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Entire Functions and Their Derivatives Share Two Finite Sets

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Abstract. In this paper, we study the uniqueness of entire functions and prove the following theorem. Let $n(\geq 5)$, k be positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \cdots, a_m\}$, where $a_1, a_2, \cdots, a_m$ are distinct nonzero constants. If two nonconstant entire functions f and g satisfy $E_f(S_1, 2) = E_g(S_1, 2)$ and $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$, then one of the following cases must occur: (1) $f = tg$, $\{a_1, a_2, \cdots, a_m\} = t\{a_1, a_2, \cdots, a_m\}$, where t is a constant satisfying $t^n = 1$; (2) $f(z) = de^{cz}$, $g(z) = \frac{t}{d}e^{-cz}$, $\{a_1, a_2, \cdots, a_m\} = (-1)^k c^{2k} t\{\frac{1}{a_1}, \cdots, \frac{1}{a_m}\}$, where t, c, d are nonzero constants and $t^n = 1$. The results in this paper improve the result given by Fang (M.L. Fang, Entire functions and their derivatives share two finite sets, Bull. Malaysian Math. Sc. Soc. 24(2001), 7-16).

1. Introduction, definitions and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane C. If for some $a \in C \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities). We assume that the reader is familiar with the notations of Nevanlinna theory that can be found, for instance, in [5] or [9].

Let S be a set of distinct elements of $C \cup \{\infty\}$ and $E_f(S) = \cup_{a \in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\cup_{a \in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. Let m be a positive integer or infinity and $a \in C \cup \{\infty\}$. We denote by $E_{m)}(a, f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. For a

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set S of distinct elements of C we define $E_{m)}(S,f)=\cup_{a\in S}E_{m)}(a,f)$. If for some $a\in C\cup\{\infty\}$, $E_{\infty)}(a,f)=E_{\infty)}(a,g)$, we say that f and g share the value a CM. We can define $\overline{E}_{m)}(a,f)$ and $\overline{E}_{m)}(S,f)$ similarly.

In 1977, Gross [4]posed the following question.

Question. Can one find two finite sets $S_j(j=1,2)$ such that any two nonconstant entire functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1,2$ must be identical ?

Yi [10] gave a positive answer to the question. He proved.

Theorem A([10]). *Let f and g be two nonconstant entire functions, $n\geq 5$ a positive integer, and let $S_1=\{z:z^n=1\}$, $S_2=\{a\}$, where $a\neq 0$ is a constant satisfying $a^{2n}\neq 1$. If $E_f(S_j)=E_g(S_j)$ for $j=1,2$, then $f\equiv g$.*

In 2001, Fang [3] investigated the question and proved the following theorems.

Theorem B([3]). *Let f and g be two nonconstant entire functions, $n\geq 5$, k two positive integers, and let $S_1=\{z:z^n=1\}$, $S_2=\{a,b,c\}$, where a, b, c are nonzero finite distinct constants satisfying $a^2\neq bc$, $b^2\neq ac$, $c^2\neq ab$. If $E_f(S_1)=E_g(S_1)$ and $E_{f^{(k)}}(S_2)=E_{g^{(k)}}(S_2)$, then $f\equiv g$.*

Theorem C([3]). *Let f and g be two nonconstant entire functions, $n\geq 5$, k two positive integers, and let $S_1=\{z:z^n=1\}$, $S_2=\{a,b\}$, where a, b are two nonzero finite distinct constants. If $E_f(S_1)=E_g(S_1)$ and $E_{f^{(k)}}(S_2)=E_{g^{(k)}}(S_2)$, then one of the following cases must occur:* (1) *$f\equiv g$;* (2) *$b=-a$, $f=e^{cz+d}$, $g=te^{-cz-d}$, where c, d, t are three constants satisfying $t^n=1$ and $(-1)^k tc^{2k}=a^2$;* (3) *$f=e^{cz+d}$, $g=te^{-cz-d}$, where c, d, t are three constants satisfying $t^n=1$ and $(-1)^k tc^{2k}=ab$;* (4) *$b=-a$, $f\equiv -g$.*

Theorem D([3]). *Let f and g be two nonconstant entire functions, $n\geq 5$, k two positive integers, and let $S_1=\{z:z^n=1\}$, $S_2=\{a\}$, where $a\neq 0,\infty$. If $E_f(S_1)=E_g(S_1)$ and $E_{f^{(k)}}(S_2)=E_{g^{(k)}}(S_2)$, then one of the following cases must occur:* (1) *$f\equiv g$;* (2) *$f=e^{cz+d}$, $g=te^{-cz-d}$, where c, d, t are three constants satisfying $t^n=1$ and $(-1)^k tc^{2k}=a^2$.*

In this paper, we consider the more general sets $S_1=\{z:z^n=1\}$, $S_2=\{a_1,a_2,\cdots,a_m\}$, where $a_1,a_2,\cdots,a_m$ are distinct nonzero constants. To state the main results of this paper, we require the following notion of weighted sharing which was introduced by I. Lahiri [6], [7].

Definition 1([6]). For a complex number $a\in C\cup\{\infty\}$, we denote by $E_k(a,f)$ the set of all a-points of f where an a-point with mutiplicity m is counted m times if $m\leq k$ and $k+1$ times if $m>k$. For a complex number $a\in C\cup\{\infty\}$, such that $E_k(a,f)=E_k(a,g)$, then we say that f and g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f-a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g-a$ with multiplicity

$m(\leq k)$ and z_0 is a zero of $f-a$ with multiplicity $m(>k)$ if and only if it is a zero of $g-a$ with multiplicity $n(>k)$, where m is not necessarily equal to n. We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 2([6]). Let S be a set of distinct elements of $C \cup \{\infty\}$ and k a nonnegative integer or ∞. We denote by $E_f(S,k)$ the set $\cup_{a\in S} E_k(a,f)$. Clearly $E_f(S) = E_f(S,\infty)$ and $\overline{E}_f(S) = E_f(S,0)$.

With the notion of weighted sharing of sets we prove the following results which improve Theorem B, Theorem C and Theorem D.

Theorem 1. *Let $n(\geq 5)$, k be positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \cdots, a_m\}$, where $a_1, a_2, \cdots, a_m$ are distinct nonzero constants. If two nonconstant entire functions f and g satisfy $E_f(S_1, 2) = E_g(S_1, 2)$ and $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$, then one of the following cases must occur:* (1) $f = tg$, $\{a_1, a_2, \cdots, a_m\} = t\{a_1, a_2, \cdots, a_m\}$, *where t is a constant satisfying $t^n = 1$;* (2) $f(z) = de^{cz}$, $g(z) = \frac{t}{d}e^{-cz}$, $\{a_1, a_2, \cdots, a_m\} = (-1)^k c^{2k} t\{\frac{1}{a_1}, \cdots, \frac{1}{a_m}\}$, *where t, c, d are nonzero constants and $t^n = 1$.*

Theorem 2. *Let $n(\geq 5)$, k be positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \cdots, a_m\}$, where $a_1, a_2, \cdots, a_m$ are distinct nonzero constants. If two nonconstant entire functions f and g satisfy $E_f(S_1, 1) = E_g(S_1, 1)$ and $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$, then one of the following cases must occur:* (1) $f = tg$, $\{a_1, a_2, \cdots, a_m\} = t\{a_1, a_2, \cdots, a_m\}$, *where t is a constant satisfying $t^n = 1$;* (2) $f(z) = de^{cz}$, $g(z) = \frac{t}{d}e^{-cz}$, $\{a_1, a_2, \cdots, a_m\} = (-1)^k c^{2k} t\{\frac{1}{a_1}, \cdots, \frac{1}{a_m}\}$, *where t, c, d are nonzero constants and $t^n = 1$.*

Theorem 3. *Let $n(\geq 8)$, k be positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \cdots, a_m\}$, where $a_1, a_2, \cdots, a_m$ are distinct nonzero constants. If two nonconstant entire functions f and g satisfy $E_f(S_1, 0) = E_g(S_1, 0)$ and $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$, then one of the following cases must occur:* (1) $f = tg$, $\{a_1, a_2, \cdots, a_m\} = t\{a_1, a_2, \cdots, a_m\}$, *where t is a constant satisfying $t^n = 1$;* (2) $f(z) = de^{cz}$, $g(z) = \frac{t}{d}e^{-cz}$, $\{a_1, a_2, \cdots, a_m\} = (-1)^k c^{2k} t\{\frac{1}{a_1}, \cdots, \frac{1}{a_m}\}$, *where t, c, d are nonzero constants and $t^n = 1$.*

Without the notion of weighted sharing of sets we prove the following theorem which also improves Theorem B, Theorem C and Theorem D.

Theorem 4. *Let $n(\geq 5)$, k be positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \cdots, a_m\}$, where $a_1, a_2, \cdots, a_m$ are distinct nonzero constants. If two nonconstant entire functions f and g satisfy $\overline{E}_{4)}(S_1, f) = \overline{E}_{4)}(S_1, g)$, $E_{2)}(S_1, f) = E_{2)}(S_1, g)$ and $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$, then one of the following cases must occur:* (1) $f = tg$, $\{a_1, a_2, \cdots, a_m\} = t\{a_1, a_2, \cdots, a_m\}$, *where t is a con-*

stant satisfying $t^n = 1$*;* (2) $f(z) = de^{cz}$, $g(z) = \frac{t}{d}e^{-cz}$, $\{a_1, a_2, \cdots, a_m\} = (-1)^k c^{2k} t\{\frac{1}{a_1}, \cdots, \frac{1}{a_m}\}$, *where* t, c, d *are nonzero constants and* $t^n = 1$.

2. Some lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 1([8]). *Let* f *be a nonconstant meromorphic function, and let* $a_0, a_1, a_2, \cdots, a_n$ *be finite complex numbers,* $a_n \neq 0$*. Then*

$$T(r, a_n f^n + \cdots + a_2 f^2 + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2([7]). *Let* H *be defined as above. If* F *and* G *share* $(1, 2)$ *and* $H \not\equiv 0$*, then*

$$T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + N_2(r, F) + N_2(r, G) + S(r, F) + S(r, G),$$

the same inequality holds for $T(r, G)$*.*

Lemma 3([2]). *Let* H *be defined as above. If* F *and* G *share* $(1, 1)$ *and* $H \not\equiv 0$*, then*

$$\begin{aligned} T(r, F) \leq\ & N_2(r, \frac{1}{F}) + N_2(r, F) + N_2(r, \frac{1}{G}) + N_2(r, G) \\ & + \frac{1}{2}\overline{N}(r, \frac{1}{F}) + \frac{1}{2}\overline{N}(r, F) + S(r, F) + S(r, G), \end{aligned}$$

the same inequality holds for $T(r, G)$*.*

Lemma 4([11]). *Let* H *be defined as above. If* $H \equiv 0$ *and*

$$\limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, F) + \overline{N}(r, G)}{T(r)} < 1, r \in I,$$

where I *is a set with infinite linear measure and* $T(r) = \max\{T(r, F), T(r, G)\}$*, then* $FG \equiv 1$ *or* $F \equiv G$*.*

Lemma 5([2]). *Let* F*,* G *be two nonconstant meromorphic functions such that* *they share* $(1,0)$*, and* $H \not\equiv 0$*. Then*

$$T(r,F) \le N_2(r,\frac{1}{F}) + N_2(r,F) + N_2(r,\frac{1}{G}) + N_2(r,G) + 2\overline{N}(r,\frac{1}{F}) + 2\overline{N}(r,F) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,G) + S(r,F) + S(r,G)\,,$$

the same inequality holds for $T(r,G)$*.*

Lemma 6([1]). *Let* F*,* G *be two nonconstant meromorphic functions such that* $\overline{E}_{4)}(1,F) = \overline{E}_{4)}(1,G)$ *and* $E_{2)}(1,F) = E_{2)}(1,G)$*, then one of the following cases holds* $(1) T(r,F)+T(r,G) \le 2\{N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+N_2(r,F)+N_2(r,G)\}+S(r,F)+S(r,G)$ *;* $(2) F \equiv G$*;* $(3) FG \equiv 1$*.*

Lemma 7([5]). *Let* f *be a nonconstant meromorphic function,* n *be a positive integer, and let* Ψ *be a function of the form* $\Psi = f^n + Q$*, where* Q *is a differential polynomial of* f *with degree* $\le n-1$*. If*

$$N(r,f) + N\left(r,\frac{1}{\Psi}\right) = S(r,f)\,,$$

then $\Psi = (f+\alpha)^n$*, where* α *is a meromorphic function with* $T(r,\alpha) = S(r,f)$*, determined by the term of degree* $n-1$ *in* Q*.*

3. Proof of theorem 1

Set $F = f^n$, $G = g^n$. From $E_f(S_1,2) = E_g(S_1,2)$, we deduce F and G share $(1,2)$. By Lemma 1, we have

$$T(r,F) = nT(r,f) + S(r,f), \qquad T(r,G) = nT(r,g) + S(r,g)\,. \tag{1}$$

Assume $H \not\equiv 0$. By Lemma 2, we have

$$\begin{aligned} T(r,F) &= nT(r,f) + S(r,f) \\ &\le N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + S(r,F) + S(r,G) \\ &\le 2T(r,f) + 2T(r,g) + S(r,f) + S(r,g)\,. \end{aligned} \tag{2}$$

Similarly, we have

$$\begin{aligned} T(r,G) &= nT(r,g) + S(r,f) \\ &\le N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + S(r,F) + S(r,G) \\ &\le 2T(r,f) + 2T(r,g) + S(r,f) + S(r,g)\,. \end{aligned} \tag{3}$$

Combining (2) and (3) together we have

$$(n-4)T(r,f) + (n-4)T(r,g) \le S(r,f) + S(r,g)\,, \tag{4}$$

which contradicts $n \geq 5$. Thus $H \equiv 0$. By Lemma 4, we have $FG \equiv 1$ or $F \equiv G$, that is $f = tg$ or $fg = t$ where t is a constant and $t^n = 1$. Next we consider the following two cases:

Case 1. $f = tg$. Then $f^{(k)} = tg^{(k)}$. By $E_{f^{(k)}}(S_2, \infty) = E_{g^{(k)}}(S_2, \infty)$, we get $\{a_1, a_2, \cdots, a_m\} = t\{a_1, a_2, \cdots, a_m\}$.

Case 2. $fg = t$. Then there exists an entire function h such that $f = e^h$ and $g = te^{-h}$. Therefore

$$f^{(i)} = \alpha_i f, g^{(i)} = \beta_i g, i = 1, 2, \cdots, \tag{5}$$

where $\alpha_1 = h'$, $\beta_1 = -h'$, and α_i, β_i satisfy the following recurrence formulas, respectively.

$$\alpha_{i+1} = \alpha_i' + \alpha_i^2, \beta_{i+1} = \beta_i' + \beta_i^2, i = 1, 2, \cdots, \tag{6}$$

Without loss of the generality, we assume that a_1 is not an exceptional value of $f^{(k)}$. Suppose $f^{(k)}(z_0) = a_1$. Then $\frac{t}{a_1}\alpha_k(z_0)\beta_k(z_0) = g^{(k)}(z_0) \in S_2$. Therefore,

$$\prod_{j=1}^{m}(\frac{t}{a_1}\alpha_k(z_0)\beta_k(z_0) - a_j) = 0\,. \tag{7}$$

Note that $\overline{N}(r, 1/(f^{(k)} - a_1)) \neq S(r, f)$. We get

$$\prod_{j=1}^{m}(\frac{t}{a_1}\alpha_k\beta_k - a_j) = 0\,, \tag{8}$$

which implies that $\alpha_k\beta_k$ is a nonzero constant. And thus α_k and β_k have no zeros. The recurrence formulas in (6) show that

$$\alpha_k = \alpha_1^k + P(\alpha_1), \beta_k = \beta_1^k + Q(\beta_1)\,, \tag{9}$$

where $P(\alpha_1)$ is a differential polynomial in α_1 of degree $k-1$, and $Q(\beta_1)$ is a differential polynomial in β_1 of degree $k-1$. If α_1 and β_1 are not constants, then by Lemma 7, we have

$$\alpha_k = \left(\alpha_1 + \frac{\gamma_1}{k}\right)^k, \beta_k = \left(\beta_1 + \frac{\gamma_2}{k}\right)^k\,, \tag{10}$$

where γ_1, γ_2 are small functions of α_1 and β_1, respectively. Note that $\alpha_1 = -\beta_1 = h'$. We conclude that $\alpha_k\beta_k$ can not be constant, which is a contradiction. Hence one of α_1 and β_1 is constant. Thus h is a linear function. Therefore, $f(z) = de^{cz}$ and $g(z) = \frac{t}{d}e^{-cz}$, where c, d are nonzero constants. Now from $E_{f^{(k)}}(S_2, \infty) =$

$E_{g^{(k)}}(S_2,\infty)$, we get $\{a_1,a_2,\cdots,a_m\}=(-1)^kc^{2k}t\{\frac{1}{a_1},\cdots,\frac{1}{a_m}\}$, which completes the proof of Theorem 1.

4. Proof of theorem 2

Set $F=f^n$, $G=g^n$. From $E_f(S_1,1)=E_g(S_1,1)$, we deduce F and G share $(1,1)$. By Lemma 1, we have

$$T(r,F)=nT(r,f)+S(r,f),\qquad T(r,G)=nT(r,g)+S(r,g)\,. \tag{11}$$

Assume $H\not\equiv 0$. By Lemma 3, we have

$$\begin{aligned} T(r,F)&=nT(r,f)+S(r,f)\\ &\le N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+\frac{1}{2}\overline{N}(r,\frac{1}{F})+S(r,F)+S(r,G)\\ &\le \frac{5}{2}T(r,f)+2T(r,g)+S(r,f)+S(r,g)\,. \end{aligned} \tag{12}$$

Similarly, we have

$$\begin{aligned} T(r,G)&=nT(r,g)+S(r,g)\\ &\le N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+\frac{1}{2}\overline{N}(r,\frac{1}{G})+S(r,F)+S(r,G)\\ &\le 2T(r,f)+\frac{5}{2}T(r,g)+S(r,f)+S(r,g)\,. \end{aligned} \tag{13}$$

Combining (12) and (13) together we have

$$(n-\frac{9}{2})T(r,f)+(n-\frac{9}{2})T(r,g)\le S(r,f)+S(r,g)\,, \tag{14}$$

which contradicts $n\ge 5$. Thus $H\equiv 0$. By Lemma 4, we have $FG\equiv 1$ or $F\equiv G$, that is $f=tg$ or $fg=t$ where t is a constant and $t^n=1$. Proceeding as in the proof of Theorem 1, we get the conclusion of Theorem 2. This completes the proof of Theorem 2.

5. Proof of theorem 3

Set $F=f^n$, $G=g^n$. From $E_f(S_1,0)=E_g(S_1,0)$, we deduce F and G share $(1,0)$. By Lemma 1, we have

$$T(r,F)=nT(r,f)+S(r,f),\qquad T(r,G)=nT(r,g)+S(r,g)\,. \tag{15}$$

Assume $H \not\equiv 0$. By Lemma 5, we have

$$\begin{aligned}(16)\quad T(r,F) &= nT(r,f) + S(r,f)\\ &\le N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + 2\overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + S(r,F) + S(r,G)\\ &\le 4T(r,f) + 3T(r,g) + S(r,f) + S(r,g)\,.\end{aligned}$$

Similarly, we have

$$\begin{aligned}(17)\qquad T(r,G) &= nT(r,g) + S(r,g)\\ &\le 3T(r,f) + 4T(r,g) + S(r,f) + S(r,g)\,.\end{aligned}$$

Combining (16) and (17) together we have

$$(18)\qquad (n-7)T(r,f) + (n-7)T(r,g) \le S(r,f) + S(r,g)\,,$$

which contradicts $n \ge 8$. Thus $H \equiv 0$. By Lemma 4, we have $FG \equiv 1$ or $F \equiv G$, that is $f = tg$ or $fg = t$ where t is a constant and $t^n = 1$. Proceeding as in the proof of Theorem 1, we get the conclusion of Theorem 3. This completes the proof of Theorem 3.

6. Proof of theorem 4

Set $F = f^n$, $G = g^n$. By Lemma 1, we have

$$(19)\qquad T(r,F) = nT(r,f) + S(r,f), \qquad T(r,G) = nT(r,g) + S(r,g)\,.$$

From $\overline{E}_{4)}(S_1,f) = \overline{E}_{4)}(S_1,g)$, $E_{2)}(S_1,f) = E_{2)}(S_1,g)$, we deduce $\overline{E}_{4)}(1,F) = \overline{E}_{4)}(1,G)$, $E_{2)}(1,F) = E_{2)}(1,G)$. Then F and G satisfy the condition of Lemma 6. We assume Case (1) in Lemma 6 holds, that is,

$$\begin{aligned}(20)\qquad T(r,F) + T(r,G) &\le 2\{N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G})\} + S(r,F) + S(r,G)\\ &\le 4T(r,f) + 4T(r,g) + S(r,f) + S(r,g).\end{aligned}$$

Combining (19) and (20) together we have

$$(21)\qquad (n-4)T(r,f) + (n-4)T(r,g) \le S(r,f) + S(r,g)\,,$$

which contradicts $n \ge 5$. Thus by Lemma 6, we get $F \equiv G$ or $FG \equiv 1$, that is, $f = tg$ or $fg = t$ where t is a constant and $t^n = 1$. Proceeding as in the proof of Theorem 1, we get the conclusion of Theorem 4. This completes the proof of Theorem 4.

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