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On Sums of Products of Horadam Numbers

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ABSTRACT. In this paper we give formulae for sums of products of two Horadam type generalized Fibonacci numbers with the same recurrence equation and with possibly different initial conditions. Analogous improved alternating sums are also studied as well as various derived sums when terms are multiplied either by binomial coefficients or by members of the sequence of natural numbers. These formulae are related to the recent work of Belbachir and Bencherif, Čerin and Čerin and Gianella.

1. Introduction

The generalized Fibonacci sequence $\{w_n\} = \{w_n(a_0, b_0; p, q)\}$ is defined by

$$w_0 = a_0, \quad w_1 = b_0, \quad w_n = p\, w_{n-1} - q\, w_{n-2} \qquad (n \geq 2),$$

where a_0, b_0, p and q are arbitrary complex numbers, with $q \neq 0$. The numbers w_n have been studied by Horadam (see, e.g. [10]). A useful and interesting special cases are $\{U_n\} = \{w_n(0, 1; p, q)\}$ and $\{V_n\} = \{w_n(2, p; p, q)\}$ that were investigated by Lucas [11].

For integers $a \geq 0$, $c \geq 0$, $j \geq 0$, $b > 0$ and $d > 0$, let $P_j = U_{a+bj}U_{c+dj}$, $Q_j = U_{a+b\,j}\, V_{c+d\,j}$ and $R_j = V_{a+b\,j}\, V_{c+d\,j}$. In [1] some formulae for the sums $\sum_{j=0}^{n} P_j$, $\sum_{j=0}^{n} Q_j$, $\sum_{j=0}^{n} R_j$, $\sum_{j=0}^{n} (-1)^j P_j$, $\sum_{j=0}^{n} (-1)^j Q_j$ and $\sum_{j=0}^{n} (-1)^j R_j$ have been discovered in the special case when $b = d = 2$ and $q = \pm 1$. Even in these restricted case they gave unification of earlier results by Čerin and by Čerin and Gianella for Fibonacci, Lucas, Pell and Pell-Lucas numbers (see [3] – [9]).

In [2] the author eliminated all restrictions from the article [1] on b, d and q (except that $q \neq 0$). Some other types of sums have also been studied like the improved alternating sums (when we multiply terms by increasing powers of a fixed complex number), the sums with binomial coefficients and sums in which we multiply terms by increasing natural numbers.

The goal in this paper is to extend these results to Horadam type generalized Fibonacci numbers. Even in this more general case these sums could be evaluated using the sum of a geometric series.

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2. Sums of products of two Horadam numbers

We first want to find the formula for the sum

$$\Psi_1 = \sum_{j=0}^{n} w_{a+b\,j}(a_0,\, b_0;\, p,\, q)\, w_{c+d\,j}(c_0,\, d_0;\, p,\, q)$$

when a_0, b_0, c_0, d_0, p and $q \neq 0$ are complex numbers and $n \geq 0$, $a \geq 0$, $c \geq 0$, $b > 0$ and $d > 0$ are integers.

Let α and β be the roots of $x^2 - p\,x + q = 0$. Then $\alpha = \frac{p+\Delta}{2}$ and $\beta = \frac{p-\Delta}{2}$, where $\Delta = \sqrt{p^2 - 4\,q}$. Moreover, $\alpha - \beta = \Delta$, $\alpha + \beta = p$, $\alpha\,\beta = q$ and the Binet forms of w_n, U_n and V_n are

$$w_n = \frac{(b_0 - a_0\beta)\,\alpha^n + (a_0\alpha - b_0)\,\beta^n}{\alpha - \beta}, \quad U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n,$$

if $\alpha \neq \beta$, and

$$w_n = \alpha^{n-1}(a_0\,\alpha + n\,(b_0 - a_0\,\alpha)), \qquad U_n = n\,\alpha^{n-1}, \qquad V_n = 2\,\alpha^n,$$

if $\alpha = \beta$.

Let $A_1 = b_0 - a_0\,\alpha$, $A_2 = d_0 - c_0\,\alpha$, $B_1 = a_0\,\beta - b_0$, $B_2 = c_0\,\beta - d_0$. Let $E = \alpha^{b+d}$, $F = \alpha^b\,\beta^d$, $G = \alpha^d\,\beta^b$ and $H = \beta^{b+d}$. Let $e = \alpha^{a+c}\,B_1\,B_2$, $f = \alpha^a\,\beta^c\,A_2\,B_1$, $g = \alpha^c\,\beta^a\,A_1\,B_2$ and $h = \beta^{a+c}\,A_1\,A_2$. When $E \neq 1$, for any integer $n \geq 0$, let $E_n = \frac{E^{n+1}-1}{E-1}$. We similarly define F_n, G_n and H_n. On the other hand, when $\alpha^b \neq \beta^b$, for any integer $n \geq 0$, let $b_n = \frac{\alpha^{b(n+1)} - \beta^{b(n+1)}}{\alpha^{b\,n}(\alpha^b - \beta^b)}$ and $b_n^* = \frac{\alpha^{b(n+1)} - \beta^{b(n+1)}}{\beta^{b\,n}(\alpha^b - \beta^b)}$. We similarly define d_n and d_n^*. For any integer $n \geq 0$, let $\lambda_n = n + 1$. Let $T = \alpha^{a+c-2}$.

Let $C_1 = A_1\,a + a_0\,\alpha$, $C_2 = A_2\,c + c_0\,\alpha$, $K_1 = b\,d\,A_1\,A_2$, $K_3 = C_1\,C_2$ and $K_2 = b\,A_1\,C_2 + d\,A_2\,C_1$. Let $K_4 = K_1 + K_2 + K_3$.

Let K, M, N and P be $n\,(2\,n+1)\,K_1 + 3\,n\,K_2 + 6\,K_3$,

$$n^2\,E^{n+3} - (2\,n^2 + 2\,n - 1)\,E^{n+2} + (n+1)^2\,E^{n+1} - E(E+1),$$

$$n\,E^{n+3} - (2\,n+1)\,E^{n+2} + (n+1)\,E^{n+1} + E(E-1),$$

$$E^{n+3} - 2\,E^{n+2} + E^{n+1} - (E-1)^2.$$

Theorem 1. (a) *When* $\Delta = 0$ *and* $E = 1$*, then* $\Psi_1 = \frac{(n+1)\,K\,T}{6}$.

(b) *When* $\Delta = 0$ *and* $E \neq 1$*, then* $\Psi_1 = \frac{T\,[\,K_1\,M + K_2\,N + K_3\,P\,]}{(E-1)^3}$.

Proof. (a) Recall that when $\Delta = 0$, then

$$w_{a+b\,j}(a_0,\, b_0;\, p,\, q) = \alpha^{a-1+b\,j}\,[\,b\,A_1\,j + C_1\,]$$

and

$$w_{c+d\,j}(c_0,\,b_0;\,p,\,q) = \alpha^{c-1+d\,j}\,[\,d\,A_2\,j + C_2\,]\,.$$

Since, $E = \alpha^{b+d} = 1$, we see that the product

$$w_{a+b\,j}(a_0,\,b_0;\,p,\,q)\,w_{c+d\,j}(c_0,\,d_0;\,p,\,q)$$

is equal to

$$T\,\left[\,K_1\,j^2 + K_2\,j + K_3\,\right].$$

From $\sum_{j=0}^{n} 1 = n+1$, $\sum_{j=0}^{n} j = \frac{n(n+1)}{2}$, and $\sum_{j=0}^{n} j^2 = \frac{n(n+1)(2\,n+1)}{6}$, it follows that Ψ_1 has the above value.

(b) Since $\Delta = 0$, the product

$$w_{a+b\,j}(a_0,\,b_0;\,p,\,q)\,w_{c+d\,j}(c_0,\,d_0;\,p,\,q)$$

is equal to

$$T\,E^j\,\left[\,K_1\,j^2 + K_2\,j + K_3\,\right].$$

From $\sum_{j=0}^{n} E^j = \frac{P}{(E-1)^3}$, $\sum_{j=0}^{n} j\,E^j = \frac{N}{(E-1)^3}$, and $\sum_{j=0}^{n} j^2\,E^j = \frac{M}{(E-1)^3}$, it follows that Ψ_1 has the above value. □

The following theorem covers for the sum Ψ_1 the cases when $\Delta \neq 0$. It uses Table 1 that should be read as follows. The symbols ■ and □ in column E mean $E \neq 1$ and $E = 1$. In column b they mean $\alpha^b \neq \beta^b$ and $\alpha^b = \beta^b$. In columns F, G, H and d they have analogous meanings. The third subcase should be read as follows: When $(\Delta \neq 0)$, $E = 1$ and $\alpha^b = \beta^b$, then $G = 1$ and $H = F$ and for $F \neq 1$ the product $\Delta^2\,\Psi_1$ is equal to $\lambda_n\,(e+g) + F_n\,(f+h)$.

Theorem 2. *When $\Delta \neq 0$, then Table 1 gives the value of $\Delta^2\,\Psi_1$. In all other cases the product $\Delta^2\,\Psi_1$ is equal to $\lambda_n\,(e+f+g+h)$.*

Proof of row 1. When $\Delta \neq 0$, we have

$$w_{a+b\,j}(a_0,\,b_0;\,p,\,q) = -\frac{1}{\Delta}\left[\,\alpha^a\,B_1\,(\alpha^b)^j + \beta^a\,A_1\,(\beta^b)^j\,\right]$$

and

$$w_{c+d\,j}(c_0,\,d_0;\,p,\,q) = -\frac{1}{\Delta}\left[\,\alpha^c\,B_2\,(\alpha^d)^j + \beta^c\,A_2\,(\beta^d)^j\,\right].$$

Hence, the product $w_{a+b\,j}(a_0,\,b_0;\,p,\,q)\,w_{c+d\,j}(c_0,\,d_0;\,p,\,q)$ is equal to

$$\frac{e\,E^j}{\Delta^2} + \frac{f\,F^j}{\Delta^2} + \frac{g\,G^j}{\Delta^2} + \frac{h\,H^j}{\Delta^2}.$$

From $\sum_{j=0}^{n} E^j = E_n$, we get $\Delta^2\,\Psi_1 = e\,E_n + f\,F_n + g\,G_n + h\,H_n$. □

	E	F	G	H	b	d	$\Delta^2\,\Psi_1$
1	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$			$E_n\,e + F_n\,f + G_n\,g + H_n\,h$
2	$\square$	$\blacksquare$		$\blacksquare$	$\blacksquare$		$\lambda_n\,e + F_n\,f + b_n\,g + H_n\,h$
3	$\square$	$\blacksquare$	$\boxtimes$	F	$\square$		$\lambda_n(e+g) + F_n(f+h)$
4	$\square$		$\blacksquare$	$\blacksquare$		$\blacksquare$	$\lambda_n\,e + d_n\,f + G_n\,g + H_n\,h$
5	$\square$	$\boxtimes$	$\blacksquare$	G		$\square$	$\lambda_n(e+f) + G_n(g+h)$
6	$\square$	$\square$				$\boxtimes$	(see 5)
7	$\square$		$\square$		$\boxtimes$		(see 3)
8	$\square$		$\square$			$\blacksquare$	$\lambda_n(e+g) + d_n\,(f+h)$
9	$\square$			$\square$		$\blacksquare$	$\lambda_n(e+h) + d_n\,f + d_n^*\,g$
10	$\blacksquare$	$\square$	$\blacksquare$		$\blacksquare$		$E_n\,e + \lambda_n\,f + G_n\,g + b_n\,h$
11	$\blacksquare$	$\square$	E	$\boxtimes$	$\square$		$E_n\,(e+g) + \lambda_n(f+h)$
12		$\square$	$\blacksquare$	$\blacksquare$		$\blacksquare$	$d_n^*\,e + \lambda_n\,f + G_n\,g + H_n\,h$
13	$\boxtimes$	$\square$	$\blacksquare$	G		$\square$	$\lambda_n(e+f) + G_n\,(g+h)$
14		$\square$	$\square$			$\blacksquare$	$d_n^*\,e + \lambda_n(f+g) + d_n\,h$
15		$\square$		$\square$	$\boxtimes$		(see 11)
16		$\square$		$\square$		$\blacksquare$	$d_n^*\,(e+g) + \lambda_n(f+h)$
17		$\blacksquare$	$\square$	$\blacksquare$	$\blacksquare$		$b_n^*\,e + F_n\,f + \lambda_n\,g + H_n\,h$
18	$\boxtimes$	$\blacksquare$	$\square$	F	$\square$		$\lambda_n(e+g) + F_n\,(f+h)$
19	$\blacksquare$	$\blacksquare$	$\square$			$\blacksquare$	$E_n\,e + F_n\,f + \lambda_n\,g + d_n\,h$
20	$\blacksquare$	E	$\square$	$\boxtimes$		$\square$	$E_n(e+f) + \lambda_n(g+h)$
21			$\square$	$\square$	$\blacksquare$	$\boxtimes$	$b_n^*\,(e+f) + \lambda_n(g+h)$
22	$\blacksquare$		$\blacksquare$	$\square$	$\blacksquare$		$E_n\,e + b_n^*\,f + G_n\,g + \lambda_n\,h$
23	$\blacksquare$	$\boxtimes$	E	$\square$	$\square$		$E_n\,(e+g) + \lambda_n(f+h)$
24	$\blacksquare$	$\blacksquare$		$\square$		$\blacksquare$	$E_n\,e + F_n\,f + d_n^*\,g + \lambda_n\,h$
25	$\blacksquare$	E	$\boxtimes$	$\square$		$\square$	$E_n\,(e+f) + \lambda_n(g+h)$

Table 1: The product $\Delta^2\,\Psi_1$ when $\Delta \neq 0$.

Proof of row 2. When $\Delta \neq 0$ and $E = \alpha^{b+d} = 1$, we get

$$w_{a+bj}(a_0, b_0; p, q)\, w_{c+dj}(c_0, d_0; p, q) = \frac{e}{\Delta^2} + \frac{fF^j}{\Delta^2} + \frac{g}{\Delta^2}\left(\frac{\beta^b}{\alpha^b}\right)^j + \frac{hH^j}{\Delta^2}.$$

From $\sum_{j=0}^n 1 = \lambda_n$, $\sum_{j=0}^n F^j = F_n$ and $\sum_{j=0}^n \left(\frac{\beta^b}{\alpha^b}\right)^j = b_n$ (for $\alpha^b \neq \beta^b$), it follows that $\Delta^2\,\Psi_1 = e\,\lambda_n + f\,F_n + g\,b_n + h\,H_n$. □

Proof of row 3. When $\Delta \neq 0$, $E = \alpha^{b+d} = 1$ and $\alpha^b = \beta^b$, then

$$G = \beta^b\,\alpha^d = \alpha^b\,\alpha^d = E = 1$$

and $H = \beta^b\,\beta^d = \alpha^b\,\beta^d = F$. Hence,

$$w_{a+bj}(a_0,\, b_0;\, p,\, q)\, w_{c+dj}(c_0,\, d_0;\, p,\, q) = \frac{e+g}{\Delta^2} + \frac{(f+h)F^j}{\Delta^2}.$$

From $\sum_{j=0}^n 1 = \lambda_n$ and $\sum_{j=0}^n F^j = F_n$ (for $F \neq 1$, of course), it follows that the product $\Delta^2\,\Psi_1$ is equal to $(e+g)\,\lambda_n + (f+h)\,F_n$. □

The missing case in the Table 1 after the third row is clearly when $E = 1$, $\alpha^b = \beta^b$ and $F = 1$. The above product is

$$w_{a+bj}(a_0,\, b_0;\, p,\, q)\, w_{c+dj}(c_0,\, d_0;\, p,\, q) = \frac{e+f+g+h}{\Delta^2},$$

so that $\Delta^2\,\Psi_1 = \lambda_n(e+f+g+h)$. The selection $p = 0$, $q = -1$, $b = 2$ and $d = 2$ shows that this case can actually happen.

Notice that $\alpha^n = \frac{V_n + \Delta\,U_n}{2}$ and $\beta^n = \frac{V_n - \Delta\,U_n}{2}$ for $\Delta \neq 0$ and $\alpha^n = \beta^n = \frac{\widetilde{U}_{n+1}}{n+1} = \frac{\widetilde{V}_n}{2}$ for $\Delta = 0$. Hence, it is clear that each of the above expressions for the sum Ψ_1 could be transformed into an expression in Lucas numbers U_n and V_n (or $\widetilde{U}_n$ and $\widetilde{V}_n$). In most cases these formulae are more complicated then the ones given above. This applies also to other sums that we consider in this paper.

3. Sum with binomial coefficients

In this section we consider the sum

$$\Psi_2 = \sum_{j=0}^{n} \binom{n}{j} w_{a+bj}(a_0,\, b_0;\, p,\, q)\, w_{c+dj}(c_0,\, d_0;\, p,\, q),$$

when a_0, b_0, c_0, d_0, p and $q \neq 0$ are complex numbers and $n \geq 0$, $a \geq 0$, $c \geq 0$, $b > 0$ and $d > 0$ are integers.

Let V and U be $n\,E\ [(n\,E+1)\,K_1 + (E+1)\,K_2] + (E+1)^2\,K_3$ and $E\,K_4 + K_3$.

Theorem 3. (a) *When* $\Delta = 0$*, then*

$$\Psi_2 = \begin{cases} T\,K_3, & \text{if } n = 0, \\ T\,U, & \text{if } n = 1, \\ T\,(E+1)^{n-2}\,V, & \text{if } n \geq 2, \end{cases}$$

(b) *When* $\Delta \neq 0$*, then*

$$\Psi_2 = \frac{(E+1)^n\,e + (F+1)^n\,f + (G+1)^n\,g + (H+1)^n\,h}{\Delta^2}.$$

Proof. (b) Since

$$\binom{n}{j}\,w_{a+b\,j}(a_0,\,b_0;\,p,\,q)\,w_{c+d\,j}(c_0,\,d_0;\,p,\,q) = \binom{n}{j}\,\frac{e\,E^j + f\,F^j + g\,G^j + h\,H^j}{\Delta^2},$$

from $\sum_{j=0}^{n}\binom{n}{j}\,E^j = (E+1)^n$, it follows that Ψ_2 indeed has the above value. □

4. The improved alternating sums, I

In this section we consider the sums obtained from the sums Ψ_1 and Ψ_2 by multiplication of their terms with the powers of a fixed complex number k. When $k = -1$ we obtain the familiar alternating sums. More precisely, we study the sums

$$\Psi_3 = \sum_{j=0}^{n} k^j\,w_{a+b\,j}(a_0,\,b_0;\,p,\,q)\,w_{c+d\,j}(c_0,\,d_0;\,p,\,q),$$

$$\Psi_4 = \sum_{j=0}^{n} k^j\,\binom{n}{j}\,w_{a+b\,j}(a_0,\,b_0;\,p,\,q)\,w_{c+d\,j}(c_0,\,d_0;\,p,\,q),$$

when a_0, b_0, c_0, d_0, p and $q \neq 0$ are complex numbers and $n \geq 0$, $a \geq 0$, $c \geq 0$, $b > 0$ and $d > 0$ are integers.

Let $E = k\,\alpha^{b+d}$, $F = k\,\alpha^b\,\beta^d$, $G = k\,\alpha^d\,\beta^b$ and $H = k\,\beta^{b+d}$. When $E \neq 1$, for any integer $n \geq 0$, let $E_n = \dfrac{E^{n+1}-1}{E-1}$. We similarly define F_n, G_n and H_n.

In this section we can assume that $k \neq 1$ and $k \neq 0$ because the case when $k = 1$ was treated earlier while for $k = 0$ all sums are equal to zero.

With this new meaning of the symbols E, F, G and H we have the following result.

Theorem 4. (a) *The values given in Theorems 1 and 2 express the sum* Ψ_3*. In particular, when* $\Delta \neq 0$*, then the Table 1 gives the values of* $\Delta^2\,\Psi_3$*. In all other cases the product* $\Delta^2\,\Psi_3$ *is equal to* $\lambda_n(e+f+g+h)$*.*
(b) *The values given in Theorem 3 for the sums* Ψ_2 *express also the sum* Ψ_4*.*

Proof. (b) Since

$$k^j \binom{n}{j} w_{a+b\,j}(a_0,\, b_0;\, p,\, q)\, w_{c+d\,j}(c_0,\, d_0;\, p,\, q) = \binom{n}{j} \frac{e\,E^j + f\,F^j + g\,G^j + h\,H^j}{\Delta^2},$$

from $\sum_{j=0}^{n} \binom{n}{j} E^j = (E+1)^n$, it follows that Ψ_4 indeed has the same expression as the sum Ψ_2. □

5. Terms multiplied by natural numbers

In this section we study the sums

$$\Psi_5 = \sum_{j=0}^{n} (j+1)\, w_{a+b\,j}(a_0,\, b_0;\, p,\, q)\, w_{c+d\,j}(c_0,\, d_0;\, p,\, q),$$

$$\Psi_6 = \sum_{j=0}^{n} (j+1) \binom{n}{j} w_{a+b\,j}(a_0,\, b_0;\, p,\, q)\, w_{c+d\,j}(c_0,\, d_0;\, p,\, q),$$

when a_0, b_0, c_0, d_0, p and $q \neq 0$ are complex numbers and $n \geq 0$, $a \geq 0$, $c \geq 0$, $b > 0$ and $d > 0$ are integers.

Let $E = \alpha^{b+d}$, $F = \alpha^b\,\beta^d$, $G = \alpha^d\,\beta^b$, $H = \beta^{b+d}$. Let $e = \alpha^{a+c}\,B_1\,B_2$, $f = \alpha^a\,\beta^c\,A_2\,B_1$, $g = \alpha^c\,\beta^a\,A_1\,B_2$, $h = \beta^{a+c}\,A_1\,A_2$. When $E \neq 1$, for any integer $n \geq 0$, let $E_n = \dfrac{(n+1)E^{n+2} - (n+2)E^{n+1} + 1}{(E-1)^2}$. We similarly define F_n, G_n and H_n. On the other hand, when $\alpha^b \neq \beta^b$, for any integer $n \geq 0$, let

$$b_n = \frac{\alpha^{b(n+2)} + (n+1)\beta^{b(n+2)} - (n+2)\alpha^b\,\beta^{n+1}}{\alpha^{b\,n}(\alpha^b - \beta^b)^2}$$

and

$$b_n^* = \frac{\beta^{b(n+2)} + (n+1)\alpha^{b(n+2)} - (n+2)\beta^b\,\alpha^{n+1}}{\beta^{b\,n}(\alpha^b - \beta^b)^2}.$$

We similarly define d_n and d_n^*. For any integer $n \geq 0$, let $\lambda_n = \dfrac{(n+1)(n+2)}{2}$.

Let M and N denote $n\,E^{n+3}\left[n\,(n+1)\,E - \left(3\,n^2 + 6\,n - 1\right)\right] + (n+2)$ $E^{n+1}\left[\left(3\,n^2 + 3\,n - 2\right)E - (n+1)^2\right] + 2\,E\,(2\,E+1)$ and $(E-1)\left[n\,(n+1)\,E^{n+3} - 2n\,(n+2)\,E^{n+2} + (n+2)\,(n+1)\,E^{n+1} - 2\,E\right]$.

Theorem 5. (a) *When $\Delta = 0$ and $E = 1$, then the sum Ψ_5 is equal to*

$$\frac{\lambda_n\,T\,\left[\,n(3\,n+1)\,K_1 + 4\,n\,K_2 + 6\,K_3\,\right]}{6}.$$

(b) *When $\Delta = 0$ and $E \neq 1$, then the sum Ψ_5 is equal to*

$$\frac{T}{(E-1)^4}\left[\,K_1\,M + K_2\,N + K_3\,(E-1)^4\,E_n\,\right].$$

Proof. (b) Since $\Delta = 0$, we have

$$\begin{aligned}(j+1)\, w_{a+b\,j}(a_0,\, b_0;\, p,\, q)\, w_{c+d\,j}(c_0,\, d_0;\, p,\, q) \\ = (j+1)\left(\alpha^{a+b\,j-1}\,\left[\, b\, A_1\, j + C_1\,\right]\right)\left(\alpha^{c+d\,j-1}\,\left[\, d\, A_2\, j + C_2\,\right]\right) \\ = (j+1)\, T\, E^j\,\left[\, K_1\, j^2 + K_2\, j + K_3\,\right].\end{aligned}$$

From $\sum_{j=0}^{n} (j+1)\, E^j = E_n$, $\sum_{j=0}^{n} j\,(j+1)\, E^j = \dfrac{N}{(E-1)^4}$, and

$$\sum_{j=0}^{n} j^2\,(j+1)\, E^j = \frac{M}{(E-1)^4},$$

it follows that Ψ_5 has the above value. □

Theorem 6. *When $\Delta \neq 0$, then the Table 1 gives the values of $\Delta^2\,\Psi_5$. In all other cases the product $\Delta^2\,\Psi_5$ is equal to $\lambda_n(e+f+g+h)$.*

Proof of row 1 in Table 1 for Ψ_5. When $\Delta \neq 0$, we have

$$\begin{aligned}(j+1)\, w_{a+b\,j}(a_0,\, b_0;\, p,\, q)\, w_{c+d\,j}(c_0,\, d_0;\, p,\, q) \\ = (j+1)\left(-\frac{1}{\Delta}\left[B_1\,\alpha^{a+b\,j} + A_1\,\beta^{a+b\,j}\right]\right)\left(-\frac{1}{\Delta}\left[B_2\,\alpha^{c+d\,j} + A_2\,\beta^{c+d\,j}\right]\right) \\ = (j+1)\left(\frac{e\,E^j}{\Delta^2} + \frac{f\,F^j}{\Delta^2} + \frac{g\,G^j}{\Delta^2} + \frac{h\,H^j}{\Delta^2}\right).\end{aligned}$$

From $\sum_{j=0}^{n} (j+1) E^j = E_n$, we get $\Delta^2\,\Psi_5 = e\,E_n + f F_n + g\,G_n + hH_n$. □

For any integer $n \geq 0$, let $E_n^* = (n+1)\,E + 1$, $E_n^{**} = E_n^*\,(E+1)^{n-1}$. We define F_n^*, G_n^*, H_n^*, F_n^{**}, G_n^{**} and H_n^{**} similarly. Let

$$M = n\,E(E+1)^{n-3}(E_{2n-2}^* + E_n^*\,E_{n-1}^*), \quad N = n\,E(E+1)^{n-2}(E_n^* + 1).$$

Theorem 7. (a) *When $\Delta = 0$, then*

$$\Psi_6 = \begin{cases} T\,K_3, & \text{if } n = 0, \\ T\,[\,2\,E\,K_4 + K_3\,], & \text{if } n = 1, \\ T\,\left[\,3\,E^2(K_4 + K_2 + 3\,K_1) + 4\,E\,K_4 + K_3\,\right], & \text{if } n = 2, \\ T\,[\,M\,K_1 + N\,K_2 + E_n^{**}\,K_3\,], & \text{if } n \geq 3. \end{cases}$$

(b) *When $\Delta \neq 0$, then $\Delta^2\,\Psi_6 = E_n^{**}\,e + F_n^{**}\,f + G_n^{**}\,g + H_n^{**}\,h$.*

Proof. (b) Since

$$\begin{aligned}(j+1)\binom{n}{j}\, w_{a+b\,j}(a_0,\, b_0;\, p,\, q)\, w_{c+d\,j}(c_0,\, d_0;\, p,\, q) \\ = \frac{1}{\Delta^2}\,(j+1)\binom{n}{j}\,\left(\,e\,E^j + f\,F^j + g\,G^j + h\,H^j\right),\end{aligned}$$

from $\sum_{j=0}^{n} (j+1) \binom{n}{j} E^j = E_n^{**}$, it follows that $\Delta^2 \Psi_6$ indeed has the above value. □

6. The improved alternating sums, II

In this section we study the following sums obtained by multiplying the terms of the sums Ψ_5 and Ψ_6 with the powers of the fixed complex number k. Of course, for $k = 1$, we get the sums Ψ_5 and Ψ_6 from the sums Ψ_7 and Ψ_8.

$$\Psi_7 = \sum_{j=0}^{n} k^j \, (j+1) \, w_{a+b\,j}(a_0, \, b_0; \, p, \, q) \, w_{c+d\,j}(c_0, \, d_0; \, p, \, q),$$

$$\Psi_8 = \sum_{j=0}^{n} k^j \, (j+1) \binom{n}{j} w_{a+b\,j}(a_0, \, b_0; \, p, \, q) \, w_{c+d\,j}(c_0, \, d_0; \, p, \, q),$$

when a_0, b_0, c_0, d_0, p and $q \neq 0$ are complex numbers and $n \geq 0$, $a \geq 0$, $c \geq 0$, $b > 0$ and $d > 0$ are integers.

Let $E = k\,\alpha^{b+d}$, $F = k\,\alpha^b\,\beta^d$, $G = k\,\alpha^d\,\beta^b$ and $H = k\,\beta^{b+d}$. When $E \neq 1$, for any integer $n \geq 0$, let $E_n = \dfrac{(n+1)E^{n+2} - (n+2)E^{n+1} + 1}{(E-1)^2}$. We similarly define F_n, G_n and H_n. On the other hand, when $\alpha^b \neq \beta^b$, for any integer $n \geq 0$, we define b_n and b_n^* as in the previous section. We similarly define d_n and d_n^*. In this section λ_n is again $\dfrac{(n+1)(n+2)}{2}$.

Theorem 8. *The expressions for Ψ_5 in Theorems 5 and 6 describe also the sum Ψ_7 (with the new meaning of E, F, G and H).*

Theorem 9. *The expressions for Ψ_6 in Theorem 7 describe also the sum Ψ_8 (with the new meaning of E, F, G and H).*

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