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## Sharp-unknotting Number of a Torus Knot

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Abstract. We give an estimation for the sharp-unknotting number of certain types of torus knots, and decide it for 39 torus knots.

### 1. Introduction

Hitoshi Murakami [3] defined a *sharp-move*, which is a change in a oriented link projection as shown in Fig. 1. He has proved that a sharp-move is an unknotting operation, that is, for any oriented knot $K$ there exists a finite sequence of sharp-moves which deform $K$ into a trivial knot. He then considered the minimum number of sharp-moves needed to transform a knot $K$ into another knot $K'$; he defined such a number the *sharp-Gordian distance* from $K$ to $K'$. In particular, the *sharp-unknottiing number* of $K$ is the sharp-Gordian distance from $K$ to a trivial knot, denoted by $\mathrm{u}^{\#}(K)$.

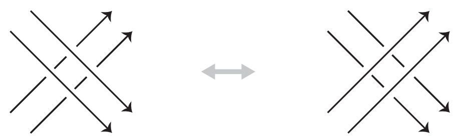

Figure 1: A sharp-move.

In this note, we estimate the sharp-unknotting number of a torus knot of type $(r, s)$ with $2 \leq s \leq 7$, $s < r$, and that of type $(p, p + 1)$ with $p \geq 2$, which enables us to decide the sharp-unknotting number for 39 torus knots listed in Table 1. In order to give a lower bound we use the unknotting number given by Kromheimer and Mrowka, and the signature of a torus knot; also we use the Arf invariant which decides the parity of the sharp-unknotting number.

This paper is organized as follows: In Sect. 2, we show a lemma to give an upper bound for the sharp-unknotting number of a torus knot. In Sect. 3, we give some formulas to give an estimation for the sharp-unknotting number of a torus knot.

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In Sect. 4, we estimate the sharp-unknotting number of the torus knots of types as mentioned above.

Table 1: $\mathrm{u}^{\#}(p,q)$.

| $\mathrm{u}^{\#}(p,q)$ | $(p,q)$ |
|---|---|
| 1 | $(2,3)$, $(2,5)$, $(3,4)$ |
| 2 | $(2,7)$, $(2,9)$, $(3,5)$, $(3,7)$ |
| 3 | $(2,11)$, $(2,13)$, $(3,8)$, $(3,10)$, $(4,5)$, $(5,6)$ |
| 4 | $(2,15)$, $(2,17)$, $(3,11)$, $(3,13)$, $(4,7)$, $(4,9)$, $(5,7)$ |
| 5 | $(2,21)$, $(3,14)$, $(3,16)$, $(4,11)$, $(5,8)$ |
| 6 | $(3,19)$, $(5,11)$, $(7,8)$ |
| 7 | $(3,22)$ , $(5,12)$ |
| 8 | $(7,10)$ |
| 9 | $(5,16)$, $(6,13)$ |
| 10 | $(7,12)$, $(9,10)$ |
| 12 | $(7,15)$ |
| 15 | $(11,12)$ |
| 21 | $(13,14)$ |
| 28 | $(15,16)$ |

## 2. Braids and torus knots

An $n$-braid is an element of the $n$-braid group $B_n$ generated by the elementary $n$-braids $\sigma_1$, $\sigma_2$, $\sigma_3, \cdots, \sigma_{n-1}$ as shown in Fig. 2. They are related by the *braid relations*:

(1) $$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| \geq 2;$$

(2) $$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad i = 1, 2, \cdots, n-2.$$

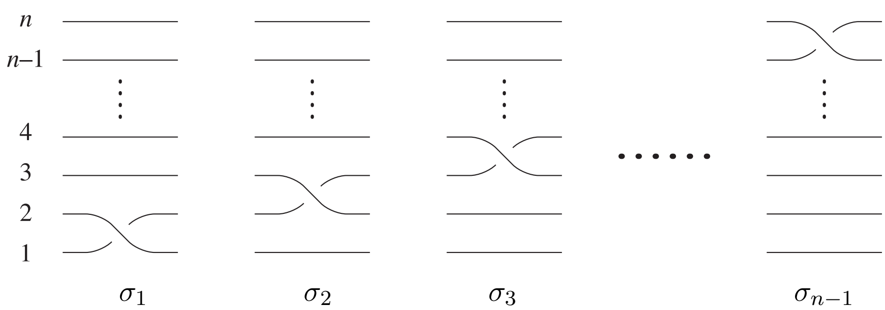


Figure 2: Elementary $n$-braids.

The *half-twist* $\Delta_n$ of an $n$-braid is given by

$$\Delta_n = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_2\sigma_1)\,(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_2)\cdots(\sigma_{n-1}\sigma_{n-2})\,\sigma_{n-1}, \tag{3}$$

and the braid $\Delta_n^2$ is called the *full-twist*, which generates the center of $B_n$. For a pair of coprime positive integers $(p,q)$, we define a torus knot of type $(p,q)$ as the closure of the $p$-braid

$$(\sigma_{p-1}\sigma_{p-2}\cdots\sigma_2\sigma_1)^q\,, \tag{4}$$

which we denote by $\mathrm{T}(p,q)$. We can deform a torus knot into a trivial knot using the twisting operations considered in the following lemma, from which we obtain an upper bound for the sharp-unknotting number of a torus knot.

**Lemma 2.1.** (i) *A full-twist of a* $(2k+1)$*-braid is deformed into a trivial braid by applying sharp-moves* $k(k+1)/2$ *times.*
(ii) *Two full-twists of a* $2k$*-braid is deformed into a trivial braid by applying sharp-moves* $k^2$ *times.*

*Proof.* (i) A full-twist of a $(2k+1)$-braid is considered as a link diagram as shown in Fig. 3(a). By applying sharp-moves $k$ times around the asterisks we may deform it into a diagram as in Fig. 3(b), which is isotopic to a $(2k+1)$-braid as in Fig. 3(c). Since it contains one full-twist of a $(2k-1)$-braid and $\sum_{i=1}^{k} i = k(k+1)/2$, we have the result; cf. [8, Fig. 2.6].

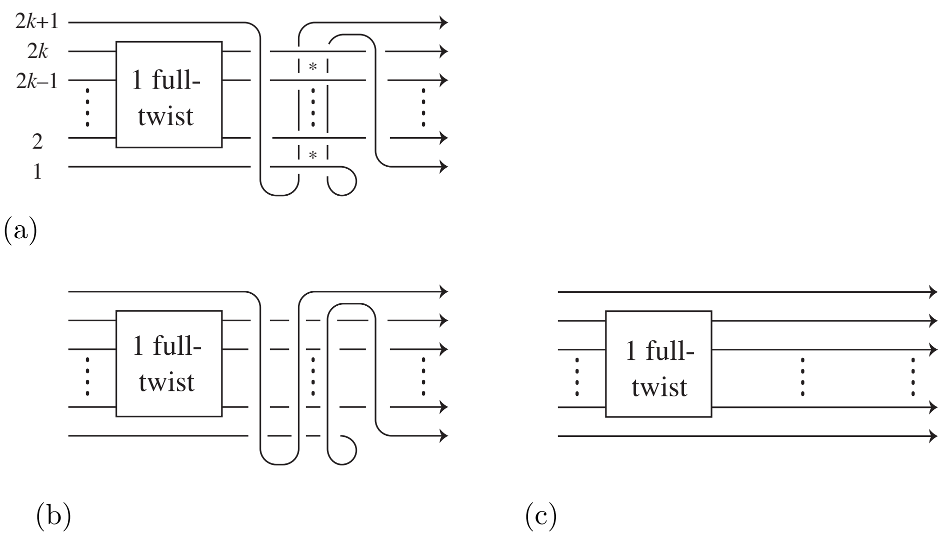


Figure 3: Sharp-moves on a full-twist of a $(2k+1)$-braid.

(ii) For $k=1$, see Fig. 4; cf. [3, Fig. 8]: The two diagrams of Figs. 4(a) and 4(b) and those of Figs. 4(c) and 4(d) are isotopic, and the diagram of Fig. 4(b) is deformed into that of Fig. 4(c) by applying one sharp-move.

A full-twist of a $2k$-braid is considered as a $2k$-braid as shown in Fig. 5(a). By applying sharp-moves $k-1$ times around the asterisks we may deform it into a $2k$-braid as shown in Fig. 5(b), which is isotopic to a $2k$-braid as shown in Fig. 5(c). It contains a full-twist of a 2-braid and one of a $(2k-2)$-braid. Thus we may deform two full-twists of a $2k$-braid into two full-twists of a 2-braid and two full-twists of a $(2k-2)$-braid by applying sharp-moves $2k-2$ times, from which we remove two full-twists of a 2-braid by applying another sharp-move. Therefore, applying sharp-moves $(2k-1)$ times on two full-twists of a $2k$-braid, we obtain two full-twists of a $(2k-2)$-braid. Since $\sum_{i=1}^{k}(2i-1)=k^2$, we obtain the result. □

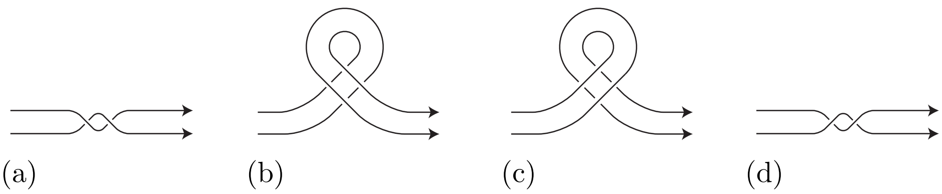


Figure 4: A twisting operation.

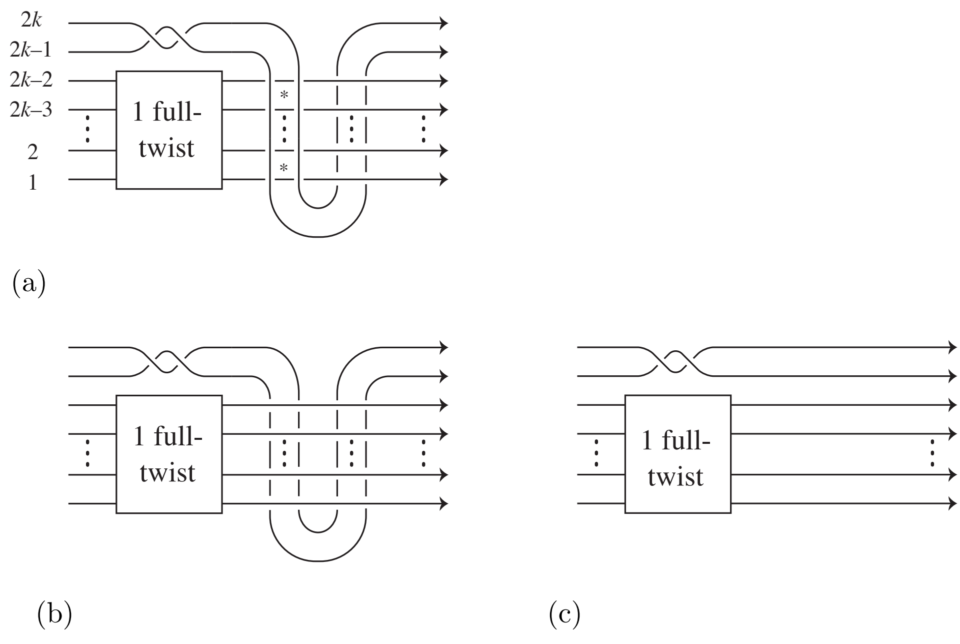


Figure 5: Sharp-moves on a full-twist of a $2k$-braid.

**Remark 2.2.** The twisting operations considered in Lemma 2.1 (i) and (ii) are called an oriented $(2k+1,1)$-move and an oriented $(2k,2)$-move, respectively, in [8].

## 3. Formulas for an estimation for the sharp-unknotting number

We give some relations to estimate the sharp-unknotting number of a torus knot. Let $\mathrm{u}(K)$ be the usual unknotting number of a knot $K$. Then from the definition, we have

$$\mathrm{u}^{\#}(K) \geq \lceil \mathrm{u}(K)/4 \rceil, \tag{5}$$

where $\lceil \mathrm{u}(K)/4 \rceil$ is the smallest integer not less than $\mathrm{u}(K)/4$. Let $\tilde{\sigma}(K)$ be the signature of a knot $K$, where $\tilde{\sigma}(\mathrm{T}(2,3)) = 2$ for the right-hand (positive) trefoil knot $\mathrm{T}(2,3)$, and $\mathrm{Arf}(K) \in \boldsymbol{Z}/2\boldsymbol{Z}$ the Arf invariant of $K$, where $\mathrm{Arf}(K) \equiv a_2(K)$ (mod 2) with $a_2(K)$ the second coefficient of the Conway polynomial of $K$. The following proposition is due to Murakami, where Eqs. (6) and (7) are given in Theorems 3.2 and 3.5 in [3], respectively.

**Proposition 3.1.** *Let $K$ and $K'$ be knots such that $K$ is obtained from $K'$ by a single sharp-move. Then the following hold:*

$$|\tilde{\sigma}(K) - \tilde{\sigma}(K')| = 2, 4, \textit{or } 6; \tag{6}$$

$$\mathrm{Arf}(K) - \mathrm{Arf}(K') \equiv 1 \pmod 2. \tag{7}$$

This implies:

**Corollary 3.2.**

$$\mathrm{u}^{\#}(K) \geq \lceil |\tilde{\sigma}(K)|/6 \rceil\, ; \tag{8}$$

$$\mathrm{u}^{\#}(K) \equiv \mathrm{Arf}(K) \pmod 2. \tag{9}$$

Note that combining $2\mathrm{u}(K) \geq |\tilde{\sigma}(K)|$ (cf. [6, Theorem 6.4.8]) and Eq. (5), we obtain $\mathrm{u}^{\#}(K) \geq \lceil |\tilde{\sigma}(K)|/8 \rceil$ for a knot $K$.

**Remark 3.3.** The sharp-unknotting number is also estimmated from below by using homology invariants from a cyclic covering of the knot space [4], which is useless in considering a torus knot.

## 4. Estimations of the sharp-unknotting numbers of torus knots

We denote the unknotting number, the sharp-unknotting number, the Arf invariant, and the signature of the torus knot $\mathrm{T}(p,q)$, $p$, $q > 0$, by $\mathrm{u}(p,q)$, $\mathrm{u}^{\#}(p,q)$,

$\mathrm{Arf}(p,q)$, and $\tilde{\sigma}(p,q)$, respectively. Then we have:

$$\mathrm{u}(p,q) = (p-1)(q-1)/2; \tag{10}$$

$$\mathrm{Arf}(p,q) \equiv (p^2-1)(q^2-1)/24 \pmod 2. \tag{11}$$

Eq. (10) is due to Kronheimer and Mrowka [2] and Eq. (11) follows the formula $a_2(\mathrm{T}(p,q)) = (p^2-1)(q^2-1)/24$ given in [7]. The signature $\tilde{\sigma}(p,q)$ can be calculated by using the recurrence formula given in [1, Theorem 5.2]; cf. [6, Theorem 7.5.1].

**4.1.** $(r,2)$**-torus knots.** From Lemma 2.1(ii), we have

$$\mathrm{u}^{\#}(4m \pm 1, 2) \le m \quad (m > 0), \tag{12}$$

which was already given by Tetsuo Shibuya; see [3, p. 45]. Since $\tilde{\sigma}(r,2) = r-1$, $r > 0$, Eq. (8) implies $\mathrm{u}^{\#}(4m+1,2) \ge \lceil 2m/3 \rceil$ and $\mathrm{u}^{\#}(4m-1,2) \ge \lceil (2m-1)/3 \rceil$.

From Eq. (11), we have $\mathrm{Arf}(4m \pm 1, 2) \equiv m \pmod 2$. Using these formulas, we obtain Table 2. In particular, we can decide the following sharp-unknotting numbers:

$$\begin{aligned} &\mathrm{u}^{\#}(3,2) = \mathrm{u}^{\#}(5,2) = 1; \quad &&\mathrm{u}^{\#}(7,2) = \mathrm{u}^{\#}(9,2) = 2; \\ &\mathrm{u}^{\#}(11,2) = \mathrm{u}^{\#}(13,2) = 3; \quad &&\mathrm{u}^{\#}(15,2) = \mathrm{u}^{\#}(17,2) = 4; \\ &\mathrm{u}^{\#}(21,2) = 5. \end{aligned} \tag{13}$$

Table 2: $\mathrm{u}^{\#}(r,2)$, $r \ge 3$.

| $r$ | $\mathrm{Arf}(r,2)$ | Lower bound | Upper bound |
|---|---|---|---|
| $24l \pm 1$ | 0 | $4l$ | $6l$ |
| $24l+3,\ 24l+5$ | 1 | $4l+1$ | $6l+1$ |
| $24l+7,\ 24l+9$ | 0 | $4l+2$ | $6l+2$ |
| $24l+11,\ 24l+13$ | 1 | $4l+3$ | $6l+3$ |
| $24l+15,\ 24l+17$ | 0 | $4l+4$ | $6l+4$ |
| $24l+19$ | 1 | $4l+3$ | $6l+5$ |
| $24l+21$ | 1 | $4l+5$ | $6l+5$ |

**4.2.** $(r,3)$**-torus knots.** From Lemma 2.1(i), we have

$$\mathrm{u}^{\#}(3m \pm 1, 3) \le m \quad (m > 0), \tag{14}$$

Since

$$\begin{aligned} &\tilde{\sigma}(6k+1,3) = 8k, \quad &&\tilde{\sigma}(6k+2,3) = 8k+2; \\ &\tilde{\sigma}(6k+4,3) = 8k+6, \quad &&\tilde{\sigma}(6k+5,3) = 8k+8, \end{aligned} \tag{15}$$

where $k \ge 0$; cf. [5, Proposition 9.1], we have

$$\begin{aligned} &\mathrm{u}^{\#}(6k+1,3) \ge 4k/3, \quad &&\mathrm{u}^{\#}(6k+2,3) \ge (4k+1)/3; \\ &\mathrm{u}^{\#}(6k+4,3) \ge (4k+3)/3, \quad &&\mathrm{u}^{\#}(6k+5,3) \ge (4k+4)/3. \end{aligned} \tag{16}$$

Using Eqs. (5) and (10), we have

(17) $$\mathrm{u}^{\#}(3m+1,3) \geq 3m/4, \qquad \mathrm{u}^{\#}(3m-1,3) \geq (3m-2)/4.$$

From Eq. (11), we have

(18) $$\mathrm{Arf}(3m \pm 1, 3) \equiv m \pmod 2.$$

Combining Eqs. (14), (16)-(18), we obtain Table 3. In particular, we can decide the following sharp-unknotting numbers:

(19) $$\begin{aligned} &\mathrm{u}^{\#}(4,3) = 1; \qquad \mathrm{u}^{\#}(5,3) = \mathrm{u}^{\#}(7,3) = 2; \\ &\mathrm{u}^{\#}(8,3) = \mathrm{u}^{\#}(10,3) = 3; \qquad \mathrm{u}^{\#}(11,3) = \mathrm{u}^{\#}(13,3) = 4; \\ &\mathrm{u}^{\#}(14,3) = \mathrm{u}^{\#}(16,3) = 5; \ \ \mathrm{u}^{\#}(19,3) = 6; \ \ \mathrm{u}^{\#}(22,3) = 7. \end{aligned}$$

Table 3: $\mathrm{u}^{\#}(r,3)$, $r \geq 4$.

| $r$ | $\mathrm{Arf}(r,3)$ | Lower bound | Upper bound |
|---|---|---|---|
| $24l-5$ | 0 | $6l$ | $8l-2$ |
| $24l \pm 1$ | 0 | $6l$ | $8l$ |
| $24l+2,\ \ 24l+4$ | 1 | $6l+1$ | $8l+1$ |
| $24l-2$ | 1 | $6l+1$ | $8l-1$ |
| $24l+5,\ \ 24l+7$ | 0 | $6l+2$ | $8l+2$ |
| $24l+8,\ \ 24l+10$ | 1 | $6l+3$ | $8l+3$ |
| $24l+11\ \ \ 24l+13$ | 0 | $6l+4$ | $8l+4$ |
| $24l+17$ | 0 | $6l+4$ | $8l+6$ |
| $24l+14,\ \ 24l+16$ | 1 | $6l+5$ | $8l+5$ |
| $24l+20$ | 1 | $6l+5$ | $8l+7$ |

**4.3.** $(p,p+1)$**-torus knots.** Suppose $m \geq 2$. From Lemma 2.1(i), by applying sharp-moves $m(m-1)/2$ times on $\mathrm{T}(2m,2m-1)$, we obtain $\mathrm{T}(1,2m-1)$, which is a trivial knot, and by applying sharp-moves $m(m+1)/2$ times on $\mathrm{T}(2m,2m+1)$, we obtain $\mathrm{T}(-1,2m+1)$, which is a trivial knot. Thus we have

(20) $$\mathrm{u}^{\#}(2m,2m+\epsilon) \leq m(m+\epsilon)/2 \quad (\epsilon = \pm 1).$$

Using the recurrence formula given in [1], and (10), we have

(21) $$\begin{aligned} &\tilde{\sigma}(2m,2m-1) = 2m^2-2, \quad \tilde{\sigma}(2m,2m+1) = 2m^2; \\ &\mathrm{u}(2m,2m-1) = (m-1)(2m-1), \quad \mathrm{u}(2m,2m+1) = m(2m-1), \end{aligned}$$

from which we have

(22) $$\lceil \mathrm{u}(2m,2m+\epsilon)/4 \rceil \geq \lceil \tilde{\sigma}(2m,2m+\epsilon)/6 \rceil .$$

Thus by Eqs. (5) and (8) we will use $\lceil \mathrm{u}(2m, 2m+\epsilon)/4\rceil$ as the lower bound of the sharp-unknotting number of $\mathrm{T}(2m, 2m+\epsilon)$, that is,

$$\begin{aligned} \mathrm{u}^{\#}(2m, 2m-1) &\geq \lceil (m-1)(2m-1)/4 \rceil\,; \\ \mathrm{u}^{\#}(2m, 2m+1) &\geq \lceil m(2m-1)/4 \rceil\,. \end{aligned} \tag{23}$$

From Eq. (11), we have

$$\mathrm{Arf}(2m, 2m+\epsilon) \equiv \frac{(m+\epsilon)(2m-1)(2m)(2m+1)}{12} \pmod 2. \tag{24}$$

Using Eqs. (20), (23), (24), we obtain Table 4. In particular, we can decide the following sharp-unknotting numbers:

$$\begin{aligned} &\mathrm{u}^{\#}(4,5) = \mathrm{u}^{\#}(5,6) = 3; \quad \mathrm{u}^{\#}(7,8) = 6; \quad \mathrm{u}^{\#}(9,10) = 10; \\ &\mathrm{u}^{\#}(11,12) = 15; \qquad \mathrm{u}^{\#}(13,14) = 21; \qquad \mathrm{u}^{\#}(15,16) = 28. \end{aligned} \tag{25}$$

Table 4: $\mathrm{u}^{\#}(p, p+1)$, $p \geq 2$.

| $p$ | $\mathrm{Arf}(p, p+1)$ | Lower bound | Upper bound |
|---|---|---|---|
| $8k-2$ | 0 | $8k^2-5k+1$ | $8k^2-2k$ |
| $8k-1$ | 0 | $8k^2-3k+1$ | $8k^2-2k$ |
| $8k$ | 0 | $8k^2-k$ | $8k^2+2k$ |
| $8k+1$ | 0 | $8k^2+k$ | $8k^2+2k$ |
| $8k+2$ | 1 | $8k^2+3k+1$ | $8k^2+6k+1$ |
| $8k+3$ | 1 | $8k^2+5k+1$ | $8k^2+6k+1$ |
| $8k+4$ | 1 | $8k^2+7k+2$ | $8k^2+10k+3$ |
| $8k+5$ | 1 | $8k^2+9k+3$ | $8k^2+10k+3$ |

**4.4.** $(r,4)$**-torus knots.** From Lemma 2.1(ii), by applying sharp-moves $4k$ times on $\mathrm{T}(8k+i, 4)$, we obtain $\mathrm{T}(i,4)$. Since $\mathrm{T}(\pm 1, 4)$ is trivial, $\mathrm{u}^{\#}(3,4) = 1$ from Eq. (19), and $\mathrm{u}^{\#}(5,4) = 3$ from Eq. (25), we have

$$\begin{aligned} &\mathrm{u}^{\#}(8k+1, 4) \leq 4k; \qquad &&\mathrm{u}^{\#}(8k+3, 4) \leq 4k+1; \\ &\mathrm{u}^{\#}(8k+5, 4) \leq 4k+3; \qquad &&\mathrm{u}^{\#}(8k+7, 4) \leq 4k+4, \end{aligned} \tag{26}$$

where $k \geq 0$. Using the recurrence formula given in [1], and (10), we have

$$\begin{aligned} &\tilde{\sigma}(4m+1, 4) = 8m, \qquad &&\tilde{\sigma}(4m-1, 4) = 8m-2; \\ &\mathrm{u}(4m+1, 4) = 6m, \qquad &&\mathrm{u}(4m-1, 4) = 6m-3, \end{aligned} \tag{27}$$

where $m > 0$, cf. [5, Proposition 9.2]. Then using Eqs. (5) and (10), we have

$$\mathrm{u}^{\#}(4m+1, 4) \geq 3m/2, \quad \mathrm{u}^{\#}(4m-1, 4) \geq (6m-3)/4. \tag{28}$$

From Eq. (11), we have

$$\mathrm{Arf}(4m \pm 1, 4) \equiv m \pmod 2. \tag{29}$$

Using Eqs. (26), (28), (29), we obtain Table 5. In particular, we can decide the following sharp-unknotting numbers:

$$\mathrm{u}^{\#}(7,4) = \mathrm{u}^{\#}(9,4) = 4; \qquad \mathrm{u}^{\#}(11,4) = 5. \tag{30}$$

Table 5: $\mathrm{u}^{\#}(r,4)$, $r \geq 5$.

| $r$ | $\mathrm{Arf}(r,4)$ | Lower bound | Upper bound |
|---|---|---|---|
| $16l \pm 1$ | 0 | $6l$ | $8l$ |
| $16l+3$ | 1 | $6l+1$ | $8l+1$ |
| $16l+5$ | 1 | $6l+3$ | $8l+3$ |
| $16l+7$ , $16l+9$ | 0 | $6l+4$ | $8l+4$ |
| $16l+11$ | 1 | $6l+5$ | $8l+5$ |
| $16l+13$ | 1 | $6l+5$ | $8l+7$ |

**4.5.** $(r,5)$**-torus knots.** From Lemma 2.1(i), by applying sharp-moves $3m$ times on $\mathrm{T}(5m+i,5)$, we obtain $\mathrm{T}(i,5)$. Since $\mathrm{T}(\pm 1,5)$ is trivial, $\mathrm{u}^{\#}(2,5) = 1$ from Eq. (13), and $\mathrm{u}^{\#}(3,5) = 2$ from Eq. (19), we have

$$\begin{aligned} &\mathrm{u}^{\#}(5m \pm 1,5) \leq 3m; \qquad \mathrm{u}^{\#}(5m+2,5) \leq 3m+1; \\ &\mathrm{u}^{\#}(5m+3,5) \leq 3m+2, \end{aligned} \tag{31}$$

where $m \geq 0$. Using the recurrence formula given in [1], and (10), we have

$$\begin{aligned} &\tilde\sigma(10k+1,5) = 24k, \qquad \tilde\sigma(10k+2,5) = 24k+4; \\ &\tilde\sigma(10k+3,5) = \tilde\sigma(10k+4,5) = 24k+8; \\ &\tilde\sigma(10k+6,5) = \tilde\sigma(10k+7,5) = 24k+16; \\ &\tilde\sigma(10k+8,5) = 24k+20, \qquad \tilde\sigma(10k+9,5) = 24k+24; \\ &\mathrm{u}(r,5) = 2(r-1), \end{aligned} \tag{32}$$

where $k \geq 0$, $r > 0$. From Eq. (11), we have

$$\mathrm{Arf}(r,5) \equiv r-1 \pmod 2. \tag{33}$$

Then from Eqs. (31)–(33), we obtain Table 6. In particular, we can decide the following sharp-unknotting numbers:

$$\begin{aligned} &\mathrm{u}^{\#}(7,5) = 4; \qquad \mathrm{u}^{\#}(8,5) = 5; \qquad \mathrm{u}^{\#}(11,5) = 6; \\ &\mathrm{u}^{\#}(12,5) = 7; \qquad \mathrm{u}^{\#}(16,5) = 9. \end{aligned} \tag{34}$$

Table 6: $\mathrm{u}^{\#}(r,5)$, $r \geq 6$.

| $r$ | $\mathrm{Arf}(r,5)$ | Lower bound | Upper bound |
|---|---|---|---|
| $10k+1$ | 0 | $5k$ | $6k$ |
| $10k+2$ | 1 | $5k+1$ | $6k+1$ |
| $10k+3$ | 0 | $5k+1$ | $6k+2$ |
| $10k+4$ | 1 | $5k+2$ | $6k+3$ |
| $10k+6$ | 1 | $5k+3$ | $6k+3$ |
| $10k+7$ | 0 | $5k+3$ | $6k+4$ |
| $10k+8$ | 1 | $5k+4$ | $6k+5$ |
| $10k+9$ | 0 | $5k+4$ | $6k+6$ |

**4.6.** $(r,6)$**-torus knots.** From Lemma 2.1(ii), by applying sharp-moves $9k$ times on $\mathrm{T}(12k+i,6)$, we obtain $\mathrm{T}(i,6)$. Since $\mathrm{T}(\pm 1,6)$ is trivial, $\mathrm{u}^{\#}(5,6)=3$ from Eq. (25), and $\mathrm{u}^{\#}(7,6)=4$ or 6 from Table 4, we have

$$\begin{aligned}(35)\qquad & \mathrm{u}^{\#}(12k\pm 1,6)\leq 9k; \quad \mathrm{u}^{\#}(12k+5,6)\leq 9k+3;\\ & \mathrm{u}^{\#}(12k+7,6)\leq 9k+6,\end{aligned}$$

where $k \geq 0$. Using the recurrence formula given in [1], and (10), we have

$$\begin{aligned}(36)\qquad & \tilde{\sigma}(6m+1,6)=18m, \qquad \tilde{\sigma}(6m+5,6)=18m+16;\\ & \mathrm{u}(12k+1,6)=30k; \qquad \mathrm{u}(12k+5,6)=30k+10;\\ & \mathrm{u}(12k+7,6)=30k+15; \qquad \mathrm{u}(12k+11,6)=30k+25,\end{aligned}$$

where $m \geq 0$, $k \geq 0$. From Eq. (11), we have

$$\begin{aligned}(37)\qquad & \mathrm{Arf}(12k\pm 1,6)\equiv \mathrm{Arf}(12k+7,6)\equiv k \pmod 2;\\ & \mathrm{Arf}(12k+5,6)\equiv k+1 \pmod 2.\end{aligned}$$

Then from Eqs. (35)–(37), we obtain Table 7. In particular, we can decide the following sharp-unknotting number:

$$(38)\qquad \mathrm{u}^{\#}(13,6)=9.$$

**4.7.** $(r,7)$**-torus knots.** From Lemma 2.1(i), by applying sharp-moves $6m$ times on $\mathrm{T}(7m+i,7)$, we obtain $\mathrm{T}(i,7)$. Since $\mathrm{T}(\pm 1,7)$ is trivial, $\mathrm{u}^{\#}(2,7)=\mathrm{u}^{\#}(3,7)=2$ from Eqs. (13) and (19) , $\mathrm{u}^{\#}(4,7)=4$, $\mathrm{u}^{\#}(6,7)=4$ or 6 from Eq. (30) and Table 4, we have

$$\begin{aligned}(39)\qquad & \mathrm{u}^{\#}(7m\pm 1,7)\leq 6m;\\ & \mathrm{u}^{\#}(7m+2,7),\ \mathrm{u}^{\#}(7m+3,7)\leq 6m+2;\\ & \mathrm{u}^{\#}(7m+4,7),\ \mathrm{u}^{\#}(7m+5,7)\leq 6m+4;\\ & \mathrm{u}^{\#}(7m+6,7)\leq 6m+6,\end{aligned}$$

Table 7: $\mathrm{u}^{\#}(r,6)$, $r \geq 7$.

| $r$ | $\mathrm{Arf}(r,6)$ | Lower bound | Upper bound |
|---|---|---|---|
| $24l+1$ | 0 | $15l$ | $18l$ |
| $24l+5$ | 1 | $15l+3$ | $18l+3$ |
| $24l+7$ | 0 | $15l+4$ | $18l+6$ |
| $24l+11$ | 1 | $15l+7$ | $18l+9$ |
| $24l+13$ | 1 | $15l+8$ | $18l+9$ |
| $24l+17$ | 0 | $15l+10$ | $18l+12$ |
| $24l+19$ | 1 | $15l+12$ | $18l+15$ |
| $24l+23$ | 0 | $15l+14$ | $18l+18$ |

where $m \geq 0$. Using the recurrence formula given in [1], and (10), we have

(40)
$$\begin{aligned}
&\tilde{\sigma}(14k+1,7) = 48k, \qquad \tilde{\sigma}(14k+2,7) = 48k+6;\\
&\tilde{\sigma}(14k+3,7) = 48k+8, \qquad \tilde{\sigma}(14k+4,7) = 48k+14;\\
&\tilde{\sigma}(14k+5,7) = 48k+16, \qquad \tilde{\sigma}(14k+6,7) = 48k+18;\\
&\tilde{\sigma}(14k+8,7) = 48k+30, \qquad \tilde{\sigma}(14k+9,7) = 48k+32;\\
&\tilde{\sigma}(14k+10,7) = 48k+34, \qquad \tilde{\sigma}(14k+11,7) = 48k+40;\\
&\tilde{\sigma}(14k+12,7) = 48k+42, \qquad \tilde{\sigma}(14k+13,7) = 48k+48;\\
&\mathrm{u}(r,7) = 3(r-1),
\end{aligned}$$

where $k \geq 0$, $r > 0$. From Eq. (11), we have

(41)
$$\mathrm{Arf}(r,7) = 0 \pmod 2.$$

Then from Eqs. (39)–(41), we obtain Table 8. In particular, we can decide the following sharp-unknotting numbers:

(42)
$$\mathrm{u}^{\#}(10,7) = 8; \quad \mathrm{u}^{\#}(12,7) = 10; \quad \mathrm{u}^{\#}(15,7) = 12.$$

## 5. Final remark

In order to decide the sharp-unknotting numbers listed in Table 1, we do not need the signature, that is, we obtain Table 1 without using Eq. (8). Remember that the unknotting number of a torus knot given in Eq. (10) cannot be obtained by only using the signature. Also, it seems that we will not be able to make a further decision of the sharp-unknotting number of a torus knot if we continue to calculate by the method in this paper.

Table 8: $\mathrm{u}^{\#}(r,7)$, $r \geq 8$.

| $r$ | $\mathrm{Arf}(r,7)$ | Lower bound | Upper bound |
|---|---|---|---|
| $28l+1$ | 0 | $21l$ | $24l$ |
| $28l+2$ | 0 | $21l+1$ | $24l+2$ |
| $28l+3$ | 0 | $21l+2$ | $24l+2$ |
| $28l+4$, $28l+5$ | 0 | $21l+3$ | $24l+4$ |
| $28l+6$ | 0 | $21l+4$ | $24l+6$ |
| $28l+8$ | 0 | $21l+6$ | $24l+6$ |
| $28l+9$ | 0 | $21l+6$ | $24l+8$ |
| $28l+10$ | 0 | $21l+7$ | $24l+8$ |
| $28l+11$ | 0 | $21l+8$ | $24l+10$ |
| $28l+12$ | 0 | $21l+9$ | $24l+10$ |
| $28l+13$ | 0 | $21l+9$ | $24l+12$ |
| $28l+15$ | 0 | $21l+11$ | $24l+12$ |
| $28l+16$, $28l+17$ | 0 | $21l+12$ | $24l+14$ |
| $28l+18$ | 0 | $21l+13$ | $24l+16$ |
| $28l+19$ | 0 | $21l+14$ | $24l+16$ |
| $28l+20$ | 0 | $21l+15$ | $24l+18$ |
| $28l+22$ | 0 | $21l+16$ | $24l+18$ |
| $28l+23$ | 0 | $21l+17$ | $24l+20$ |
| $28l+24$ | 0 | $21l+18$ | $24l+20$ |
| $28l+25$ | 0 | $21l+18$ | $24l+22$ |
| $28l+26$ | 0 | $21l+19$ | $24l+22$ |
| $28l+27$ | 0 | $21l+20$ | $24l+24$ |

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