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On a Relation to Hilbert's Integral Inequality and a Hilbert-Type Inequality

Bicheng Yang
Department of Mathematics, Guangdong Education Institute, Guangzhou, Guangdong 510303, P. R. China
e-mail : bcyang@pub.guangzhou.gd.cn

Abstract. In this paper, by introducing some parameters and using the way of weight function, a new integral inequality with a best constant factor is given, which is a relation between Hilbert's integral inequality and a Hilbert-type inequality. As applications, the equivalent form, the reverse forms and some particular inequalities are considered.

1. Introduction

If $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then the well known Hilbert's integral inequality is as follows[1]:

$$(1) \qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}},$$

where the constant factor π is the best possible. Following the same condition, we also have a Hilbert-type integral inequality as follows[1]

$$(2) \qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < 4 \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}},$$

where the constant factor 4 is still the best possible. Inequalities (1) and (2) are important in analysis and its applications[1,2]. In recent years, by introducing some parameters and estimating the weight function, a number of extensions of (1) and (2) were given by Yang et al. [3], [4], [5]. In 2006, Li et al. [6] gave the following inequality with a kernel coupling (1) and (2):

$$(3) \qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dxdy < c \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{\frac{1}{2}},$$

where the constant factor $c = 2\sqrt{2}\arctan\frac{1}{\sqrt{2}}$ is the best possible. In 2007, Xie [7] gave a best extension of (3) by introducing some parameters, and Guo et al. [8], [9]

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gave a similar form of (3) as

$$(4) \qquad \int_0^\infty \int_0^\infty \frac{f(x)g(y)dxdy}{x+y+\min\{x,y\}} < \widetilde{c}\left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right)^{\frac{1}{2}},$$

where the constant factor $\widetilde{c} = 2\sqrt{2}\arctan\sqrt{2}$ is the best possible.

In this paper, by introducing some parameters and using the way of weight function, we give a new extended inequality of (3) and (4) with a best constant factor, which is a relation between (1) and (2). The equivalent form the reverse forms and some particular inequalities are obtained.

2. Some lemmas

Lemma 1. *If* $\lambda > 0, B, C \geq 0, A > -\min\{B, C\}$, *setting* $k_\lambda(A,B,C) := \int_0^\infty \frac{1}{A\max\{u^\lambda, 1\} + Bu^\lambda + C} u^{\frac{\lambda}{2}-1}du$, *then*

(5)

$$k_\lambda(A,B,C) = \begin{cases} \frac{2}{\lambda}\left[\frac{\arctan\sqrt{\frac{B}{A+C}}}{\sqrt{B(A+C)}} + \frac{\arctan\sqrt{\frac{C}{A+B}}}{\sqrt{C(A+B)}}\right], B, C > 0, A > -\min\{B, C\} \\ \frac{2}{\lambda}\left[\frac{1}{A+C} + \frac{2}{\sqrt{AC}}\arctan\sqrt{\frac{C}{A}}\right], \ B = 0, C, A > 0 \\ \frac{2}{\lambda}\left[\frac{1}{A+B} + \frac{2}{\sqrt{AB}}\arctan\sqrt{\frac{B}{A}}\right], \ B > 0, C = 0, A > 0 \\ \frac{4}{\lambda A}, \ B = C = 0, A > 0. \end{cases}$$

Proof. Setting $v = u^\lambda$, we find

$$(6) \qquad \begin{aligned} k_\lambda(A,B,C) &= \frac{1}{\lambda}\left[\int_0^1 \frac{v^{-\frac{1}{2}}}{A+Bv+C}dv + \int_1^\infty \frac{v^{-\frac{1}{2}}}{(A+B)v+C}dv\right] \\ &= \frac{1}{\lambda}\left[\int_0^1 \frac{u^{-\frac{1}{2}}}{Bu+(A+C)}du + \int_0^1 \frac{u^{-\frac{1}{2}}}{Cu+(A+B)}du\right]. \end{aligned}$$

(a) For $B, C > 0, A > -\min\{B, C\}$, we obtain

$$\begin{aligned} k_\lambda(A,B,C) &= \frac{1}{\lambda}\left[\frac{2}{\sqrt{B(A+C)}}\int_0^1 \frac{1}{\frac{B}{A+C}u+1} d\sqrt{\frac{B}{A+C}u}\right. \\ &\quad \left. + \frac{2}{\sqrt{C(A+B)}}\int_0^1 \frac{1}{\frac{C}{A+B}u+1} d\sqrt{\frac{C}{A+B}u}\right] \\ &= \frac{2}{\lambda}\left[\frac{\arctan\sqrt{B/(A+C)}}{\sqrt{B(A+C)}} + \frac{\arctan\sqrt{C/(A+B)}}{\sqrt{C(A+B)}}\right]; \end{aligned}$$

(b) for $B=0, C, A>0$, by (6), we find

$$\begin{aligned} k_\lambda(A,0,C) &= \frac{1}{\lambda}\int_0^1 \left(\frac{u^{-\frac{1}{2}}}{A+C}+\frac{u^{-\frac{1}{2}}}{Cu+A}\right)du \\ &= \frac{2}{\lambda}\left(\frac{1}{A+C}+\frac{1}{\sqrt{AC}}\arctan\sqrt{\frac{C}{A}}\right); \end{aligned}$$

(c) similarly, for $A, B>0, C=0$, $k_\lambda(A,B,0)=\frac{2}{\lambda}\left(\frac{1}{A+B}+\frac{\arctan\sqrt{B/A}}{\sqrt{AB}}\right)$;

(d) for $B=C=0, A>0$, we find $k_\lambda(A,0,0)=\frac{2}{\lambda}\int_0^1\frac{1}{A}u^{-\frac{1}{2}}du=\frac{4}{\lambda A}$.

Hence Expression (5) is valid. The lemma is proved.

Lemma 2. *Assume that* $p>0 (p\neq 1), |q|>0, \lambda>0, B, C\geq 0, A>-\min\{B,C\}$ *and* $0<\varepsilon<\frac{\lambda}{2}\min\{p,|q|\}$, *then we have*

$$\int_0^1 \frac{u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}du}{Bu^\lambda+(A+C)} = \int_0^1 \frac{u^{\frac{\lambda}{2}-1}du}{Bu^\lambda+(A+C)}+o_1(1)(\varepsilon\to 0^+); \tag{7}$$

$$\int_1^\infty \frac{u^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}du}{(A+B)u^\lambda+C} = \int_1^\infty \frac{u^{\frac{\lambda}{2}-1}du}{(A+B)u^\lambda+C}+o_2(1)(\varepsilon\to 0^+); \tag{8}$$

$$\int_0^1 \frac{u^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}du}{Bu^\lambda+(A+C)} = \int_0^1 \frac{u^{\frac{\lambda}{2}-1}du}{Bu^\lambda+(A+C)}+o_3(1)(\varepsilon\to 0^+). \tag{9}$$

Proof. In view of the assumption, for $\varepsilon\to 0^+$, we find

$$\begin{aligned} 0 &< \int_0^1 \frac{u^{\frac{\lambda}{2}-1}du}{Bu^\lambda+(A+C)}-\int_0^1 \frac{u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}du}{Bu^\lambda+(A+C)} \\ &\leq \int_0^1 \frac{u^{\frac{\lambda}{2}-1}(1-u^{\frac{\varepsilon}{q}})du}{A+C}=\frac{1}{A+C}\left(\frac{2}{\lambda}-\frac{1}{\frac{\lambda}{2}+\frac{\varepsilon}{q}}\right)\to 0; \\ 0 &< \int_1^\infty \frac{u^{\frac{\lambda}{2}-1}du}{(A+B)u^\lambda+C}-\int_1^\infty \frac{u^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}du}{(A+B)u^\lambda+C} \\ &\leq \int_1^\infty \frac{u^{\frac{-\lambda}{2}-1}(1-u^{-\frac{\varepsilon}{p}})du}{A+B}=\frac{1}{A+B}\left(\frac{2}{\lambda}-\frac{1}{\frac{\lambda}{2}+\frac{\varepsilon}{p}}\right)\to 0; \\ 0 &< \int_0^1 \frac{u^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}du}{Bu^\lambda+(A+C)}-\int_0^1 \frac{u^{\frac{\lambda}{2}-1}du}{Bu^\lambda+(A+C)} \\ &\leq \int_0^1 \frac{u^{\frac{\lambda}{2}-1}(u^{-\frac{\varepsilon}{p}}-1)du}{A+C}=\frac{1}{A+C}\left(\frac{1}{\frac{\lambda}{2}-\frac{\varepsilon}{p}}-\frac{2}{\lambda}\right)\to 0. \end{aligned}$$

Hence Expressions (7), (8) and (9) are valid. The lemma is proved.

3. Main results and applications

Theorem 1. *If* $p > 0 (p \neq 1), \frac{1}{p} + \frac{1}{q} = 1, \lambda > 0, B, C \geq 0, A > -\min\{B, C\}$, $\phi_r(x) = x^{r(1-\frac{\lambda}{2})-1} (r = p, q)$, $f, g \geq 0, 0 < ||f||_{p,\phi_p} = \{\int_0^\infty x^{p(1-\frac{\lambda}{2})-1} f^p(x)dx\}^{\frac{1}{p}}$ $< \infty$ *and* $0 < ||g||_{q.\phi_q} = \{\int_0^\infty x^{q(1-\frac{\lambda}{2})-1} g^q(x)dx\}^{\frac{1}{q}} < \infty$, *then*
(a) *for* $p > 1$,*we have the following equivalent inequalities:*

(10)
$$I_\lambda := \int_0^\infty y^{\frac{p\lambda}{2}-1} \left(\int_0^\infty \frac{f(x)dx}{A\max\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} \right)^p dy < k_\lambda^p(A, B, C)||f||_{p,\phi_p}^p;$$

$$(11) \quad J_\lambda := \int_0^\infty \int_0^\infty \frac{f(x)g(y)dxdy}{A\max\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} < k_\lambda(A, B, C)||f||_{p,\phi_p}||g||_{q.\phi_q};$$

(b) *for* $0 < p < 1$, *we have the reverse equivalent forms of* (10) *and* (11).

Proof. (a) Setting $u = x/y$, we find

$$(12) \qquad \varpi_\lambda(y) \ : = \int_0^\infty \frac{y^{\frac{\lambda}{2}} x^{\frac{\lambda}{2}-1}}{A\max\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} dx = k_\lambda(A, B, C);$$

$$(13) \qquad \omega_\lambda(x) \ : = \int_0^\infty \frac{x^{\frac{\lambda}{2}} y^{\frac{\lambda}{2}-1}}{A\max\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} dy = k_\lambda(A, B, C).$$

By Hölder's inequality with weight[10] and (12), for $y \in (0, \infty)$,we obtain

$$\begin{aligned}
(14) \quad & \left(\int_0^\infty \frac{f(x)dx}{A\max\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} \right)^p \\
= \ & \left\{ \int_0^\infty \frac{1}{A\max\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} \left[\frac{x^{(1-\frac{\lambda}{2})/q}}{y^{(1-\frac{\lambda}{2})/p}} f(x) \right] \left[\frac{y^{(1-\frac{\lambda}{2})/p}}{x^{(1-\frac{\lambda}{2})/q}} \right] dx \right\}^p \\
\leq \ & \int_0^\infty \frac{x^{(1-\frac{\lambda}{2})(p-1)} y^{\frac{\lambda}{2}-1} f^p(x)dx}{A\max\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} \left\{ \int_0^\infty \frac{y^{(1-\frac{\lambda}{2})(q-1)} x^{\frac{\lambda}{2}-1} dx}{A\max\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} \right\}^{p-1} \\
= \ & k_\lambda^{p-1}(A, B, C) y^{1-\frac{p\lambda}{2}} \int_0^\infty \frac{x^{(1-\frac{\lambda}{2})(p-1)} y^{\frac{\lambda}{2}-1} f^p(x)}{A\max\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} dx.
\end{aligned}$$

By (14) and (13), in view of Fubini's Theorem[11], it follows

$$\begin{aligned}
(15) \ I_\lambda \ \leq \ & k_\lambda^{p-1}(A, B, C) \int_0^\infty \int_0^\infty \frac{x^{(1-\frac{\lambda}{2})(p-1)} y^{\frac{\lambda}{2}-1} f^p(x)}{A\max\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} dxdy \\
= \ & k_\lambda^{p-1}(A, B, C) \int_0^\infty \left[\int_0^\infty \frac{x^{(1-\frac{\lambda}{2})(p-1)} y^{\frac{\lambda}{2}-1} dy}{A\max\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} \right] f^p(x)dx \\
= \ & k_\lambda^{p-1}(A, B, C) \int_0^\infty \omega_\lambda(x)\phi_p(x) f^p(x)dx = k_\lambda^p(A, B, C)||f||_{p,\phi_p}^p.
\end{aligned}$$

If there exists $y \in (0,\infty)$, such that (14) takes the form of equality, then[10], there exist constants A and B, satisfying they are not all zero and $Ax^{(1-\frac{\lambda}{2})(p-1)}y^{\frac{\lambda}{2}-1}f^p(x) = By^{(1-\frac{\lambda}{2})(q-1)}x^{\frac{\lambda}{2}-1}$ *a.e.* in $(0,\infty)$. It follows $Ax^{p(1-\frac{\lambda}{2})}f^p(x) = By^{q(1-\frac{\lambda}{2})}$ *a.e.* in $(0,\infty)$. We affirm that $A \neq 0$, otherwise $B = A = 0$. Hence we obtain $x^{p(1-\frac{\lambda}{2})-1}f^p(x) = [By^{q(1-\frac{\lambda}{2})}]/(Ax)$ *a.e.* in $(0,\infty)$, which contradicts the fact that $0 < ||f||_{p,\phi_p} < \infty$. Then (14) keeps the form of strict inequality for any $y \in (0,\infty)$; so does (15). And (10) is valid .

By Hölder's inequality[10], we find

$$(16)\quad J_\lambda = \int_0^\infty \left[\int_0^\infty \frac{y^{\frac{-1}{p}+\frac{\lambda}{2}}f(x)dx}{A\max\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}\right][\phi_q^{\frac{1}{q}}(y)g(y)]dy \le I_\lambda^{\frac{1}{p}}||g||_{q.\phi_q}.$$

In view of (10), we have (11).

On the other-hand, suppose (11) is valid. Since $||f||^p_{p,\phi_p} > 0$, there exists $n_0 \in N$, such that for $n \ge n_0, \int_{1/n}^n \phi_p(x)[f(x)]_n^p dx > 0$, where $[f(x)]_n = n$, for $f(x) \ge n; [f(x)]_n = f(x)$, for $f(x) < n$. For $n \ge n_0$, setting

$$(17)\qquad g_n(y) := y^{\frac{p\lambda}{2}-1}\left[\int_{1/n}^n \frac{[f(x)]_n dx}{A\max\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}\right]^{p-1}, y \in (0,n],$$

by (11), we find

$$\begin{aligned}(18)\quad 0 &< \int_{1/n}^n \phi_q(y)g_n^q(y)dy\\ &= \int_{1/n}^n y^{\frac{p\lambda}{2}-1}\left[\int_{1/n}^n \frac{[f(x)]_n dx}{A\max\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}\right]^p dy\\ &= \int_{1/n}^n\int_{1/n}^n \frac{[f(x)]_n g_n(y)}{A\max\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dxdy\\ &< \widetilde{K}_\lambda(A)\left\{\int_{1/n}^n \phi_p(x)[f(x)]_n^p dx\right\}^{\frac{1}{p}}\left\{\int_{1/n}^n \phi_q(y)g_n^q(y)dy\right\}^{\frac{1}{q}} < \infty;\end{aligned}$$

$$(19)\qquad \left\{\int_{1/n}^n \phi_q(y)g_n^q(y)dy\right\}^{\frac{1}{p}} < \widetilde{K}_\lambda(A)\left\{\int_0^\infty \phi_p(x)f^p(x)dx\right\}^{\frac{1}{p}}.$$

Hence $0 < \int_0^\infty \phi_q(y)g_\infty^q(y)dy < \infty$, and then (18) and (19) are valid for $n \to \infty$ by using (11). Therefore we have (10), which is equivalent to (11).

(b) By the reverse Hölder's inequality and the same way, we can obtain the reverse forms of (10) and (16). And then we deduce the reverse form of (11). On the other-hand, suppose that the reverse form of (11) is valid. Setting $g_n(y)$ as

(17), by the reverse form of (11), we find the reverse form of (18) and (19), and then deduce the reverse form of (10), which is equivalent to the reverse form of (11). The theorem is proved. □

Theorem 2. *As the assumption of Theorem 1, all the constant factors in* (10), (11) *and the reverse forms are the best possible.*

Proof. For $0 < \varepsilon < \frac{\lambda}{2}\min\{p, |q|\}$, setting $f_\varepsilon, g_\varepsilon$ as: $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for $x \in (0, 1)$; $f_\varepsilon(x) = x^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}, g_\varepsilon(x) = x^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}$, for $x \in [1, \infty)$.

(a) For $p > 1$, if there exists constant $0 < k \le k_\lambda(A, B, C)$, such that (11) is still valid as we replace $k_\lambda(A, B, C)$ by k, then in particular, we have

$$\begin{aligned} k &= \varepsilon k||f_\varepsilon||_{p,\phi_p}||g_\varepsilon||_{q.\phi_q} > \varepsilon \int_0^\infty \int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)dxdy}{A\max\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda} \\ &= \varepsilon \int_1^\infty x^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}\left[\int_1^\infty \frac{y^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}}{A\max\{x^\lambda, y^\lambda\} + Bx^\lambda + Cy^\lambda}dy\right]dx. \end{aligned}$$

Setting $u = x/y$ in the above integral, by Fubini's Theorem, we obtain

$$\begin{aligned} k &> \varepsilon \int_1^\infty x^{-\varepsilon-1}\left[\int_0^x \frac{u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}}{A\max\{u^\lambda, 1\} + Bu^\lambda + C}du\right]dx \\ &= \varepsilon\left\{\int_1^\infty x^{-\varepsilon-1}\left[\int_0^1 \frac{u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}du}{Bu^\lambda + (A+C)} + \int_1^x \frac{u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}du}{(A+B)u^\lambda + C}\right]dx\right\} \\ &= \int_0^1 \frac{u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}du}{Bu^\lambda + (A+C)} + \varepsilon\int_1^\infty \frac{(\int_u^\infty x^{-\varepsilon-1}dx)u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}}{(A+B)u^\lambda + C}du \\ &= \int_0^1 \frac{u^{\frac{\lambda}{2}+\frac{\varepsilon}{q}-1}du}{Bu^\lambda + (A+C)} + \int_1^\infty \frac{u^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}du}{(A+B)u^\lambda + C}. \end{aligned}$$

For $\varepsilon \to 0^+$, in view of (7) and (8), we obtain $k \ge k_\lambda(A, B, C)$. Hence $k = k_\lambda(A, B, C)$ is the best constant factor of (11). If the constant factor in (10) is not the best possible, then by (16), we may get a contradiction that the constant factor in (11) is not the best possible.

(b) For $0 < p < 1$, if there exists $K \ge k_\lambda(A, B, C)$, such that the reverse form of (11) is valid as we replace $k_\lambda(A, B, C)$ by K, then we have

$$\begin{aligned}
K &= \varepsilon K||f_\varepsilon||_{p,\phi_p}||g_\varepsilon||_{q.\phi_q} < \varepsilon\int_0^\infty\int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)dxdy}{A\max\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda} \\
&= \varepsilon\int_1^\infty y^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}\left[\int_1^\infty \frac{x^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}}{A\max\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dx\right]dy \\
&\le \varepsilon\int_1^\infty y^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}\left[\int_0^\infty \frac{x^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}}{A\max\{x^\lambda,y^\lambda\}+Bx^\lambda+Cy^\lambda}dx\right]dy \\
&= \int_0^1 \frac{u^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}du}{A\max\{u^\lambda,1\}+Bu^\lambda+C}+\int_1^\infty \frac{u^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}du}{A\max\{u^\lambda,1\}+Bu^\lambda+C} \\
&\le \int_0^1 \frac{u^{\frac{\lambda}{2}-\frac{\varepsilon}{p}-1}}{Bu^\lambda+(A+C)}du+\int_1^\infty \frac{u^{\frac{\lambda}{2}-1}}{(A+B)u^\lambda+C}du.
\end{aligned}$$

For $\varepsilon \to 0^+$, in view of (9), we obtain $K \le k_\lambda(A,B,C)$. Hence $K = k_\lambda(A,B,C)$ is the best constant factor of the reverse form of (11). If the constant factor in the reverse form of (10) is not the best possible, then by the reverse form of (16), we may get a contradiction that the constant factor in the reverse form of (11) is not the best possible. The theorem is proved. □

In the following corollaries, some words that $p > 0 (p \ne 1), \frac{1}{p}+\frac{1}{q} = 1, \lambda > 0, f, g \ge 0, 0 < ||f||_{p,\phi_p} = \{\int_0^\infty x^{p(1-\frac{\lambda}{2})-1}f^p(x)dx\}^{\frac{1}{p}} < \infty$,$0 < ||g||_{q,\phi_q} = \{\int_0^\infty x^{q(1-\frac{\lambda}{2})-1}g^q(x)dx\}^{\frac{1}{q}} < \infty$, and the constant factors are the best possible are omitted. We obviously have the formula that

$$|x-y| = 2\max\{x,y\}-x-y;\ \ \min\{x,y\} = x+y-\max\{x,y\}.$$

Since $\max\{x,y\}-a|x-y| = (1-2a)\max\{x,y\}+ax+ay$, setting $B = C = a \ge 0, A = 1-2a > -\min\{B,C\} = -a$, in (5), we find $0 \le a < 1$ and

$$K_\lambda(a) := k_\lambda(1-2a,a,a) = \begin{cases} \dfrac{4}{\lambda\sqrt{a(1-a)}}\arctan\sqrt{\dfrac{a}{1-a}}, 0 < a < 1 \\ \dfrac{4}{\lambda},\ a = 0. \end{cases}$$

By Theorem1, it follows

Corollary 1. *For* $p > 1$, $0 \le a < 1$, *we have the equivalent forms as:*

$$\int_0^\infty y^{\frac{p\lambda}{2}-1}\left(\int_0^\infty \frac{f(x)dx}{\max\{x^\lambda,y^\lambda\}-a|x^\lambda-y^\lambda|}\right)^p dy < K_\lambda^p(a)||f||_{p,\phi_p}^p; \tag{20}$$

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda,y^\lambda\}-a|x^\lambda-y^\lambda|}dxdy < K_\lambda(a)||f||_{p,\phi_p}||g||_{q,\phi_q}. \tag{21}$$

Since $\max\{x,y\}+a\min\{x,y\}=(1-a)\max\{x,y\}+ax+ay$, setting $B=C=a\geq 0, A=1-a>-\min\{B,C\}=-a$, in (5), we find $a\geq 0$ and

$$K'_\lambda(a):=k_\lambda(1-a,a,a)=\begin{cases}\dfrac{4}{\lambda\sqrt{a}}\arctan\sqrt{a},\ a>0\\ \dfrac{4}{\lambda},\ a=0.\end{cases}$$

By Theorem1, it follows[12]

Corollary 2. *For $p>1$, $a\geq 0$, we have the following equivalent forms:*

$$\int_0^\infty y^{\frac{p\lambda}{2}-1}\left(\int_0^\infty \frac{f(x)dx}{\max\{x^\lambda,y^\lambda\}+a\min\{x^\lambda,y^\lambda\}}\right)^p dy<K'^{\,p}_\lambda(a)||f||^p_{p,\phi_p};\tag{22}$$

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)dxdy}{\max\{x^\lambda,y^\lambda\}+a\min\{x^\lambda,y^\lambda\}}<K'_\lambda(a)||f||_{p,\phi_p}||g||_{q,\phi_q}.\tag{23}$$

Since $ax+by+c|x-y|=2c\max\{x,y\}+(a-c)x+(b-c)y$, setting $B=a-c\geq 0, C=b-c\geq 0$, $A=2c>-\min\{B,C\}=-\min\{a,b\}+c$, in (5), we find $-\min\{a,b\}<c\leq\min\{a,b\}$ and

$$\widetilde{K}_\lambda(a,b,c):=k_\lambda(2c,a-c,b-c)$$
$$=\begin{cases}\dfrac{2}{\lambda}\left[\dfrac{\arctan\sqrt{\frac{a-c}{b+c}}}{\sqrt{(a-c)(b+c)}}+\dfrac{\arctan\sqrt{\frac{b-c}{a+c}}}{\sqrt{(b-c)(a+c)}}\right],-\min\{a,b\}<c<\min\{a,b\}\\ \dfrac{2}{\lambda}\left[\dfrac{1}{b+c}+\dfrac{1}{\sqrt{2c(b-c)}}\arctan\sqrt{\dfrac{b-c}{2c}}\right],\ a=c,b>c>0\\ \dfrac{2}{\lambda}\left[\dfrac{1}{a+c}+\dfrac{1}{\sqrt{2c(a-c)}}\arctan\sqrt{\dfrac{a-c}{2c}}\right],\ b=c,a>c>0\\ \dfrac{2}{\lambda c},\ a=b=c>0.\end{cases}$$

By Theorem1, it follows

Corollary 3. *For $p>1$, $-\min\{a,b\}<c\leq\min\{a,b\}$, we have the following equivalent forms:*

$$\int_0^\infty y^{\frac{p\lambda}{2}-1}\left(\int_0^\infty \frac{f(x)dx}{ax^\lambda+by^\lambda+c|x^\lambda-y^\lambda|}\right)^p dy<\widetilde{K}^p_\lambda(a,b,c)||f||^p_{p,\phi_p};\tag{24}$$

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{ax^\lambda+by^\lambda+c|x^\lambda-y^\lambda|}dxdy<\widetilde{K}_\lambda(a,b,c)||f||_{p,\phi_p}||g||_{q,\phi_q}.\tag{25}$$

Since $ax + by - c\min\{x, y\} = c\max\{x, y\} + (a - c)x + (b - c)y$, setting $B = a - c, C = b - c \geq 0$, $A = c > -\min\{B, C\} = -\min\{a - c, b - c\} = -\min\{a, b\} + c$, in (5), we find $c \leq \min\{a, b\}, a, b > 0$ and

$$
\begin{aligned}
& k'_\lambda(a, b, c) := k_\lambda(c, a - c, b - c) \\
= & \begin{cases}
\frac{2}{\lambda}\left[\frac{\arctan\sqrt{\frac{a-c}{b}}}{\sqrt{(a-c)b}} + \frac{\arctan\sqrt{\frac{b-c}{a}}}{\sqrt{(b-c)a}}\right], a, b > 0, c < \min\{a, b\} \\
\frac{2}{\lambda}\left[\frac{1}{b} + \frac{1}{\sqrt{c(b-c)}}\arctan\sqrt{\frac{b-c}{c}}\right], \ a = c, 0 < c < b \\
\frac{2}{\lambda}\left[\frac{1}{a} + \frac{1}{\sqrt{c(a-c)}}\arctan\sqrt{\frac{a-c}{c}}\right], \ b = c, 0 < c < a \\
\frac{4}{\lambda c}, \ a = b = c > 0
\end{cases}
\end{aligned}
$$

By Theorem1, it follows

Corollary 4. *For* $p > 1$, $c \leq \min\{a, b\}, a, b > 0$, *we have the following equivalent forms:*

$$
\int_0^\infty y^{\frac{p\lambda}{2}-1}\left(\int_0^\infty \frac{f(x)dx}{ax^\lambda + by^\lambda - c\min\{x^\lambda, y^\lambda\}}\right)^p dy < k'^{\,p}_\lambda(a, b, c)||f||^p_{p,\phi_p}; \tag{26}
$$

$$
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{ax^\lambda + by^\lambda - c\min\{x^\lambda, y^\lambda\}}dxdy < k'_\lambda(a, b, c)||f||_{p,\phi_p}||g||_{q,\phi_q}. \tag{27}
$$

Remarks. (i) For $0 < p < 1$, we have the reverse forms of (20) -(27) with the best constant factors. (ii) For $p = q = 2, \lambda = 1$ in (11), if $A = B = C = 1$, we obtain (3); if $A = -1, B = C = 2$, we obtain (4); if $A = 0, B = C = 1$, we obtain (1); if $A = 1, B = C = 0$, we obtain (2). Hence we give a new inequality (11) with a best constant factor, which is a relation between (1) and (2). Also it is an extension of (3) and (4).

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