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A New Dual Hardy-Hilbert's Inequality with some Parameters and its Reverse

Wuyi Zhong
Department of Mathematics, Guangdong Institute of Education, Guangzhou, Guangdong, 510303, P. R. China
e-mail : zwy@gdei.edu.cn

Abstract. By using the improved Euler-Maclaurin summation formula and estimating the weight coefficients in this paper, a new dual Hardy-Hilbert's inequality and its reverse form are obtained, which are all with two pairs of conjugate exponents $(p, q), (r, s)$ and a independent parameter λ. In addition, some equivalent forms of the inequalities are considered. We also prove that the constant factors in the new inequalities are all the best possible. As a particular case of our results, we obtain the reverse form of a famous Hardy-Hilbert's inequality.

1. Introduction and preliminaries

If $a_n,\ b_n \geq 0$, and $0 < \sum\limits_{n=0}^{\infty} a_n^2 < \infty$, $0 < \sum\limits_{n=0}^{\infty} b_n^2 < \infty$, then the well know Hilbert's inequality is written in the following form(see Hardy et al. [1. Ch.9]):

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \pi\{\sum_{n=0}^{\infty} a_n^2 \sum_{m=0}^{\infty} b_n^2\}^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible. We also have a classical extension of Hilbert's inequality with a pair of conjugate exponents as follows [1]:

If $a_n,\ b_n \geq 0$, $p > 1$, $\dfrac{1}{p} + \dfrac{1}{q} = 1((p, q)$ is called a pair of conjugate exponents), such that $0 < \sum\limits_{n=0}^{\infty} a_n^p < \infty$, $0 < \sum\limits_{n=0}^{\infty} b_n^q < \infty$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\frac{\pi}{p})}\{\sum_{n=0}^{\infty} a_n^p\}^{\frac{1}{p}}\{\sum_{m=0}^{\infty} b_n^q\}^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\dfrac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible.

Inequality (1.2) is the famous Hardy-Hilbert's inequality, which is important

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in analysis and its applications [2]. In recent years, many results [3]-[9] have been obtained in the research of Hardy-Hilbert's inequalities and Hilbert-type inequalities. In 2006, by introducing a pair of conjugate exponents (p, q) and a independent parameter λ, Yang [10] gave a dual Hardy-Hilbert's inequality, which is a new extension of inequality (1.1):

If $a_n, b_n \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leq 1$, $0 < \sum_{n=0}^{\infty}(2n+1)^{p-1-\lambda}a_n^p < \infty$, $0 < \sum_{n=0}^{\infty}(2n+1)^{q-1-\lambda}b_n^q < \infty$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(2m+1)^\lambda + (2n+1)^\lambda} < \frac{\pi}{2\lambda\sin(\frac{\pi}{p})}\{\sum_{n=0}^{\infty}(2n+1)^{p-1-\lambda}a_n^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(2n+1)^{q-1-\lambda}b_n^q\}^{\frac{1}{q}}, \tag{1.3}$$

where the constant factor $\frac{\pi}{2\lambda\sin(\frac{\pi}{p})}$ is the best possible. (1.3) has a equivalent form as follows:

$$\sum_{n=0}^{\infty}(2n+1)^{\lambda(p-1)-1}\{\sum_{m=0}^{\infty}\frac{a_m}{(2m+1)^\lambda + (2n+1)^\lambda}\}^p < [\frac{\pi}{2\lambda\sin(\frac{\pi}{p})}]^p\sum_{n=0}^{\infty}(2n+1)^{p-1-\lambda}a_n^p, \tag{1.4}$$

where the constant factor $[\frac{\pi}{2\lambda\sin(\frac{\pi}{p})}]^p$ is still the best possible. Letting $\lambda = 1$, we can find that (1.2) and (1.3) are all the extensions of (1.1). They are all related to the same best constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ and (p, q)-parameters, but different.

By using the improved Euler-Maclaurin summation formula and Beta function to estimate the weight coefficients, the main objective of This paper is to build a new extension of (1.3) and its reverse form. Which are all with two pairs of conjugate exponents $(p, q), (r, s)$ and a independent parameter λ. In addition, some equivalent forms are considered. We also prove that the constant factors of the inequalities in this paper are all the best possible. As a particular case of our results, we obtain the reverse forms of inequality (1.2).

For these purposes, we introduce the improved Euler-Maclaurin summation formula [11] and Hölder′s inequality [12] as follows:

The improved Euler-Maclaurin summation formula: If for $i = 0, 1, 2, 3$, $(-1)^i f^{(i)}(x) > 0$, $x \in [0, \infty)$, $f^{(i)}(\infty) = 0$, and $\int_0^\infty f(x)dx < \infty$, then we have

$$\sum_{n=0}^{\infty} f(n) < \int_0^\infty f(x)dx + \frac{1}{2}f(0) - \frac{1}{12}f^{'}(0), \tag{1.5}$$

$$\sum_{n=0}^{\infty} f(n) > \int_0^{\infty} f(x)dx + \frac{1}{2}f(0). \tag{1.6}$$

Hölder$^{'}$s inequality: Assume that $p > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $F, G \geq 0$ and $F \in L^p(E)$, $G \in L^q(E)$. We have the following Hölder's inequalities:

(1) If $p > 1$, then

$$\int_E F(t)G(t)dt \leq (\int_E F^p(t)dt)^{\frac{1}{p}} (\int_E G^q(t)dt)^{\frac{1}{p}}; \tag{1.7}$$

(2) if $0 < p < 1$, then

$$\int_E F(t)G(t)dt \geq (\int_E F^p(t)dt)^{\frac{1}{p}} (\int_E G^q(t)dt)^{\frac{1}{p}}, \tag{1.8}$$

where the equalities hold if and only if there exist real numbers A and B($A^2 + B^2 \neq 0$) such that $AF^p(t) = BG^q(t)\, a.e.$ in E.

Lemma 1.1. *If* $s > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $0 < \lambda \leq 1$,

$$f_m(x) := \frac{(2x+1)^{\frac{\lambda}{s}-1}}{(2x+1)^{\lambda} + (2m+1)^{\lambda}}, \qquad x \in [0, \infty), \tag{1.9}$$

then $f_m(x)$ *satisfies* (1.5) *and* (1.6).

Proof. Set $g(u) = \frac{1}{u}$, and $u_m(x) = (2x+1)^{\lambda} + (2m+1)^{\lambda}$, $h_m(x) = g(u_m(x))$, by $0 < \lambda \leq 1$, we have $h_m(x) \in C^3[0, \infty)$ and $(-1)^i h_m^{(i)}(x) > 0$ for i=0, 1, 2, 3. And by $\frac{\lambda}{s} - 1 < 0$, so $f_m(x) = h_m(x) \cdot (2x+1)^{\frac{\lambda}{s}-1}$ satisfies the inequalities (1.5) and (1.6), the lemma is proved. □

Lemma 1.2. *If* $m \in N_0$, $s > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $0 < \lambda \leq 1$, *defining* $R_{\lambda}(m, r)$ *as follows:*

$$R_{\lambda}(m,r) := \frac{1}{2\lambda} \int_0^{\frac{1}{(2m+1)^{\lambda}}} \frac{y^{\frac{1}{s}-1}}{1+y} dy + \frac{(2m+1)^{\frac{\lambda}{r}} (\frac{\frac{\lambda}{s}-1}{3} - 1)}{2[1 + (2m+1)^{\lambda}]} - \frac{(2m+1)^{\frac{\lambda}{r}} \lambda}{6[1 + (2m+1)^{\lambda}]^2}, \tag{1.10}$$

then we have $R_{\lambda}(m, r) > 0$.

Proof. Using integration by parts, we have

$$\int_0^{\frac{1}{(2m+1)^{\lambda}}} \frac{y^{\frac{1}{s}-1}}{1+y} dy = \frac{s(2m+1)^{\lambda - \frac{\lambda}{s}}}{1 + (2m+1)^{\lambda}} + \frac{s^2}{1+s} \int_0^{\frac{1}{(2m+1)^{\lambda}}} \frac{dy^{\frac{1}{s}+1}}{(1+y)^2}$$

$$> \frac{s(2m+1)^{\frac{\lambda}{r}}}{1 + (2m+1)^{\lambda}} + \frac{s^2 (2m+1)^{\frac{\lambda}{r}}}{(1+s)[1 + (2m+1)^{\lambda}]^2}. \tag{1.11}$$

Then in view of (1.10) and $s > 1$, for $0 < \lambda \leq 1$, we have

$$R_\lambda(m,r) > \frac{(2m+1)^{\frac{\lambda}{r}}}{2[1+(2m+1)^\lambda]}\left(\frac{s}{\lambda}+\frac{\frac{\lambda}{s}-1}{3}-1\right)+\frac{(2m+1)^{\frac{\lambda}{r}}}{2[1+(2m+1)^\lambda]^2}\left(\frac{s^2}{\lambda(1+s)}-\frac{\lambda}{3}\right),$$

by

$$\frac{s}{\lambda}+\frac{\frac{\lambda}{s}-1}{3}-1=\frac{(3s-\lambda)(s-\lambda)}{3s\lambda}>0,$$
$$\frac{s^2}{\lambda(1+s)}-\frac{\lambda}{3}=\frac{3s^2-\lambda^2(1+s)}{3\lambda(1+s)}\geq\frac{3s^2-(1+s)}{3\lambda(1+s)}>0.$$

The lemma is proved. □

Lemma 1.3. *If $m \in N_0$, $r > 1$, $\frac{1}{r}+\frac{1}{s}=1$, and $0 < \lambda \leq 1$, defining $\omega_\lambda(m,r)$ and $\omega_\lambda(n,s)$ as follows:*

$$\omega_\lambda(m,r) := \sum_{n=0}^{\infty}\frac{(2m+1)^{\frac{\lambda}{r}}}{(2m+1)^\lambda+(2n+1)^\lambda}\frac{1}{(2n+1)^{1-\frac{\lambda}{s}}}, \tag{1.12}$$

$$\omega_\lambda(n,s) := \sum_{m=0}^{\infty}\frac{(2n+1)^{\frac{\lambda}{s}}}{(2m+1)^\lambda+(2n+1)^\lambda}\frac{1}{(2m+1)^{1-\frac{\lambda}{r}}}. \tag{1.13}$$

Then we have

$$(1)\,\omega_\lambda(m,r)=\frac{1}{2m+1}\sum_{n=0}^{\infty}\frac{[(\frac{2n+1}{2m+1})^\lambda]^{\frac{1}{s}-\frac{1}{\lambda}}}{(\frac{2n+1}{2m+1})^\lambda+1}:=\frac{1}{2m+1}\sum_{n=0}^{\infty}h_m(n,\frac{1}{s}), \tag{1.14}$$

$$(2)\,\omega_\lambda(m,r)<\frac{1}{2\lambda}B(\frac{1}{s},\frac{1}{r})=\frac{\pi}{2\lambda\sin(\frac{\pi}{s})}:=k_\lambda(s), \tag{1.15}$$

$$(3)\,\omega_\lambda(n,s)<\frac{\pi}{2\lambda\sin(\frac{\pi}{r})}:=k_\lambda(r)=k_\lambda(s). \tag{1.16}$$

Proof. (1) $\omega_\lambda(m,r)=(2m+1)^{\frac{\lambda}{r}}\sum_{n=0}^{\infty}\frac{(\frac{2n+1}{2m+1})^{\frac{\lambda}{s}-1}(2m+1)^{\frac{\lambda}{s}-1-\lambda}}{(\frac{2n+1}{2m+1})^\lambda+1}=\frac{1}{2m+1}\sum_{n=0}^{\infty}h_m(n,\frac{1}{s})$,
(2) By the definition of (1.9) and using Lemma 1.1 and (1.5), we have

$$\omega_\lambda(m,r)=(2m+1)^{\frac{\lambda}{r}}\sum_{n=0}^{\infty}f_m(n)$$
$$<(2m+1)^{\frac{\lambda}{r}}[\int_0^\infty f_m(x)dx+\frac{1}{2}f_m(0)-\frac{1}{12}f_m'(0)]. \tag{1.17}$$

Setting $y = (\frac{2x+1}{2m+1})^{\lambda}$, we have $d(\frac{2x+1}{2m+1}) = \frac{1}{\lambda} y^{\frac{1}{\lambda}-1} dy$ and

$$\int_0^{\infty} f_m(x)dx = \frac{1}{2\lambda(2m+1)^{\frac{\lambda}{r}}} \int_{\frac{1}{(2m+1)^{\lambda}}}^{\infty} \frac{y^{\frac{1}{s}-1}}{1+y} dy$$

$$= \frac{1}{2\lambda(2m+1)^{\frac{\lambda}{r}}} [B(\frac{1}{s}, \frac{1}{r}) - \int_0^{\frac{1}{(2m+1)^{\lambda}}} \frac{y^{\frac{1}{s}-1}}{1+y} dy]. \tag{1.18}$$

In view of

$$f_m(0) = \frac{1}{1+(2m+1)^{\lambda}}, \tag{1.19}$$

$$f_m'(0) = \frac{-2\lambda}{[1+(2m+1)^{\lambda}]^2} + \frac{2(\frac{\lambda}{s}-1)}{1+(2m+1)^{\lambda}},$$

by (1.17), (1.10) and Lemma 1.2, we find

$$\omega_{\lambda}(r,m) = (2m+1)^{\frac{\lambda}{r}} \sum_{n=0}^{\infty} f_m(n) < \frac{B(\frac{1}{s}, \frac{1}{r})}{2\lambda} - R_{\lambda}(m,r) < \frac{B(\frac{1}{s}, \frac{1}{r})}{2\lambda}.$$

we have (1.15), so does (1.16). The lemma is proved. □

Lemma 1.4. *Let* $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $0 < \lambda \le 1$. *And let* $\omega_{\lambda}(m,r)$, $k_{\lambda}(s)$ *be defined by* (1.12) *and* (1.15), *respectively. Let*

$$\eta_{\lambda}(m) := \frac{1}{2k_{\lambda}(s)} \{ \frac{1}{\lambda} \int_0^{\frac{1}{(2m+1)^{\lambda}}} \frac{y^{\frac{1}{s}-1}}{1+y} dy - \frac{(2m+1)^{\frac{\lambda}{r}}}{1+(2m+1)^{\lambda}} \}, \tag{1.20}$$

then we have

(1.21) (1) $\omega_{\lambda}(m,r) > k_{\lambda}(s)[1-\eta_{\lambda}(m)]$,

(1.22) (2) $0 < \eta_{\lambda}(m) < \theta_{\lambda}(r) < 1, (\theta_{\lambda}(r) = \frac{1}{2\lambda k_{\lambda}(s)} \int_0^1 \frac{y^{\frac{1}{s}-1}}{1+y} dy)$,

(1.23) (3) $\eta_{\lambda}(m) = O(\frac{1}{(2m+1)^{\frac{\lambda}{s}}}) \, (m \to \infty)$.

Proof. By forms (1.9), (1.12), Lemma 1.1 and forms (1.6), (1.18) and (1.19), we have

$$\omega_{\lambda}(m,r) = (2m+1)^{\frac{\lambda}{r}} \sum_{n=0}^{\infty} f_m(n) > (2m+1)^{\frac{\lambda}{r}} [\int_0^{\infty} f_m(x)dx + \frac{1}{2} f_m(0)]$$

$$= \frac{1}{2\lambda} \frac{\pi}{\sin(\frac{\pi}{s})} - \frac{1}{2\lambda} \int_0^{\frac{1}{(2m+1)^{\lambda}}} \frac{y^{\frac{1}{s}-1}}{1+y} dy + \frac{(2m+1)^{\frac{\lambda}{r}}}{2[1+(2m+1)^{\lambda}]}.$$

(1.21) is valid. And in view of (1.11), by

$$\int_0^{\frac{1}{(2m+1)^\lambda}} \frac{y^{\frac{1}{s}-1}}{1+y}dy = \frac{s(2m+1)^{\lambda-\frac{\lambda}{s}}}{1+(2m+1)^\lambda} + \frac{s^2}{1+s}\int_0^{\frac{1}{(2m+1)^\lambda}} \frac{dy^{\frac{1}{s}+1}}{(1+y)^2},$$

we have

$$0 < \frac{(2m+1)^{\frac{\lambda}{r}}}{2\lambda k_\lambda(s)[1+(2m+1)^\lambda]}(s-\lambda) < \eta_\lambda(m) < \frac{1}{2\lambda k_\lambda(s)}\int_0^1 \frac{y^{\frac{1}{s}-1}}{1+y}dy, \tag{1.24}$$

and

$$\eta_\lambda(m) < \frac{1}{2\lambda k_\lambda(s)}\int_0^{\frac{1}{(2m+1)^\lambda}} y^{\frac{1}{s}-1}dy = \frac{s}{2\lambda k_\lambda(s)}\frac{1}{(2m+1)^{\frac{\lambda}{s}}}. \tag{1.25}$$

The forms (1.22) and (1.23) are valid. The lemma is proved. □

Lemma 1.5. *Let* $p > 1$, $r > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{r} + \frac{1}{s} = 1$, *and* $0 < \lambda \le 1$, $0 < \varepsilon < \frac{p\lambda}{r}$, *and let* $k_\lambda(s)$ *be defined by* (1.15). *Set that* $\widetilde{a}_n := (2n+1)^{\frac{\lambda}{r}-\frac{\varepsilon}{p}-1}$, $\widetilde{b}_n := (2n+1)^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}$,

$$I_1 := \varepsilon\{\sum_{n=0}^{\infty}(2n+1)^{p(1-\frac{\lambda}{r})-1}\widetilde{a}_n^p\}^{\frac{1}{p}} \cdot \{\sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}\widetilde{b}_n^q\}^{\frac{1}{q}},$$

$$I_2 := \varepsilon\int_0^\infty\int_0^\infty \frac{(2x+1)^{\frac{\lambda}{r}-\frac{\varepsilon}{p}-1}(2y+1)^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}}{(2x+1)^\lambda+(2y+1)^\lambda}dxdy,$$

then we have

$$(1)\ I_1 < \varepsilon(1+\frac{1}{2\varepsilon}), \tag{1.26}$$

$$(2)\ I_2 \ge \frac{1}{2}k_\lambda(s) + \circ(1)\ (\varepsilon \to 0^+). \tag{1.27}$$

Proof. By the definitions of $\widetilde{a}_n$ and $\widetilde{b}_n$, using the strictly monotone decrement of the sequence "$(2n+1)^{-1-\varepsilon}$, $n \in N$", we have

$$\begin{aligned} I_1 &= \varepsilon\{\sum_{n=0}^{\infty}(2n+1)^{-1-\varepsilon}\}^{\frac{1}{p}} \cdot \{\sum_{n=0}^{\infty}(2n+1)^{-1-\varepsilon}\}^{\frac{1}{q}} = \varepsilon\sum_{n=0}^{\infty}(2n+1)^{-1-\varepsilon} \\ &= \varepsilon\{1+\sum_{n=1}^{\infty}(2n+1)^{-1-\varepsilon}\} < \varepsilon\{1+\int_0^\infty (2x+1)^{-1-\varepsilon}dx\} \\ &= \varepsilon\{1-\frac{1}{2\varepsilon}(2x+1)^{-\varepsilon}|_0^\infty\}. \end{aligned}$$

(1.26) holds. Setting $u = (\frac{2x+1}{2y+1})^{\lambda}$, by $0 < \varepsilon < \frac{p\lambda}{r}$, we have

$$\begin{aligned} I_2 &= \frac{\varepsilon}{2\lambda}\int_0^{\infty}(2y+1)^{-1-\varepsilon}[\int_{\frac{1}{(2y+1)^{\lambda}}}^{\infty}\frac{u^{\frac{1}{r}-\frac{\varepsilon}{p\lambda}-1}}{1+u}du]dy \\ &= \frac{\varepsilon}{2\lambda}\int_0^{\infty}(2y+1)^{-1-\varepsilon}[\int_0^{\infty}\frac{u^{\frac{1}{r}-\frac{\varepsilon}{p\lambda}-1}}{1+u}du - \int_0^{\frac{1}{(2y+1)^{\lambda}}}\frac{u^{\frac{1}{r}-\frac{\varepsilon}{p\lambda}-1}}{1+u}du]dy \\ &\geq \frac{1}{4\lambda}B(\frac{1}{r}-\frac{\varepsilon}{p\lambda},\frac{1}{s}+\frac{\varepsilon}{p\lambda}) - \frac{\varepsilon}{2\lambda}\int_0^{\infty}(2y+1)^{-1}[\int_0^{\frac{1}{(2y+1)^{\lambda}}}u^{\frac{1}{r}-\frac{\varepsilon}{p\lambda}-1}du]dy \\ &= \frac{1}{4\lambda}B(\frac{1}{r}-\frac{\varepsilon}{p\lambda},\frac{1}{s}+\frac{\varepsilon}{p\lambda}) - \frac{\varepsilon}{4\lambda^2(\frac{1}{r}-\frac{\varepsilon}{p\lambda})^2}. \end{aligned}$$

Letting $\varepsilon \to 0^+$, we have (1.27). The lemma is proved. □

2. Main results

Theorem 2.1. *If* $p > 1$, $r > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\frac{1}{r}+\frac{1}{s} = 1$, $0 < \lambda \leq 1$, *and* a_n, $b_n \geq 0$, *such that* $0 < \sum_{n=0}^{\infty}(2n+1)^{p(1-\frac{\lambda}{r})-1}a_n^p < \infty$, $0 < \sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q < \infty$, *setting*

$$H_{\lambda}(a_m, b_n) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(2m+1)^{\lambda}+(2n+1)^{\lambda}}, \tag{2.1}$$

then we have

$$H_{\lambda}(a_m, b_n) < k_{\lambda}(s)\{\sum_{m=0}^{\infty}(2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p\}^{\frac{1}{p}} \cdot \{\sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}, \tag{2.2}$$

where, the constant factor $k_{\lambda}(s) = \frac{\pi}{2\lambda\sin(\frac{\pi}{s})}$ *is the best possible.*

Proof. By $p > 1$, using Hölder's inequality (1.7), then we have

$$\begin{aligned} &H_{\lambda}(a_m, b_n) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^{\lambda}+(2n+1)^{\lambda}}[\frac{(2m+1)^{(1-\frac{\lambda}{r})/q}}{(2n+1)^{(1-\frac{\lambda}{s})/p}}a_m][\frac{(2n+1)^{(1-\frac{\lambda}{s})/p}}{(2m+1)^{(1-\frac{\lambda}{r})/q}}b_n] \\ &\leq \{\sum_{m=0}^{\infty}[\sum_{n=0}^{\infty}\frac{(2m+1)^{\frac{\lambda}{r}}}{(2m+1)^{\lambda}+(2n+1)^{\lambda}}\frac{1}{(2n+1)^{1-\frac{\lambda}{s}}}](2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p\}^{\frac{1}{p}} \\ &\quad\times\{\sum_{n=0}^{\infty}[\sum_{m=0}^{\infty}\frac{(2n+1)^{\frac{\lambda}{s}}}{(2m+1)^{\lambda}+(2n+1)^{\lambda}}\frac{1}{(2m+1)^{1-\frac{\lambda}{r}}}](2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}} \end{aligned}$$

$$= \{\sum_{m=0}^{\infty}\omega_{\lambda}(m,r)(2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p\}^{\frac{1}{p}} \cdot \{\sum_{n=0}^{\infty}\omega_{\lambda}(n,s)(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}, \tag{2.3}$$

where $\omega_\lambda(m,r)$, $\omega_\lambda(n,s)$ are defined by (1.12) and (1.13), respectively. In view of (1.15) and (1.16), we have (2.2).

If there exists a positive number K $\le k_\lambda(r)$, such that (2.2) is still valid when we replace $k_\lambda(r)$ by K, in particular, for $0<\varepsilon<\frac{p\lambda}{r}$, setting: $\widetilde{a}_m := (2m+1)^{\frac{\lambda}{r}-\frac{\varepsilon}{p}-1}$, $\widetilde{b}_n := (2n+1)^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}$, $m,n\in N$, we have

$$\varepsilon H_\lambda(\widetilde{a}_m,\widetilde{b}_n) < K\varepsilon\{\sum_{n=0}^{\infty}(2n+1)^{p(1-\frac{\lambda}{r})-1}\widetilde{a}_n^p\}^{\frac{1}{p}}\cdot\{\sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}\widetilde{b}_n^q\}^{\frac{1}{q}} = KI_1.$$

But by (1.26), (1.27), we have

$$\begin{aligned} K\cdot(\varepsilon+\frac{1}{2}) &> KI_1 > \varepsilon H_\lambda(\widetilde{a}_m,\widetilde{b}_n) \\ &\ge \varepsilon\int_0^\infty\int_0^\infty \frac{(2x+1)^{\frac{\lambda}{r}-\frac{\varepsilon}{p}-1}(2y+1)^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}}{(2x+1)^\lambda+(2y+1)^\lambda}dxdy \\ &= I_2 \ge \frac{1}{2}k_\lambda(r)+\circ(1), (\varepsilon\to 0^+). \end{aligned}$$

Letting $\varepsilon\to 0^+$, we have $K\ge k_\lambda(r)$, it follows that $K=k_\lambda(r)$. Hence the constant factor $k_\lambda(r)$ in (2.2) is the best possible. The theorem is proved. □

Theorem 2.2. *Let $p>1$, $r>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{r}+\frac{1}{s}=1$, $0<\lambda\le 1$, and $a_m\ge 0$, such that $0<\sum_{m=0}^{\infty}(2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p<\infty$, and let $k_\lambda(r)$ be defined by (1.16), then we have*

$$\sum_{n=0}^{\infty}(2n+1)^{\frac{p\lambda}{s}-1}[\sum_{m=0}^{\infty}\frac{a_m}{(2m+1)^\lambda+(2n+1)^\lambda}]^p < k_\lambda^p(r)\sum_{m=0}^{\infty}(2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p, \tag{2.4}$$

where the constant factor $k_\lambda^p(r)$ is the best possible. Inequality (2.4) is equivalent to (2.2).

Proof. Since $0<\sum_{m=0}^{\infty}(2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p<\infty$, then there exists $k_0\in N$, such for any $K>k_0$, that $0<\sum_{m=0}^{K}(2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p<\infty$, then by setting

$$b_n(K) = (2n+1)^{\frac{p\lambda}{s}-1}[\sum_{m=0}^{K}\frac{a_m}{(2m+1)^\lambda+(2n+1)^\lambda}]^{p-1},$$

and using Hölder′s inequality (1.7) as in (2.3), we have

$$0 < \sum_{n=0}^{K}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q(K)$$
$$= \sum_{m=0}^{K}(2n+1)^{\frac{p\lambda}{s}-1}[\sum_{m=0}^{K}\frac{a_m}{(2m+1)^\lambda+(2n+1)^\lambda}]^p$$
$$= \sum_{n=0}^{K}\sum_{m=0}^{K}\frac{a_m b_n(K)}{(2m+1)^\lambda+(2n+1)^\lambda}$$
$$< k_\lambda(r)\{\sum_{n=0}^{K}(2n+1)^{p(1-\frac{\lambda}{r})-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=0}^{K}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q(K)\}^{\frac{1}{q}}. \tag{2.5}$$

Hence we have

$$\{\sum_{n=0}^{K}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q(K)\}^{\frac{1}{p}} = \{\sum_{m=0}^{K}(2n+1)^{\frac{p\lambda}{s}-1}[\sum_{m=0}^{K}\frac{a_m}{(2m+1)^\lambda+(2n+1)^\lambda}]^p\}^{\frac{1}{p}}$$
$$< k_\lambda(r)\{\sum_{n=0}^{K}(2n+1)^{p(1-\frac{\lambda}{r})-1}a_n^p\}^{\frac{1}{p}}. \tag{2.6}$$

Letting $K\to\infty$, it follows that $0<\sum\limits_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q(\infty)<\infty$. Hence by (2.2), (2.5) keeps the form of strict inequality when $K\to\infty$, So does (2.6). Thus inequality (2.4) is valid.

On the other hand, if (2.4) is valid, by Hölder′s inequality (1.7), we have

$$H_\lambda(a_m,b_n)$$
$$= \sum_{n=0}^{\infty}[(2n+1)^{\frac{\lambda}{s}-\frac{1}{p}}\sum_{m=0}^{\infty}\frac{a_m}{(2m+1)^\lambda+(2n+1)^\lambda}][(2n+1)^{\frac{1}{p}-\frac{\lambda}{s}}b_n]$$
$$\le \{\sum_{n=0}^{\infty}(2n+1)^{\frac{p\lambda}{s}-1}[\sum_{m=0}^{\infty}\frac{a_m}{(2m+1)^\lambda+(2n+1)^\lambda}]^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}, \tag{2.7}$$

where, the notation $H_\lambda(a_m,b_n)$ is defined by (2.1). Then by (2.4), we obtain (2.2). Inequality (2.4) is equivalent to (2.2).

Since the constant factor in (2.2) is the best possible, we may show that the constant factor in (2.4) is also the best possible by (2.7). The theorem is proved. □

Theorem 2.3. *If* $0<p<1$, $r>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{r}+\frac{1}{s}=1$, $0<\lambda\le 1$, *and* a_n,

$b_n \geq 0$, such that $0 < \sum_{m=0}^{\infty}(2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p < \infty$, $0 < \sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q < \infty$, then we have the reverse inequality:

(2.8)

$$H_\lambda(a_m,b_n) > k_\lambda(r)\{\sum_{m=0}^{\infty}[1-\eta_\lambda(m)](2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}},$$

where, the notation $H_\lambda(a_m,b_n)$, $\eta_\lambda(m)$ are defined by (2.1) *and* (1.20)*, respectively. And the factor $\eta_\lambda(m)$ satisfies* (1.22) *and* (1.23)*. The constant factor $k_\lambda(r) = \frac{\pi}{2\lambda\sin(\frac{\pi}{r})}$ is the best possible.*

Proof. By $0 < p < 1$, using the reverse Hölder$'s$ inequality (1.8), we have

$$\begin{aligned}
&H_\lambda(a_m,b_n)\\
&\quad=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^\lambda+(2n+1)^\lambda}[\frac{(2m+1)^{(1-\frac{\lambda}{r})/q}}{(2n+1)^{(1-\frac{\lambda}{s})/p}}a_m][\frac{(2n+1)^{(1-\frac{\lambda}{s})/p}}{(2m+1)^{(1-\frac{\lambda}{r})/q}}b_n]\\
&\quad\geq\{\sum_{m=0}^{\infty}[\sum_{n=0}^{\infty}\frac{(2m+1)^{\frac{\lambda}{r}}}{(2m+1)^\lambda+(2n+1)^\lambda}\frac{1}{(2n+1)^{1-\frac{\lambda}{s}}}](2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p\}^{\frac{1}{p}}\\
&\qquad\times\{\sum_{n=0}^{\infty}[\sum_{m=0}^{\infty}\frac{(2n+1)^{\frac{\lambda}{s}}}{(2m+1)^\lambda+(2n+1)^\lambda}\frac{1}{(2m+1)^{1-\frac{\lambda}{r}}}](2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}
\end{aligned}$$

$$(2.9) = \{\sum_{m=0}^{\infty}\omega_\lambda(m,r)(2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p\}^{\frac{1}{p}}\cdot\{\sum_{n=0}^{\infty}\omega_\lambda(n,s)(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}},$$

where, factors $\omega_\lambda(m,r)$, $\omega_\lambda(n,s)$ are defined by (1.12) and (1.13), respectively. In view of (1.21) and (1.16), by $q < 0(0 < p < 1)$, (2.8) is valid.

If there exists a positive number K $\geq k_\lambda(r)$, such that (2.8) is still valid when we replace $k_\lambda(r)$ by K, in particular, for $0 < \varepsilon < -\frac{q\lambda}{r}$, setting: $\widetilde{a}_m = (2m+1)^{\frac{\lambda}{r}-\frac{\varepsilon}{p}-1}$, $\widetilde{b}_n = (2n+1)^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}, m,n \in N$, using (1.23), we have

$$\begin{aligned}
&H_\lambda(\widetilde{a}_m,\widetilde{b}_n)\\
&\quad> K\{\sum_{m=0}^{\infty}[1-\eta_\lambda(m)](2m+1)^{p(1-\frac{\lambda}{r})-1}\widetilde{a}_m^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}\widetilde{b}_n^q\}^{\frac{1}{q}}\\
&\quad= K\{\sum_{m=0}^{\infty}[1-\eta_\lambda(m)](2m+1)^{-1-\varepsilon}\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(2n+1)^{-1-\varepsilon}\}^{\frac{1}{q}}\\
&\quad= K\{\sum_{m=0}^{\infty}\frac{1}{(2m+1)^{1+\varepsilon}}-\sum_{m=0}^{\infty}O(\frac{1}{(2m+1)^{1+\varepsilon+\frac{\lambda}{s}}})\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{1+\varepsilon}}\}^{\frac{1}{q}}
\end{aligned}$$

$$(2.10) = K\sum_{m=0}^{\infty}\frac{1}{(2m+1)^{1+\varepsilon}}\{1-[\sum_{m=0}^{\infty}\frac{1}{(2m+1)^{1+\varepsilon}}]^{-1}\sum_{m=0}^{\infty}O(\frac{1}{(2m+1)^{1+\varepsilon+\frac{\lambda}{s}}})\}^{\frac{1}{p}},$$

On the other hand, by (1.14) and (1.15), we have

$$\begin{aligned} H_\lambda(\widetilde{a}_m,\widetilde{b}_n) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2m+1)^{\frac{\lambda}{r}-\frac{\varepsilon}{p}-1}(2n+1)^{\frac{\lambda}{s}-\frac{\varepsilon}{q}-1}}{(2m+1)^\lambda+(2n+1)^\lambda} \\ &= \sum_{m=0}^{\infty}(2m+1)^{-\varepsilon-2}\sum_{n=0}^{\infty}\frac{[(\frac{2n+1}{2m+1})^\lambda]^{\frac{1}{s}-\frac{\varepsilon}{q\lambda}-\frac{1}{\lambda}}}{(\frac{2n+1}{2m+1})^\lambda+1} \\ &= \sum_{m=0}^{\infty}(2m+1)^{-\varepsilon-1}\frac{1}{2m+1}\sum_{n=0}^{\infty}h_m(n,\frac{1}{s}-\frac{\varepsilon}{q\lambda}) \\ &< \frac{1}{2\lambda}B(\frac{1}{s}-\frac{\varepsilon}{q\lambda},\frac{1}{r}+\frac{\varepsilon}{q\lambda})\sum_{m=0}^{\infty}\frac{1}{(2m+1)^{1+\varepsilon}} \end{aligned} \tag{2.11}$$

In view of (2.10) and (2.11), we have

$$\frac{1}{2\lambda}B(\frac{1}{s}-\frac{\varepsilon}{q\lambda},\frac{1}{r}+\frac{\varepsilon}{q\lambda}) > K\{1-[\sum_{m=0}^{\infty}\frac{1}{(2m+1)^{1+\varepsilon}}]^{-1}\sum_{m=0}^{\infty}O(\frac{1}{(2m+1)^{1+\varepsilon+\frac{\lambda}{s}}})\}^{\frac{1}{p}},$$

letting $\varepsilon \to 0^+$, we have $K \le k_\lambda(r)$, then $K = k_\lambda(r)$. That is what the constant factor $k_\lambda(r)$ is the best possible. The theorem is proved. □

Theorem 2.4. *Let* $0 < p < 1$, $r > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{r}+\frac{1}{s}=1$, $0<\lambda\le 1$, *then we have*

(1) *If* $a_n \ge 0$, $0 < \sum_{m=0}^{\infty}(2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p < \infty$, *then*

$$\begin{aligned} &\sum_{n=0}^{\infty}(2n+1)^{\frac{p\lambda}{s}-1}[\sum_{m=0}^{\infty}\frac{a_m}{(2m+1)^\lambda+(2n+1)^\lambda}]^p \\ &\quad > k_\lambda^p(r)\sum_{m=0}^{\infty}[1-\eta_\lambda(m)](2m+1)^{\frac{p\lambda}{s}-1}a_m^p, \end{aligned} \tag{2.12}$$

(2) *If* $b_n \ge 0$, $0 < \sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q < \infty$, *then*

$$\begin{aligned} &\sum_{m=0}^{\infty}[\frac{(2m+1)^{1-p(1-\frac{\lambda}{r})}}{1-\eta_\lambda(m)}]^{q-1}[\sum_{n=0}^{\infty}\frac{b_n}{(2m+1)^\lambda+(2n+1)^\lambda}]^q \\ &\quad < k_\lambda^q(r)\sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q, \end{aligned} \tag{2.13}$$

where, inequalities (2.12) *and* (2.13) *are both equivalent to inequality* (2.8). *The factor* $\eta_\lambda(m)$ *is defined by* (1.20), $k_\lambda(r)$ *is defined by* (1.16). *The constant factors* $k_\lambda^p(r)$ *and* $k_\lambda^q(r)$ *are both the best possible.*

Proof. (1) Letting $b_n := (2n+1)^{\frac{p\lambda}{s}-1}[\sum\limits_{m=0}^{\infty}\frac{a_m}{(2m+1)^\lambda+(2n+1)^\lambda}]^{p-1}$, using the notation (2.1), by (2.9), (1.21), (1.15) and $q<0$, we have

$$\sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q$$

$$= \sum_{n=0}^{\infty}(2n+1)^{\frac{p\lambda}{s}-1}[\sum_{m=0}^{\infty}\frac{a_m}{(2m+1)^\lambda+(2n+1)^\lambda}]^p = H_\lambda(a_m,b_n)$$

$$\geq k_\lambda(r)\{\sum_{m=0}^{\infty}[1-\eta_\lambda(m)](2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}. \tag{2.14}$$

It follows

$$\sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q \geq k_\lambda^p(r)\sum_{m=0}^{\infty}[1-\eta_\lambda(m)](2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p. \tag{2.15}$$

If $\sum\limits_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q = \infty$, by $0 < \sum\limits_{m=0}^{\infty}(2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p < \infty$ and (1.22), (2.15) takes the strict inequality, then inequality (2.12) holds. If $\sum\limits_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q < \infty$, (2.14) is a strict inequality by (2.8), so does (2.15). we have (2.12). And it follows that (2.8) implies (2.12). On the other hand, by (2.12) and $q<0(0<p<1)$, using Hölder$'s$ inequality (1.8), we have

$$\begin{aligned} H_\lambda(a_m,b_n) &= \sum_{n=0}^{\infty}[(2n+1)^{\frac{\lambda}{s}-\frac{1}{p}}\sum_{m=0}^{\infty}\frac{a_m}{(2m+1)^\lambda+(2n+1)^\lambda}][(2n+1)^{\frac{1}{p}-\frac{\lambda}{s}}b_n] \\ &\geq \{\sum_{n=0}^{\infty}(2n+1)^{\frac{p\lambda}{s}-1}[\sum_{m=0}^{\infty}\frac{a_m}{(2m+1)^\lambda+(2n+1)^\lambda}]^p\}^{\frac{1}{p}} \\ &\quad \times\{\sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}. \end{aligned} \tag{2.16}$$

then (2.8) holds . It follows that (2.8) and (2.12) are equivalent.

(2) letting $a_m = [\frac{(2m+1)^{1-p(1-\frac{\lambda}{r})}}{1-\eta_\lambda(m)}\sum\limits_{n=0}^{\infty}\frac{b_n}{(2m+1)^\lambda+(2n+1)^\lambda}]^{q-1}(>0)$, using Hölder$'s$ inequality (1.8), we have

$$\begin{aligned} 0 &< \sum_{m=0}^{\infty}[1-\eta_\lambda(m)](2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p \\ &= \sum_{m=0}^{\infty}[\frac{(2m+1)^{1-p(1-\frac{\lambda}{r})}}{1-\eta_\lambda(m)}]^{q-1}[\sum_{n=0}^{\infty}\frac{b_n}{(2m+1)^\lambda+(2n+1)^\lambda}]^q = H_\lambda(a_m,b_n) \\ &\geq k_\lambda(r)\{\sum_{m=0}^{\infty}[1-\eta_\lambda(m)](2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q\}^{\frac{1}{q}}. \end{aligned} \tag{2.17}$$

In view of $0 < p < 1, q < 0$, we find

$$\sum_{m=0}^{\infty}[1-\eta_\lambda(m)](2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p \le k_\lambda^q(r)\sum_{n=0}^{\infty}(2n+1)^{q(1-\frac{\lambda}{s})-1}b_n^q < \infty. \tag{2.18}$$

By (1.22), it follows

$$0 < \sum_{m=0}^{\infty}(2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p \le \frac{1}{1-\theta_\lambda(r)}\sum_{m=0}^{\infty}[1-\eta_\lambda(m)](2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p < \infty,$$

and shows that (2.8) is valid, hence (2.17) takes the strict forms. So does (2.18). (2.13) holds . So does that (2.8) implies (2.13). On the other hand, by $q < 0(0 < p < 1)$, using the reverse Hölder$'$s inequality (1.8), we have

$$\begin{aligned} &H_\lambda(a_m, b_n) \\ &\quad= \sum_{m=0}^{\infty}\{[\frac{1-\eta_\lambda(m)}{(2m+1)^{1-p(1-\frac{\lambda}{r})}}]^{\frac{1}{p}}a_m\}\{[\frac{(2m+1)^{1-p(1-\frac{\lambda}{r})}}{1-\eta_\lambda(m)}]^{\frac{1}{p}} \\ &\qquad\times\sum_{n=0}^{\infty}\frac{b_n}{(2m+1)^\lambda+(2n+1)^\lambda}\} \\ &\quad\ge \{\sum_{m=0}^{\infty}[1-\eta_\lambda(m)](2m+1)^{p(1-\frac{\lambda}{r})-1}a_m^p\}^{\frac{1}{p}} \qquad (2.19)\\ &\qquad\times\{\sum_{m=0}^{\infty}[\frac{(2m+1)^{1-p(1-\frac{\lambda}{r})}}{1-\eta_\lambda(m)}]^{q-1}[\sum_{n=0}^{\infty}\frac{b_n}{(2m+1)^\lambda+(2n+1)^\lambda}]^q\}^{\frac{1}{q}}. \end{aligned}$$

By (2.13), we have (2.8). It follows that (2.8) is equivalent to (2.13).

If the constant factor $k_\lambda^p(r)$ or $k_\lambda^q(r)$ in (2.12) or (2.13) is not the best possible, by (2.16) or (2.19), we can get a contradiction that the constant factor $k_\lambda(r)$ in (2.8) is not the best possible. The theorem is proved. □

3. A particular case

For $r = s = 2$, $\lambda = 1$, by (2.8), (2.12) and (2.13), we have

Corollary 3.1. *If $0 < p < 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then we have following equivalent inequalities*

(3.1)
$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} > \pi\{\sum_{m=0}^{\infty}[1-\eta_1(m)](2m+1)^{\frac{p}{2}-1}a_m^p\}^{\frac{1}{p}}\{\sum_{n=0}^{\infty}(2n+1)^{\frac{q}{2}-1}b_n^q\}^{\frac{1}{q}},$$

$$\sum_{n=0}^{\infty}(2n+1)^{\frac{p}{2}-1}[\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1}]^p > \pi^p\sum_{m=0}^{\infty}[1-\eta_1(m)](2m+1)^{\frac{p}{2}-1}a_m^p, \tag{3.2}$$

$$\sum_{m=0}^{\infty}[\frac{(2m+1)^{1-\frac{p}{2}}}{1-\eta_1(m)}]^{q-1}[\sum_{n=0}^{\infty}\frac{b_n}{m+n+1}]^q < \pi^q\sum_{n=0}^{\infty}(2n+1)^{\frac{q}{2}-1}b_n^q, \tag{3.3}$$

where the constant factors π, π^p, π^q *are all the best possible. By* (1.20), (1.24) *and* (1.25), *the factor* $\eta_1(m) = \frac{1}{\pi}\{\int_0^{\frac{1}{2m+1}}\frac{y^{-\frac{1}{2}}}{1+y}dy - \frac{(2m+1)^{\frac{1}{2}}}{2(m+1)}\}$ *satisfies inequality:* $\frac{(2m+1)^{\frac{1}{2}}}{2\pi(m+1)} < \eta_1(m) < \frac{2}{\pi(2m+1)^{\frac{1}{2}}}$. *Inequalities* (3.1) *is a reverse forms of* (1.2). (3.2) *and* (3.3) *are both equivalent to* (3.1).

Remark 3.2. For $r = p$, $s = q$, inequality (2.2) and (2.4) reduce to (1.3) and (1.4). It follows that inequalities (2.2) and (2.4) are the generalizations of (1.3) and (1.4), respectively.

References

[1] G. H. Hardy, J. E. Littiewood and G. Pariólya, Inequalities, London: Cambridge University Press, 1952.

[2] D. S. Mintrinovic, J. E. Pecaric and A. M. Fink, Inequalities involving functions and their integrals and derivatives, Boston: Kluwer Academic Publishers, 1991.

[3] B. Yang, *On a generalization of Hilbert's double series theorem,* Mathematical Inequalities and Applications, **5(2)**(2002), 197-204.

[4] M. Gao and B. Yang, *On extended Hilbert's inequality,* Proceedings of the American Mathematical Society, **126(3)**(1998), 751-759, .

[5] I. Brnetic and J. Pecaric, *Generalization of Inequality of Hardy-Hilbert's Type Inequalities,* Mathematical Inequalities and Applications, **7(2)**(2004), 217-225.

[6] W. Zhong, *A reverse Hilbert's type integral inequality,* International Journal of Pure and Applied Mathematics, **36(3)**(2007), 353-360.

[7] W. Zhong and B. Yang, *On the extended form on the reverse Hardy-Hilbert's integral inequalities,* Journal of Southwest China Normal University(Natural Science), **29(4)**(2007), 44-48.

[8] W. Zhong and B. Yang, *On a Multiple Hilbert-type Integral Inequality with the Symmetric Kernel,* Journal of Inequalities and Applications, 2007, Article ID 27962, 1-17.

[9] C. Zhao and L. Debnath, *Some new inverse type integral Hilbert integral inequalities,* Journal Mathematical Analysis and Applications, **245**(2001), 248-265.

[10] B. Yang, *A Dual Hardy-Hilbert's Type Inequality and Generalizations,* Advances in Mathematics, **35(1)**(2006), 102-108.

[11] B. Yang and Y. Zhu, *Inequalities on the Hurwitz Zeta-Function Restricted to the Axis of Positive Reals,* Acta Sunyatseni Universitatis(Natu.Scie.), **36(3)**(1997), 30-35.

[12] J. Kuang, Applied inequalities, Jinan: Shangdong Science and Technology Press, 2004, 5-20.