{1}{2}}Wx\| &= \sup\{|(B^{-\frac{1}{2}}Wx, v)|; \|v\| = 1\} \\ &= \sup\{|(B^{-\frac{1}{2}}Wx, B^{\frac{1}{2}}u)|; \|B^{\frac{1}{2}}u\| = 1\} \\ &= \sup\{|(Wx, u)|; (Bu, u) = 1\}. \end{aligned}$$

On the other hand, since

$$\left(\begin{pmatrix} B & W \\ W^* & A \end{pmatrix}\begin{pmatrix} u \\ tx \end{pmatrix}, \begin{pmatrix} u \\ tx \end{pmatrix}\right) = |t|^2(Ax, x) + 2\mathrm{Re}\; t(Wx, u) + (Bu, u) \geq 0$$

for all scalars t, we have

$$|(Wx, u)|^2 \leq (Ax, x)(Bu, u).$$

Hence it follows that

$$\|B^{-\frac{1}{2}}Wx\|^2 = \sup\{|(Wx, u)|^2; (Bu, u) = 1\} \leq (Ax, x),$$

which implies that $X = B^{-\frac{1}{2}}W$ is bounded and $A \geq X^*X$. □

Theorem 11. *Let B, C be positive operators. Then*

$$B \,!\, C = 2(B - [(B + C)^{-\frac{1}{2}}B]^*[(B + C)^{-\frac{1}{2}}B]).$$

In particular, if $B + C$ is invertible, then

$$B \,!\, C = 2(B - B(B + C)^{-1}B) = 2B(B + C)^{-1}C.$$

Proof. First of all, the inequality $\begin{pmatrix} 2B & 0 \\ 0 & 2C \end{pmatrix} \geq \begin{pmatrix} X & X \\ X & X \end{pmatrix}$ is equivalent to

$$\begin{pmatrix} 2(B + C) & -2B \\ -2B & 2B - X \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2B - X & -X \\ -X & 2C - X \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \geq 0.$$

Then it follows from Lemma 5 that $D = [2(B+C)]^{-\frac{1}{2}}(-2B)$ is bounded and $D^*D \leq 2B - X$. Therefore we have the explicit expression of $B\ !\ C$ even if both B and C are non-inbertible:

$$B\ !\ C = \max\{X \geq 0; D^*D \leq 2B - X\} = 2B - D^*D.$$

In particular, if $B + C$ is invertible, then

$$B\ !\ C = 2B - D^*D = 2(B - B(B+C)^{-1}B) = 2B(B+C)^{-1}C.$$

□

5. Riccati inequality for projections

Finally we consider the set

$$\mathcal{F}_E = \{X \in B(H); X^*EX \leq E\}$$

for a projection E, where $B(H)$ is the set of all bounded linear operators on H.

Lemma 12. *Let E be a projection. Then*

$$\mathcal{F}_E = \left\{\begin{pmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{pmatrix} \text{ on } EH \oplus (1-E)H;\ \|X_{11}\| \leq 1\right\}.$$

Proof. If $X \in \mathcal{F}_E$, then $EX^*EXE \leq E$ and so EXE is a contraction. On the other hand, since $(1-E)X^*EX(1-E) = 0$, we have $EX(1-E) = 0$.

Conversely suppose that

$$X = \begin{pmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{pmatrix} \text{ on } EH \oplus (1-E)H \text{ and } \|X_{11}\| \leq 1.$$

Then

$$X^*EX = \begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = E.$$

□

Consequently we have the following:

Theorem 13. *Let E be a projection. Then*

(1) *A positive operator X belongs to $\mathcal{F}_E$ if and only if $X = X_1 \oplus X_2$ on $EH \oplus (1-E)H$ and $X_1 \leq 1$.*

(2) *A projection F belongs to $\mathcal{F}_E$ if and only if F commutes with E.*

(3) *A projection F satisfies $FEF = E$ if and only if $F \leq E$.*

Proof. (1) follows from the preceding lemma, and (2) does from (1). For (3), first suppose that a projection F satisfies FEF. Then F commutes with E by (2), so that $FE = FEF = E$. The converse is clear. □

Acknowledgment. The authors would like to express their thanks to Professor J.

Tomiyama for his valuable comment on Theorem 7 and to Professor H.Kosaki who informed that Schmitt's paper [9] is related to our discussion.

References

[1] W. N. Anderson, Jr. and G. E. Trapp, Operator means and electrical networks, Proc. 1980 IEEE International Symposium on Circuits and systems, 1980, 523-527.

[2] T. Ando, Topics on Operator Inequalities, Lecture Note, Sapporo, 1978.

[3] R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc., **17**(1966), 413-415.

[4] J. I. Fujii and M. Fujii, *Some remarks on operator means*, Math. Japon., **24**(1979), 335-339.

[5] S. Izumino and M. Nakamura, Wigner's weakly positive operators, Sci. Math. Japon., **65**(2007), 61-67.

[6] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann., **246**(1980), 205-224.

[7] R. Nakamoto, *On the operator equation* $THT = K$, Math. Japon., **24**(1973), 251-252.

[8] G. K. Pedersen and M. Takesaki, *The operator equation* $THT = K$, Proc. Amer. Math. Soc., **36**(1972), 311-312.

[9] L. M. Schmitt, *The Radon-Nikodym theorem for* L^p*-sapces of* W^**-algebras*, Publ. RIMS, Kyoto Univ., **22**(1986), 1025-1034.

[10] Ju. L. Šmul'jan, *An operator Hellinger integral*, Mat. Sb., **91**(1959), 381-430. (English translation: *A Hellinger operator integral*, Amer. Math. Soc. Translations, Ser. 2, **22**(1962), 289-338.)

[11] G. E. Trapp, *Hermitian semidefinite matrix means and related matrix inequalities - An introduction*, Linear Multilinear Alg., **16**(1984), 113-123.

[12] E. P. Wigner, *On weakly positive operators*, Canadian J. Math., **15**(1965), 313-317.