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# On the Envelopes of Homotopies

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Abstract. This paper is indented to explain a dynamics on homotopies on the compact metric space, by the envelopes of homotopies. It generalizes the notion of not only the envelopes of maps in discrete geometry ([3]), but the envelopes of flows in continuous geometry ([5]). Certain distinctions among the homotopy geometry, the flow geometry and the discrete geometry will be illustrated. In particular, it is shown that any $\omega$-limit set, as well as any attractor, for an envelope of homotopies is an empty set (provided the homotopies that we treat are not trivial), whereas it is nonempty in general in discrete case.

## 1. Introduction

The purpose of this paper is to explain the dynamics of homotopies on a compact metric space. Classically, following the work of Auslander-Kolyada-Snoha[3], the phase space $S(X)$ is the semigroup of continuous self-maps on $X$. If one fixes a self-map $f$ on $X$, then we have the classical dynamical system $(X, f)$ and the functional envelope $(S(X), F_f)$, where $F_f$ is a self-map (continuous with respect to the uniform metric) on $S(X)$ by

$$F_f(g) = f \circ g.$$

The functional envelope of the classical dynamical system was soon generalized by the authors in [5]. Let $\gamma$ be a continuous flow on $X$. If the phase space is $\mathcal{F}(X)$ the set of all flows, then one might think $(\mathcal{F}(X), F_\gamma)$ as the induced envelope, where $F_\gamma$ is a continuous map from $\mathcal{F}(X) \times \mathbb{R}$ mapping a flow $\alpha$ to

$$F_\gamma(\alpha, t)(x, s) = \gamma(\alpha(x, s), st). \tag{1.1}$$

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This definition is sufficiently canonical in the sense that if one puts $f = \gamma(x,1), g = \alpha(x,1)$ then $F_\gamma(\alpha, n)(x,1) = \gamma(\alpha(x,1), n) = f^n \circ g(x) = F_f^n(g)(x)$ for $n \in \mathbb{N}$, which realizes the classical dynamical system and the functional envelope. However, in contrast to the classical dynamical system (i.e., the functional dynamical systems), one should be careful for that the target space of $F_\gamma$ does not lie in $\mathcal{F}(X)$. Here is where one considers the set of the homotopies (Definition 2.1). In [5], the authors suggest two distinct ways of resolutions for this phenomenon, as follows:

(1) restrict $\mathcal{F}(X)$ to a maximal subset in which $F_\gamma$ is well-defined;

(2) enlarge $\mathcal{F}(X)$ to a bigger set $\mathcal{H}(X)$ the set of homotopies on $X$.

For (1), by [5, Proposition 2.2], $\mathcal{F}_\gamma(X)$ the set of the flows commutative with $\gamma$, is such a subset of $\mathcal{F}(X)$. A flow $\alpha$ on $X$ is commutative with $\gamma$ if

$$\gamma(\alpha(x,s),t) = \alpha(\gamma(x,t),s) \text{ for all } s,t \in \mathbb{R}.$$

As a result, we have $F_\gamma : \mathcal{F}_\gamma(X) \times \mathbb{R} \to \mathcal{F}_\gamma(X)$ by the assignment in (1.1). Furthermore, by [5, Proposition 2.5], $F_\gamma$ is a flow on $\mathcal{F}_\gamma(X)$. For (2), as was noted in [5, Remark 6.6], the $F_\gamma$ extends to $\mathcal{H}(X)$, which will be investigated in this paper.

Let us give an explanation on the motivation of our paper. One aspect, in a most applicable viewpoint in the topological dynamics, is on $\omega$-limit of dynamics. The $\omega$-limit sets, as an effective prober of topological dynamical structures, sit in a significant in Dynamical Systems. In particular, these frequently provide a rather precise description the time behavior of given dynamical systems, as well as a localization of the complexity.

Another aspect is on enveloping semigroup of dynamics. The theory for the enveloping semigroup of a dynamical system, which was introduced by R. Ellis in [8], is a kind of compactification theory of the acting group. It has turned out to be a fundamental tool in the abstract theory of topological dynamical systems and also to be extremely useful in the study of dynamical properties of transformation groups. R. Ellis, thereafter, obtains series of valuable results concerning the theory for the enveloping semigroup of a dynamical system including the applications to the ergodic theory [11], the algebraic theory for the structure of projective flows [9] and the recurrence theory for semigroup actions [10].

A third aspect, while the above two aspects arise purely from the Dynamics, is on the homotopy theory and the Morse theory. Recall that, in [5], it was proven that the set of flows $\mathcal{F}(X)$ does not admit a nonempty $\omega$-limit set. It is closely related to Sacks-Uhlenbeck's bubbling[17], as an embedded disk need not have convergent conformally embedded disks, but does those up to bubblings (but, in fact, for the purpose, we use a different metric on a phase space, say a Sobolev metric). This partly suggests what should be done in a near future; the convergence will be studied modulo a certain "bubbling" phenomenon. In fact, such flavors have been widely reflected in many geometry theories as the Donaldson theory on the instanton moduli in 4-manifold topology(ref. [7]), the Floer theory on the relative Donaldson theory in 3-manifolds and the Lagrangian moduli in Symplectic topology(ref. [13]) and the Kodaira-Spencer theory on the deformation space in Algebraic geometry(ref. [12]). In short, to have a proper meaning of convergence, those theories adopted some sort of splitting (corresponding to the bubblings). A very tiny note on this will be told in Remark 3.5.

A few more words, getting back to Dynamical systems, are added. In the last few years, it is proceeded to describe some new connections between three seemingly unrelated topics: the theory of enveloping semigroups, the theory of chaotic behavior, and the representation theory of dynamical systems on Banach spaces[14]. Various applications of enveloping semigroup of maps or flows have been subsequently investigated by many authors. (see [1], [2], [5], [14], [19].)

*Contents of the paper.* In §3, we define the envelope $(F_\gamma, \mathcal{H}(X))$ of homotopies on compact metric space and study basic properties of it.

In §4, we want to investigate the $\omega$-limit sets of $\mathcal{H}(X)$. A quick observation from [5, Theorem 3.4] is that a limit homotopy is far harder to exist than a limit map does in case of functional envelope, because point-wise consideration is never sufficient. Indeed, Theorem 4.7 certainly tells us the emptiness of the $\omega$-limit sets and the attractors.

In §5, some examples of the envelopes will be illustrated. These show the distinctions among the envelopes of maps, flows and homotopies.

## 2. Preliminaries

We set up some notations. Let $X$ be a compact metric space throughout the paper.

**Definition 2.1.** A *homotopy* on $X$, here, is a continuous map $\beta : X \times \mathbb{R} \to X$ such that $\beta(x, 0) = x$ for all $x \in X$.

The *phase space* of this paper's interest is

$$\mathcal{H}(X) = \text{the set of homotopies on } X.$$

**Definition 2.2.** Let $X, Y$ be a compact metric space. A *homotopy of a (continuous) map* $f : X \to Y$ (or an *$f$-homotopy*, sometimes referred as a *relative homotopy*) is a continuous map $\beta_f : X \times \mathbb{R} \to Y$ such that $\beta_f(x, 0) = f(x)$ for all $x \in X$. The induced set of $f$ homotopies is

$$\mathcal{H}_f(X, Y) = \text{the set of } f\text{-homotopies from } X \text{ to } Y\ .$$

Thus, $\mathcal{H}(X) = \mathcal{H}_{\mathrm{Id}_X}(X, X)$. Further, without fixing $f : X \to Y$, we may define

$$\mathcal{H}(X, Y) = \text{the set of } f\text{-homotopies from } X \text{ to } Y \text{ for some } f : X \to Y.$$

Hence, we have a fibration $\mathcal{H}(X, Y) \to \mathcal{C}(X, Y)$ with the fibre $\mathcal{H}_f(X, Y)$ at $f \in \mathcal{C}(X, Y)$ where $\mathcal{C}(X, Y)$ is the space of continuous maps from $X$ to $Y$.

**Remark 2.3.** A flow is a special case of homotopies as will be seen. Any map can be seen as a homotopy if it is homotopic to an identity map. Therefore, the study on $\mathcal{H}(X)$ amounts to the connected component of the set of continuous endomorphisms on $X$.

**Example 2.4.** A *flow* on $X$ is an example of homotopies, which is defined to be a continuous map $\alpha : X \times \mathbb{R} \to X$ with the properties

$$\alpha(x,0) = x,\ \alpha(x, t_1 + t_2) = \alpha(\alpha(x,t_1), t_2) \tag{2.1}$$

for $x \in X,\ t_1, t_2 \in \mathbb{R}$. Let us define

$$\mathcal{F}(X) = \text{ the set of flows on } X.$$

As was observed in Remark 2.3, a homotopy on $X$ can be seen as a natural generalization (especially in the dynamics) of both a flow and a map on $X$.

**Definition 2.5.** By a *dynamical system* $(X, \gamma)$, we mean a pair of a compact metric space $X$ and a homotopy $\gamma : X \times \mathbb{R} \to X$. The *envelope of homotopies* on $(X, \gamma)$ is the dynamical system $(\mathcal{H}(X), F_\gamma)$ where $F_\gamma : \mathcal{H}(X) \times \mathbb{R} \to \mathcal{H}(X)$ is defined by

$$(F_\gamma(\alpha, t))(x, s) = \gamma(\alpha(x, s), st),$$

where $x \in X$ and $s, t \in \mathbb{R}$.

**Definition 2.6.** Given two continuous maps $\beta_1, \beta_2 : X \times \mathbb{R} \to X$, the *uniform metric* $d(\beta_1, \beta_2)$ is defined as

$$d(\beta_1, \beta_2) = \sup_{x \in X, t \in \mathbb{R}} d_X(\beta_1(x,t), \beta_2(x,t)).$$

**Remark 2.7.** Indeed, the metric is well-defined since $X$ is a compact space with the metric $d_X$. The uniform metric in Definition 2.6, defines the induced metric on $\mathcal{H}(X)$. The compactness of $X$ in Definition 2.1 is not merely indispensable, hence we may define a homotopy on the non-compact space $\mathcal{H}(X)$. This can be viewed as a canonical extension of the discrete case of [3] and the flow case of [5].

## 3. Envelopes of homotopies

In this section, we start the study of the dynamics of the envelopes of homotopies.

**Proposition 3.1.** (a) *Let $\beta_f$ be an $f$-homotopy from $X$ to $Y$. Then, this gives rise to an induced homotopy from $\mathcal{H}(X)$ to $\mathcal{H}(X, Y)$, by assigning $(\alpha, t) \mapsto \beta_f(\cdot, t) \circ \alpha$ where $\alpha \in \mathcal{H}(X)$ and $t \in \mathbb{R}$.*
(b) *Furthermore, if $\beta_f$ is an isotopy (i.e., $\beta_f(\cdot, t)$ is a bi-continuous map from $X$ to $Y$), then the induced homotopy from $\mathcal{H}(X)$ to $\mathcal{H}(X, Y)$ is an isotopy.*

(c) *With the assumption of* (b) *and* $\gamma \in \mathcal{H}(X)$*, we have equivalent envelopes in the sense that*

$$\begin{array}{ccc} \mathcal{H}(X) \times \mathbb{R} & \xrightarrow{F_\gamma} & \mathcal{H}(X) \\ \big\downarrow & & \big\downarrow \\ \mathcal{H}(Y) \times \mathbb{R} & \xrightarrow{F_{\gamma_Y}} & \mathcal{H}(Y) \end{array} \tag{3.1}$$

*commutes, where* $F_{\gamma_Y} = f \circ \gamma \circ (f^{-1} \times \mathrm{Id}_{\mathbb{R}})$*, the right vertical map sends* $\alpha \in \mathcal{H}(X)$ *to* $\alpha_Y = f \circ \alpha \circ (f^{-1} \times \mathrm{Id}_{\mathbb{R}})$ *and the left one is the obvious extension of* $\alpha \mapsto \alpha_Y$ *by the identity on* $\mathbb{R}$.

*Proof.* The functorialities in the statements directly follow from the computations, which we omit. □

As we can see in Definition 2.5 or in the upper horizontal map in (3.1), $F_\gamma : \mathcal{H}(X) \times \mathbb{R} \to \mathcal{H}(X)$ is defined; further nicely, it is a homotopy on $\mathcal{H}(X)$ as below. This enables the towering argument.

**Proposition 3.2.** *The map* $F_\gamma$ *is a homotopy on the phase space* $\mathcal{H}(X)$*. Furthermore, if* $\gamma$ *is a flow on* $X$*, then* $F_\gamma$ *is a flow on* $\mathcal{H}(X)$*.*

*Proof.* For any homotopy $\alpha$ on $X$, we have $F_\gamma(\alpha, 0)(x, s) = \gamma(\alpha(x, s), 0) = \alpha(x, s)$. Therefore, we obtain $F_\gamma(\alpha, 0) = \alpha$.

The continuity of $F_\gamma$ comes from that if a sequence of homotopies $\alpha_i$ converges to $\alpha$, then $\lim_{i\to\infty} \gamma(\alpha_i(x, s), ts) = \gamma(\alpha(x, s), ts)$ for any $x \in X$ and $t, s \in \mathbb{R}$. Consequently $F_\gamma : \mathcal{H}(X) \times \mathbb{R} \to \mathcal{H}(X)$ is a homotopy.

Finally, if $\gamma$ is a flow, then we show $F_\gamma(\alpha, t_1 + t_2) = F_\gamma(F_\gamma(\alpha, t_1), t_2)$. This follows from the direct calculation $\gamma(\alpha(x, s), (t_1 + t_2)s) = \gamma(\gamma(\alpha(x, s), t_1 s), t_2 s)$ since $\gamma$ is a flow. □

**Remark 3.3.** Unfortunately, even if $\gamma$ is flow on $X$, the image of $\mathcal{F}(X) \subset \mathcal{H}(X)$ under $F_\gamma$ does not lie in $\mathcal{F}(X)$ again.

For the remedy of this, in [5], a phase space $\mathcal{F}_\gamma(X)$ is defined by the set of all flows $\alpha$ on $X$ with the *commutative property*

$$\mathcal{F}_\gamma(X) = \{\alpha | \gamma(\alpha(x, s), t) = \alpha(\gamma(x, t), s) \text{ for all } s, t \in \mathbb{R}\}.$$

Then, by [5, Proposition], $F_\gamma$ can be restricted to $F_\gamma : \mathcal{F}_\gamma(X) \times \mathbb{R} \to \mathcal{F}_\gamma(X)$.

**Definition 3.4.** In the same vein with the above remark, for any homotopy $\gamma$, one can define

$$\mathcal{H}_\gamma(X) = \{\alpha \in \mathcal{H}(X) | \gamma(\alpha(x, s), t) = \alpha(\gamma(x, t), s) \text{ for all } s, t \in \mathbb{R}\}.$$

For those $\gamma, \alpha$, we call *commutative homotopies.* If $\gamma$ is commutative with itself, $\gamma$ is called a *self-commutative homotopy.*

**Remark 3.5.** Note that if $\gamma$ is a flow, then it is obviously self-commutative. This is where our model is similar to the deformation-obstruction theories in geometry, e.g., the Donaldson theory (on the gauge theory[7]). A homotopy $\gamma$ corresponds to some local 1-form-valued matrix part $A$ of a connection on a closed 4-manifold with $dA = 0$, and the self-commutativity of $\gamma$ corresponds to the unobstructed-ness (to the local smoothness of the moduli of instantons) of $A$, i.e., $[A, A] = 0$. If $B$ denotes the corresponding 1-form-valued matrix (in a connection) corresponding to a homotopy $\alpha$, then $[A, B] = 0$ amounts to the commutativity of $\alpha, \gamma$. This model, in fact, is more suitable for the parallel theory (with ours) of the phase space of homotopies $X \times I \to X$ where $I$ is the interval $[0, 1]$. It is because there is a correlation between the addition of $A$ and $B$ and the composition of homotopies. But, we shall not continue this subject here, since we are pursuing a generalization of envelopes of flows in this paper.

**Proposition 3.6.** *If $\gamma$ is self-commutative and $\gamma, \alpha$ are commutative homotopies on $X$, then the image $F_\gamma(\alpha, t)$ is an element of $\mathcal{H}_\gamma(X)$ for any $t \in \mathbb{R}$. Hence, we have the restriction $F_\gamma : \mathcal{H}_\gamma(X) \times \mathbb{R} \to \mathcal{H}_\gamma(X)$ for a self-commutative homotopy.*

*Proof.* We have to show $F_\gamma(\alpha, t) \in \mathcal{H}_\gamma(X)$, i.e., the commutativity

$$\gamma(F_\gamma(\alpha, t)(x, s), \tau) = F_\gamma(\alpha, t)(\gamma(x, \tau), s)$$

should be satisfied. By direct calculation, we have

$$\gamma(F_\gamma(\alpha, t)(x, s), \tau) = \gamma(\gamma(\alpha(x, s), st), \tau)$$

and

$$\begin{aligned} F_\gamma(\gamma(\alpha, t)(x, \tau), s) &= \gamma(\alpha(\gamma(x, \tau), s), st) \\ &= \gamma(\gamma(\alpha(x, s), \tau), st). \end{aligned}$$

Since $\gamma(\gamma(\alpha(x, s), st), \tau) = \gamma(\gamma(\alpha(x, s), \tau), st)$ by the self-commutativity of $\gamma$, the proof is completed. □

**Remark 3.7.** Note that $\mathcal{H}(X)$, as well as $\mathcal{H}_\gamma(X), \mathcal{F}(X)$ and $\mathcal{F}_\gamma(X)$, is a complete metric space (with respect to the uniform metric) whenever $X$ is a complete metric space. In the next section, we will see any sequence $F_\gamma(\alpha, n)$ ($n \in \mathbb{Z}_+$) cannot be a Cauchy sequence, unless $\gamma$ is a trivial homotopy.

## 4. Some dynamical properties for $F_\gamma$ in the envelope

The envelope of homotopies $(\mathcal{H}_\gamma(X), F_\gamma)$ has properties parallel to the functional envelope of [3, §4] and $(\mathcal{F}_\gamma(X), F_\gamma)$, concerning the theories related to orbit closures, $\omega$-limits, etc. However, as was shown in [5, §3] in case $(\mathcal{F}_\gamma(X), F_\gamma)$, while the $\omega$-limit in the functional envelope, if any, results from point-wise convergence, the $\omega$-limit in our envelope is not enough to consider the point-wise convergence. The main theorem in this section proves that the $\omega$-limit set of $F_\gamma$ is an empty set.

The theorem may look very similar to [5, Theorem 3.4]. A point that one should be cautious, is that the convergence used in this paper strictly enlarges the classical notion of convergence (usually it has been the convergence by the iterated composition of a map or the transport along a flow by time-shifting).

**Definition 4.1.** Let us consider a dynamical system $(X, \gamma)$ where $\gamma$ is a homotopy. Let $\alpha$ be a homotopy on $X$ (i.e., $\alpha \in \mathcal{H}(X)$). The *$\omega$-limit set of* $\alpha$ is the set of homotopies $\beta \in \mathcal{H}(X)$ which are all limits of some sequences $F_\gamma(\alpha, t_i)$ for $t_i \to \infty$ $(i = 1, 2, \cdots)$. Here, the metric on the homotopy space is given by the uniform metric (Definition 2.6).

**Remark 4.2.** If we define $f(x) = \gamma(x, 1), g(x) = \alpha(x, 1)$, then the $\omega$-limit set of $g$ is the set of limit maps of sequences $F_f^{n_i}(g)$ for $n_i \to \infty$ $(i = 1, 2, \cdots)$ where $n_i \in \mathbb{N}$. Therefore, a limit homotopy in an $\omega$-limit set, if any, would express a limit map by putting $n_i$ to the time variable $t$. If $\gamma$ is trivial, i.e., $\gamma(x, t) = x$ for any $t$, then the $\omega$-limit set of any flow $\alpha$ is the singleton set $\{\alpha\}$. But, if $\gamma$ is not trivial, the existence of a limit homotopy will be denied in Theorem 4.7.

Before going to the theorem, we may guess out an evidence of the nonexistence of a limit homotopy, by using a finer metric, say, the uniform $\mathcal{C}^1$-metric.

**Remark 4.3.** Let $\mathcal{H}^1(X)$ be the set of $\mathcal{C}^1$-homotopies on $X \subset \mathbb{R}^n$. By a $\mathcal{C}^1$-homotopy, we mean a homotopy $\alpha$ such that for any $x \in X$, $\alpha(x, \bullet) : \mathbb{R} \to X$ is $\mathcal{C}^1$-differentiable.

Let $\gamma \in \mathcal{H}^1(X)$. Assume that $\frac{\partial}{\partial t}\gamma(x_0, t)|_{t=0} \neq 0$ for some fixed $x_0 \in X$. This amounts to non-triviality of $\gamma$. Let us put $\dot{\gamma}_0 = \frac{\partial}{\partial t}\gamma(x_0, t)|_{t=0}$. Let $\dot{\alpha}_0 = \frac{\partial}{\partial t}\alpha(x_0, t)|_{t=0}$. Suppose there exists a limit homotopy of $\alpha$ in the uniform $\mathcal{C}^1$-metric (i.e., the first order derivatives, as well as the continuously differentiable homotopies $F_\gamma(\alpha, t_i)$, converge). By the chain rule, we have

$$\begin{aligned} \frac{\partial}{\partial s}F_\gamma(\alpha, t_i)(x_0, s)|_{s=0} &= \frac{\partial}{\partial s}\gamma(\alpha(x_0, s), t_i s)|_{s=0} \\ &= \frac{\partial}{\partial x}\gamma(x_0, 0) \cdot \dot{\alpha}_0 + t_i\dot{\gamma}_0, \end{aligned}$$

which cannot be convergent because $t_i\dot{\gamma}_0$ does not converge as $t_i \to \infty$. This is a contradiction.

**Definition 4.4.** We say $\gamma \in \mathcal{H}(X)$ is *trivial* if $\gamma(x, s) = x$ for every $x \in X$ and every $t \in \mathbb{R}$.

**Definition 4.5.** An *attractor* of $(F_\gamma, \mathcal{H}(X))$ is a proper closed subset $\mathcal{A}$ of $\mathcal{H}(X)$ such that there exists an open set $\mathcal{U}$ containing $\mathcal{A}$ satisfying

$$\mathcal{A} = \bigcap_{t \geq 0} F_\gamma(\mathcal{U}, t).$$

**Remark 4.6.** There are several definitions of attractors. The above one is a slight, but direct and natural generalization of that of Smale[18, pp.786]. In fact, to have the asymptotic stability, one notice papers by Milnor by which various implications between attractors are studied. See [15], especially [16, pp.517]. In [16], it is shown that our definition coincides with the asymptotically stable attractor if it is compact. The compactness assumption is not satisfactory in many applications to non-compact space, e.g., the phase space $\mathcal{H}(X)$ is not compact. With only the compact boundary of a given attractor, the attractor can be proven to be asymptotically stable (ref. [6]).

Now we have a main theorem. The proof is presented after a lemma.

**Theorem 4.7.** *For any nontrivial $\gamma \in \mathcal{H}(X)$, every $\omega$-limit set for $F_\gamma$ is an empty set.*

**Corollary 4.8.** *For any nontrivial $\gamma \in \mathcal{H}(X)$, there does not exist an attractor.*

The following lemma comes from [5, Lemma 3.5]. The proof is short, so we reproduce it for the convenience's sake.

**Corollary 4.9.** *Let $\gamma \in \mathcal{H}(X)$. Assume that there is $x \in X$ such that $\gamma(x, s) = x$ for not every $t \in \mathbb{R}$. Then, there exist $r, s_0, \epsilon > 0$ such that*

$$d_X(B_X(x,r), \gamma(B_X(x,r), s_0)) \geq \epsilon,$$

*where $B_X(x,r) = \{y \in X | d_X(y,x) \leq r\}$ and $d_X(A,B) = \inf\{d_X(a,b) | a \in A, b \in B\}$ for two subsets $A, B$ of $X$.*

*Proof.* There exists $s_0 > 0$ such that $x \neq \gamma(x, s_0)$ and we can take an $\epsilon$ with $d_X(x, \gamma(x, s_0)) > 3\epsilon$. From the continuity, there is a positive real number $r$ with $\epsilon > r$ such that $\gamma(B_X(x,r), s_0)) \subseteq B_X(\gamma(x, s_0), \epsilon)$, which completes the proof. □

The assumption of Lemma 4.9 is equivalent to the non-triviality of $\gamma$.

*Proof of Theorem 4.7.* Suppose the contrary; we have a homotopy $\beta$ which is an element of the $\omega$-limit set of some homotopy $\alpha$. Then, we have $t_i \to \infty$ $(i = 1, 2, \cdots)$ such that $F_\gamma(\alpha, t_i) \to \beta$. Thus, given any $\epsilon > 0$,

$$d_X(\gamma(\alpha(x,s), t_i s), \beta(x,s)) < \epsilon \tag{4.1}$$

for any $x \in X$, $s \in \mathbb{R}$ and all sufficiently large $i$. Now, let us choose $x \in X$ such that the assumption of the above lemma is satisfied. Let $r, s_0, \epsilon > 0$ as in Lemma 4.9. Let $s_i = \frac{s_0}{t_i}$. For a sufficiently large $i$, $\alpha(x, s_i), \beta(x, s_i)$ are contained in $B_X(x,r)$ because $s_i \to 0$. Since $\gamma(\alpha(x,s_i), t_i s_i) = \gamma(\alpha(x, s_i), s_0) \in \gamma(B_X(x,r), s_0)$, and $\beta(x, s_i) \in B_X(x,r)$, by Lemma 4.9, we obtain

$$d_X(\gamma(\alpha(x,s_i), s_0), \beta(x, s_i)) \geq \epsilon,$$

which contradicts (4.1). □

**5. Some examples of the envelopes**

We illustrate some examples of the functional envelopes, the envelope of flows and the envelope of homotopies.

**Example 5.1** ([5], Example 3.6)**.** Let $X$ be a 2-dimensional unit sphere $S^2$. Let $\gamma$ be a periodic flow whose trajectories are latitudes with the simultaneous periodicity $\gamma(x,t_0) = \gamma(x)$ for any $x \in X$ and a fixed $t_0 > 0$, say $t_0 = 100$. Let $\alpha$ be a flow whose trajectories are longitudes. By Theorem 4.7, the $\omega$-limit set of $\alpha$ is empty. But if we put $f(x) = \gamma(x,1),\ g(x) = \alpha(x,1)$ as in Remark 4.2, $g$ is a limit map of $g$, i.e. $g$ itself is an element of the $\omega$-limit set of $g$. Indeed, by the periodicity of $f$, $f^n(g) = g$ for all $n \in 100\mathbb{Z}$.

**Example 5.2.** The above phenomenon is observed in a simpler example. Let $X$ be a unit circle $S^1$. Let $\gamma$ be any nontrivial homotopy with $\gamma(x,n) = x$ for any $x \in X$ and any $n \in \mathbb{Z}$. For instance, $\gamma$ is a rotation with the period 1. By the same logic with the above example, any $\omega$-limit set should be empty. But the function $f : X \to X$ defined by $f(x) = \gamma(x,1) = x$, is in the $\omega$-limit set for $F_f$.

The last example again recycles the dynamics on $S^1$ in the above example, in order to show a difference between the envelope of homotopies and flows. The example is dealt with in [4, §1] during their study on diffeomorphisms of $S^1$ with non-trivial centralizers.

**Example 5.3.** Let $\mathcal{D}$ be the set of $\mathcal{C}^\infty$ Morse-Smale diffeomorphisms $f$ such that the $p$-time composite $f^p$ of $f$ is a time 1 map of a Morse-Smale vector field $X$ on $S^1$ where $p$ is the period (=the least common multiple of the periods of finite periodic points). By [4, Proposition 1.10], $\mathcal{D}$ is dense in $Diff^1(S^1)$. By the parallel method of [4], one can show that $\mathcal{F}(S^1)$ is dense in $\mathcal{H}(S^1)$ in $\mathcal{C}^1$-topology. Let $\alpha_s \in \mathcal{F}(S^1)$ be a family of flows parametrized by $s \in \mathbb{R}^1$. For instance, $\alpha_s : S^1 \times \mathbb{R} \to S^1$ by $(x,t) \mapsto x + st$ for the canonical identification $S^1 \cong \mathbb{R}^1/\mathbb{Z}$. Define $\alpha' \in \mathcal{H}(S^1)$ by $\alpha'(x,t) = \alpha_t(x,t)$. Obviously, $\alpha' \notin \mathcal{F}(S^1)$. One can realize that this is always the case whenever a non-trivial family of flows is given. Also, such a phenomenon is reflected in $Diff^1(S^1)$ as $\mathcal{D}$ is dense but not open. In fact, if we denote the time $\varepsilon$ map by $T_\varepsilon : \mathcal{H}^1(S^1) \to Diff^1(S^1)$ where $0 < \varepsilon \ll 1$ and $\mathcal{H}^1(S^1)$ is the subset of $\mathcal{H}(S^1)$ of $\mathcal{C}^1$-homotopies, then $T_\varepsilon$ is an open map (while $T_\varepsilon|_{\mathcal{F}^1(S^1)}$ is a dense image).

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