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E-Inversive Γ-Semigroups

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Abstract. Let $S = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two nonempty sets. S is called a Γ-semigroup if $a\alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. An element $e \in S$ is said to be an α-idempotent for some $\alpha \in \Gamma$ if $e\alpha e = e$. A Γ-semigroup S is called an E-inversive Γ-semigroup if for each $a \in S$ there exist $x \in S$ and $\alpha \in \Gamma$ such that $a\alpha x$ is a β-idempotent for some $\beta \in \Gamma$. A Γ-semigroup is called a right E- Γ-semigroup if for each α-idempotent e and β-idempotent f, $e\alpha f$ is a β-idempotent. In this paper we investigate different properties of E-inversive Γ-semigroup and right E-Γ-semigroup.

1. Introduction

Let S be a semigroup. According to Catino and Miccoli [1] S is E-inversive if for every $a \in S$ there exists $x \in S$ such that ax is idempotent. They proved that S is E-inversive if and only if $W(a) \neq \phi$ for all $a \in S$ where $W(a) = \{x \in S : xax = x\}$. The elements of $W(a)$ are called weak inverse element of a. S is E-semigroup if the set $E(S)$ of idempotents of S forms a subsemigroup. Basic properties of E-inversive semigroup and E-semigroups are studied by Catino and Miccoli [1], Mitsch [4], Weipoltshammer [9]. In this paper we introduce this notion in Γ-semigroup and study the structures. We now recall some definitions and results of Γ-semigroups.

Definition 1.1. Let $S = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two nonempty sets. S is called a Γ-*semigroup*, if

(i) $a\alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and

(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

S is said to be a Γ-*semigroup with zero* if there exists an element $0 \in S$ such that $0\alpha a = a\alpha 0 = 0$ for all $\alpha \in \Gamma$.

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Let S be an arbitrary semigroup. Let 1 be a symbol not representing any element of S. Let us extend the binary operation defined on S to $S \cup \{1\}$ by defining $11 = 1$ and $1a = a1$ for all $a \in S$. It can be shown that $S \cup \{1\}$ is a semigroup with identity element 1. Let $\Gamma = \{1\}$. If we take $ab = a1b$, it can be shown that the semigroup S is a $\Gamma-$semigroup where $\Gamma = \{1\}$. Thus a semigroup can be considered to be a Γ-semigroup.

Let S be a Γ-semigroup and x be a fixed element of Γ. We define $a.b = axb$ for all $a, b \in S$. We can show that $(S, .)$ is a semigroup and we denote this semigroup by S_x.

Definition 1.2 ([7]). Let S be a Γ-semigroup. An element $a \in S$ is said to be *regular*, if $a \in a\Gamma S\Gamma a$, where $a\Gamma S\Gamma a = \{a\alpha b\beta a : b \in S, \alpha, \beta \in \Gamma\}$. S is said to be *regular* if every element of S is regular. We now describe some examples of regular Γ-semigroup.

Example 1.3. Let S be the set of all 3×2 matrices and Γ be the set of all 2×3 matrices over a field. Then for $A, B \in S$, the product AB can not be defined i.e., S is not a semigroup under the usual matrix multiplication. But for all $A, B, C \in S$ and $P, Q \in \Gamma$ we have $APB \in S$ and since the matrix multiplication is associative, we have $(APB)QC = AP(BQC)$. Hence S is a Γ-semigroup. Moreover it is regular shown in [7].

Example 1.4. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. S denotes the set of all mappings from A to B. Here members of S will be described by the images of the elements 1, 2, 3. For example the map $1 \to 4$, $2 \to 5$, $3 \to 4$ will be written as $(4, 5, 4)$ and $(5, 5, 4)$ denotes the map $1 \to 5$, $2 \to 5$, $3 \to 4$. A map from B to A will be described in the same fashion. For example $(1, 2)$ denotes $4 \to 1$, $5 \to 2$. Now $S = \Big\{(4,4,4), (4,4,5), (4,5,4), (4,5,5), (5,5,5), (5,4,5), (5,4,4), (5,5,4)\Big\}$ and let $\Gamma = \{(1,1), (1,2), (2,3), (3,1)\}$. Let $f, g \in S$ and $\alpha \in \Gamma$. We define $f\alpha g$ by $(f\alpha g)(a) = f\alpha\Big(g(a)\Big)$ for all $a \in A$. So $f\alpha g$ is a mapping from A to B and hence $f\alpha g \in S$ and we can show that $(f\alpha g)\beta h = f\alpha(g\beta h)$ for all $f, g, h \in S$ and $\alpha, \beta \in \Gamma$. We can show that each element x of S is an α-idempotent for an $\alpha \in \Gamma$ and hence each element is regular. Thus S is a regular Γ-semigroup.

Example 1.5. Let T be a semigroup, I, Λ be two index sets and Γ be the collection of some $\Lambda \times I$ matrices over T. Then the set $S = I \times T \times \Lambda$ is a Γ-semigroup with respect to the multiplication $(i, a, \lambda)P(j, b, \mu) = (i, ap_{\lambda j}b, \mu)$ for $(i, a, \lambda), (j, b, \mu) \in S$ and $P = (p_{\lambda i}) \in \Gamma$. This Γ-semigroup is called the Rees matrix Γ-semigroup over T with the set Γ of sandwich matrices and it is denoted by $S = \mathcal{M}(I, T, \Lambda, \Gamma)$. Let T^0 denote the semigroup T with a zero element adjoint. Let Γ be a set of some $\Lambda \times I$ matrices over T^0. Then the set $S = (I \times T \times \Lambda) \cup \{0\}$ is a Γ-semigroup with respect to the multiplication

$$(i, a, \lambda)P(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu)\ , & \text{if} \quad p_{\lambda j} \neq 0 \\ 0\ , & \text{if} \quad p_{\lambda j} = 0 \end{cases}$$

and $0\Gamma(i,a,\lambda) = (i,a,\lambda)\Gamma 0 = 0\Gamma 0 = \{0\}$ for all $(i,a,\lambda), (j,b,\mu) \in S$ and $P = (p_{\lambda i}) \in \Gamma$. This Γ-semigroup is called the Rees matrix Γ-semigroup over T^0 with the set Γ of sandwich matrices and we denote it by $\mathcal{M}^0(I,T,\Lambda,\Gamma)$.

In [8] author studied Rees matrix Γ-semigroup over a group.

Definition 1.6 ([8]). The set Γ of sandwich matrices is called *regular*, if for each $i \in I$ there exists a matrix $P \in \Gamma$ and for each $\lambda \in \Lambda$ there exists a matrix $Q \in \Gamma$ such that P has at least one nonzero entry in the i-th column and Q has at least one nonzero entry in the λ-th row.

Theorem 1.7 ([8]). *Rees $I \times \Lambda$ matrix Γ-semigroup $\mathcal{M}^0(G,I,\Lambda,\Gamma)$ over G^0, a group with zero is regular if and only if Γ is regular.*

Definition 1.8 ([7]). Let S be a Γ-semigroup and $\alpha \in \Gamma$. Then $e \in S$ is said to be an *α-idempotent*, if $e\alpha e = e$. The set of all α-idempotents is denoted by E_α and we denote $\bigcup_{\alpha \in \Gamma} E_\alpha$ by $E(S)$. The elements of $E(S)$ are called idempotent element of S.

Definition 1.9 ([7]). Let S be a Γ-semigroup and $a, b \in S$, $\alpha, \beta \in \Gamma$. b is said to be an (α,β)-*inverse* of a, if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$. This is denoted by $b \in V_\alpha^\beta(a)$.

Definition 1.10 ([7]). A nonempty subset I of a Γ-semigroup S is called an Γ-*ideal*, if $I\Gamma S \subseteq I$ and $S\Gamma I \subseteq I$ where for subsets U, V of S and Γ_1 of Γ, $U\Gamma_1 V = \{u\alpha v : u \in U, v \in V, \alpha \in \Gamma_1\}$.

In a Γ-semigroup S, the Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{J}$ on S are defined as follows:

Definition 1.11 ([5]). Let S be a Γ-*semigroup.* For $a, b \in S$,

$a\mathcal{L}b$ if $S\Gamma a \cup \{a\} = S\Gamma b \cup \{b\}$,

$a\mathcal{R}b$ if $a\Gamma S \cup \{a\} = b\Gamma S \cup \{b\}$,

$a\mathcal{H}b$ if $a\mathcal{L}b$ and $a\mathcal{R}b$,

$a\mathcal{D}b$ if $a\mathcal{L}c$ and $c\mathcal{R}b$ for some $c \in S$ and

$a\mathcal{J}b$ if $a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S \cup \{a\} = b\Gamma S \cup S\Gamma b \cup S\Gamma b\Gamma S \cup \{b\}$.

Theorem 1.12 ([5]). *Let S be a Γ-semigroup and $a \in S$. Let D_a denote the $\mathcal{D}$-class of S containing a. If a is regular, then every element of D_a is regular.*

2. E-inversive Γ-semigroup

We see that the Γ-semigroup given in example 1.4 is regular. We now take the same set S and modify Γ as $\Gamma = \{(1,1),(1,2)\}$. Then S is a Γ-semigroup under the same operation defined in the example but the elements $(4,4,5)$ and $(5,5,4)$ are not regular. Thus S is not a regular Γ-semigroup. But in this example we see that for each element $a \in S$ there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x \in E_\beta$. In this section we study such type of Γ-semigroups.

Definition 2.1. Let S be a Γ-semigroup. An element $a \in S$ is called *E-inversive*, if there exist $x \in S, \alpha, \beta \in \Gamma$ such that $a\alpha x \in E_\beta$. S is called *E-inversive* Γ-semigroup, if every $a \in S$ is E-inversive.

Let us take an E-inversive element $a \in S$ and let $a\alpha x \in E_\beta$ for some $x \in S$ and $\alpha, \beta \in \Gamma$. If we take $y = x\beta a\alpha x$ then $a\alpha y = a\alpha x\beta a\alpha x = a\alpha x \in E_\beta$ and $(y\beta a)\alpha(y\beta a) = x\beta a\alpha x\beta a\alpha x\beta a\alpha x\beta a = x\beta(a\alpha x\beta a\alpha x\beta a\alpha x)\beta a = x\beta(a\alpha x)\beta a = (x\beta a\alpha x)\beta a = y\beta a$. Hence $y\beta a \in E_\alpha$. Hence we see that if a is an E-inversive element of S then there exist $y \in S$ and $\alpha,\ \beta \in \Gamma$ such that $a\alpha y \in E_\beta$ and $y\beta a \in E_\alpha$.

Definition 2.2. Let S be a Γ-semigroup with zero. A nonzero element $a \in S$ is called *E^*-inversive*, if there exist $x \in S, \alpha, \beta \in \Gamma$ such that $0 \neq a\alpha x \in E_\beta$. S is called E^*-inversive Γ-semigroup if every nonzero element $a \in S$ is E^*-inversive.

Example 2.3. Let $I = \{1, 2\}$ and $\Lambda = \{1, 2, 3\}$ be two index sets. Let us consider the group $G = \{1, w, w^2\}$ and let $\Gamma = \left\{ \begin{pmatrix} 0 & 0 \\ w & 0 \\ 1 & w^2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ w^2 & w \\ 0 & 1 \end{pmatrix} \right\}$. From the Theorem 1.7 the Rees $I \times \Lambda$ matrix Γ-semigroup $S = \mathcal{M}^0(G, I, \Lambda, \Gamma)$ is not regular since Γ is not regular. Let us now consider an arbitrary element (i, a, λ) of S. If there exists a matrix P such that $p_{\lambda k} = 0$ then $(i, a, \lambda)P(k, b, \mu) = 0$ which is P-idempotent. If $p_{\lambda j} \neq 0$ for all $j \in I$ then we have $(i, a, \lambda)P(i, p_{\lambda i}^{-1}a^{-1}p_{\lambda i}^{-1}, \lambda)$ is P-idempotent. Hence S is an E-inversive Γ-semigroup.

Clearly every regular Γ-semigroup is E-inversive Γ-semigroup but from the above example we see that the converse is not true. In the introduction we have pointed out that every semigroup can be considered as a Γ-semigroup. Hence every E-inversive semigroup can be considered as an E-inversive Γ-semigroup. The aim of this paper is to extend different interesting results of E-inversive semigroups to E-inversive Γ-semigroups.

Definition 2.4. For a Γ-semigroup S, $a \in S$ and $\alpha, \beta \in \Gamma$ the set $W_\alpha^\beta(a)$ is defined by $W_\alpha^\beta(a) = \{x \in S : x\beta a\alpha x = x\}$. The elements of $W_\alpha^\beta(a)$ are called *weak inverse element* of a.

Theorem 2.5. *An element a of a Γ-semigroup S is E-inversive if and only if $W_\alpha^\beta(a) \neq \phi$ for some $\alpha, \beta \in \Gamma$.*

Proof. Let a be E-inversive. Hence we have $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x \in E_\beta$ i.e, $(a\alpha x)\beta(a\alpha x) = a\alpha x$ which implies that $(a\alpha x)\beta(a\alpha x)\beta(a\alpha x) = a\alpha x$ i.e, $x\beta(a\alpha x\beta a\alpha x\beta a\alpha x) = x\beta a\alpha x$. This shows that $(x\beta a\alpha x)\beta a\alpha(x\beta a\alpha x) = x\beta a\alpha x$. Thus $x\beta a\alpha x \in W_\alpha^\beta(a)$ and hence $W_\alpha^\beta(a) \neq \phi$.

Conversely let $W_\alpha^\beta(a) \neq \phi$ for some $\alpha, \beta \in \Gamma$ and let $x \in W_\alpha^\beta(a)$. Then $x\beta a\alpha x = x$. Now $(a\alpha x)\beta(a\alpha x) = a\alpha(x\beta a\alpha x) = a\alpha x$ i.e, $a\alpha x \in E_\beta$. Hence a is E-inversive. □

From the above theorem we can conclude that a Γ-semigroup is E-inversive if and only if for $a \in S$, $W_\alpha^\beta(a) \neq \phi$ for some $\alpha, \beta \in \Gamma$.

Theorem 2.6. *The set I of all non E-inversive elements of a Γ-semigroup S is either empty or an Γ-ideal.*

Proof. Suppose $I \neq \phi$. Let $a \in I$, $b \in S$ and $\alpha \in \Gamma$. If possible let $a\alpha b$ be an E-inversive element. Then there exist $\beta \in \Gamma$ and $x \in S$ such that $(a\alpha b)\beta x$ is γ-idempotent for some $\gamma \in \Gamma$. Thus we have $a\alpha(b\beta x) \in E_\gamma$, and then a is E-inversive. This is a contradiction. Again if $b\alpha a$ is E-inversive then by Theorem 2.5, there exist $\gamma, \delta \in \Gamma$ and $y \in S$ such that $y \in W_\gamma^\delta(b\alpha a)$. Thus $a\gamma y\delta b \in E_\alpha$. Which implies a is E-inversive, which is also a contradiction. Thus the result follows. □

Theorem 2.7. *In a Γ-semigroup S the following conditions are equivalent:*

(i) *for two E-inversive elements $a, b \in S$, $a\alpha b$ is E-inversive element for some $\alpha \in \Gamma$,*

(ii) *for e, $f \in E(S)$, $e\alpha_1 f$ is an E-inversive element of S for some $\alpha_1 \in \Gamma$.*

Proof. Clearly (i) implies (ii) since every idempotent element is E-inversive. Conversely suppose that (ii) holds. Let x and y be two E-inversive elements of S. Then there exist $x', y' \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x' \in W_\alpha^\beta(x)$ and $y' \in W_\gamma^\delta(y)$. Thus $x'\beta x \in E_\alpha$ and $y\gamma y' \in E_\delta$. Thus by (ii) $(x'\beta x)\alpha_1(y\gamma y')$ is E-inversive for some $\alpha_1 \in \Gamma$. i.e, there exist $z \in S$ and $p, q \in \Gamma$ such that $z \in W_p^q((x'\beta x)\alpha_1(y\gamma y'))$. Let $w = y'pzqx'$. Then $w\beta(x\alpha_1 y)\gamma w = (y'pzqx')\beta(x\alpha_1 y)\gamma(y'pzqx') = y'p(zq(x'\beta x)\alpha_1(y\gamma y')pz)qx' = y'pzqx' = w$. This shows that $W_\gamma^\beta(x\alpha_1 y) \neq \phi$. i.e., $x\alpha_1 y$ is E-inversive. Hence the proof. □

The following theorem shows that E-inversive property of Rees matrix Γ-semigroup over a semigroup T^0 depends not only on the semigroup T but also on the set of sandwich matrices.

Theorem 2.8. *Let T be a semigroup without zero. Then $S = \mathcal{M}^0(I, T, \Lambda, \Gamma)$ is E^*- inversive Γ-semigroup if and only if T is E-inversive and Γ is regular.*

Proof. Let T be an E-inversive semigroup, Γ be regular and $(i, a, \lambda) \in S$. Then there exist matrices $P = (p_{\nu k})$ and $Q = (q_{\nu k})$ such that $p_{\lambda j} \neq 0$ and $q_{\mu i} \neq 0$. Hence $0 \neq p_{\lambda j}aq_{\mu i} \in T$. Since T is E-inversive, there exists $x \in T$ such that $x(p_{\lambda j}aq_{\mu i})x = x$. Thus we have $((i, a, \lambda)P(j, x, \mu))Q((i, a, \lambda)P(j, x, \mu)) = (i, ap_{\lambda j}xq_{\mu i}ap_{\lambda j}x, \mu) = (i, ap_{\lambda j}x, \mu) = (i, a, \lambda)P(j, x, \mu)$. Hence $0 \neq (i, a, \lambda)P(j, x, \mu)$ is Q-idempotent. Thus S is E^*-inversive.

Conversely let S be E^*-inversive. Let $i \in I$, $\lambda \in \Lambda$ and $a \in T$. Now $(i, a, \lambda) \in S$. Since S is E^*-inversive, there exist $(j, x, \mu) \in S$, $P = (p_{\nu k})$, $Q = (q_{\nu k}) \in \Gamma$ such that $0 \neq (i, a, \lambda)P(j, x, \mu) = (i, ap_{\lambda j}x, \mu)$ is Q-idempotent. Hence P has nonzero entry in the λ-th row and $0 \neq (i, ap_{\lambda j}x, \mu) = (i, ap_{\lambda j}x, \mu)Q(i, ap_{\lambda j}x, \mu)$ shows that Q has nonzero entry in the i-th column. Hence Γ is regular. Also from $((i, a, \lambda)P(j, x, \mu))Q((i, a, \lambda)P(j, x, \mu)) = (i, a, \lambda)P(j, x, \mu)$ we find that $(i, ap_{\lambda j}xq_{\mu i}ap_{\lambda j}x, \mu) = (i, ap_{\lambda j}x, \mu)$ and then $(a(p_{\lambda j}xq_{\mu i}))(a(p_{\lambda j}xq_{\mu i})) = (a(p_{\lambda j}xq_{\mu i}))$ and then $ap_{\lambda j}xq_{\mu i}$ is an idempotent element in T for $a \in T$. Thus it follows that T is E-inversive. □

The following example shows that in a Γ-semigroup S,

(i) for some $\alpha \in \Gamma$, S_α may be E^*-inversive semigroup but there may exist $\beta \in \Gamma$ such that S_β is not an E^*-inversive semigroup and
(ii) S_α may not be an E^*-inversive semigroup for some $\alpha \in \Gamma$, but S may be an E^*-inversive Γ-semigroup.

Example 2.9. Let us consider a Rees matrix semigroup $S = \mathcal{M}^0(I, T, \Lambda, \Gamma)$ over the semigroup $T = \{e, a, f, b, \}$ with Cayley table

	e	a	f	b
e	e	a	f	b
a	a	e	b	f
f	f	b	f	b
b	b	f	b	f

where $I = \{1, 2\}$ and $\Lambda = \{1, 2, 3\}$ and $\Gamma = \{\alpha, \beta\}$ where $\alpha = \begin{pmatrix} 0 & 0 \\ a & e \\ b & f \end{pmatrix}$ and $\beta = \begin{pmatrix} b & e \\ f & b \\ a & a \end{pmatrix}$. Now we see that T is an E-inversive semigroup and Γ is regular. Hence by Theorem 2.8 S is E^*-inversive. It is to be noted here that S_β is E^*-inversive but S_α is not E^*-inversive since for $(1, a, 1)$ there is no (i, b, λ) such that $(1, a, 1)\alpha(i, b, \lambda) \neq 0$.

3. Right E-Γ-semigroup

In this section we study some particular type of Γ-semigroup which is a generalization of right orthodox Γ-semigroup.

Definition 3.1. Let S be a Γ-semigroup. S is called a *right (resp. left) E-Γ-semigroup*, if for any α-idempotent e and β-idempotent f of S, $e\alpha f$ (resp. $f\alpha e$) is a β-idempotent in S.

Proceeding as in the proof of Proposition 5.2([9]), we prove the following result in Γ-semigroups.

Theorem 3.2. *Let T be a semigroup without zero. Then $S = \mathcal{M}^0(I, T, \Lambda, \Gamma)$ is right E- Γ-semigroup if and only if for all $i, j \in I$, $\lambda, \mu \in \Lambda$: $W(p_{\lambda i})p_{\lambda j}W(q_{\mu j}) \subseteq W(q_{\mu i})$.*

Proof. Let $S = \mathcal{M}^0(T, I, \Lambda, \Gamma)$ and $W(t)$ denote the set of all weak inverses of t in T^0. Let $P \in \Gamma$ and (i, a, λ) be a nonzero P-idempotent in S. Then we have $(i, ap_{\lambda i}a, \lambda) = (i, a, \lambda)$. Since (i, a, λ) is nonzero we have $p_{\lambda i} \neq 0$ and $a \in W(p_{\lambda i})$. Hence $E_P(S) \subseteq \{(i, p'_{\lambda i}, \lambda) \in S : p_{\lambda i} \neq 0, p'_{\lambda i} \in W(p_{\lambda i})\} \cup \{0\}$. Again for $i \in I$, $\lambda \in \Lambda$ with $p_{\lambda i} \neq 0$, $(i, p'_{\lambda i}, \lambda)$ is P-idempotent for $p'_{\lambda i} \in W(p_{\lambda i})$. Since the zero element is P-idempotent we can conclude that $E_P(S) = \{(i, p'_{\lambda i}, \lambda) \in S : p_{\lambda i} \neq 0, p'_{\lambda i} \in W(p_{\lambda i})\} \cup \{0\}$. Let S be a right E-Γ-semigroup. Now for $i, j \in I$, $\lambda, \mu \in \Lambda$, $p'_{\lambda i} \in W(p_{\lambda i}), q'_{\mu j} \in W(q_{\mu j})$. If one of $p'_{\lambda i}$, $p_{\lambda j}$, $q'_{\mu j}$ is the zero in T^0, then $p'_{\lambda i}p_{\lambda j}q'_{\mu j} = 0 \in W(q_{\mu i})$. Suppose none of $p'_{\lambda i}$, $p_{\lambda j}$, $q'_{\mu j}$ is zero. Then $(i,\, p'_{\lambda i},\, \lambda) \in E_P$, $(j,\, q'_{\mu j},\, \mu) \in E_Q$. Since S is right E-Γ-semigroup, $(i,\, p'_{\lambda i}p_{\lambda j}q'_{\mu j},\, \mu) = (i,\, p'_{\lambda i},\, \lambda)P(j,\, q'_{\mu j},\, \mu) \in E_Q$. This implies $p'_{\lambda i}p_{\lambda j}q'_{\mu j} \in W(q_{\mu i})$ i.e, $W(p_{\lambda i})p_{\lambda j}W(q_{\mu j}) \subseteq W(q_{\mu i})$.

Conversely, let the condition hold. Suppose $(i,\, a,\, \lambda)$ be a nonzero P-idempotent and $(j,\, b,\, \mu)$ be a nonzero Q-idempotent for $P, Q \in \Gamma$. Then $a \in W(p_{\lambda i})$ and $b \in W(q_{\mu j})$. If $p_{\lambda j} = 0$, then $(i,\, a,\, \lambda)P(j,\, b,\, \mu) = 0 \in E_Q$. Let $p_{\lambda j} \neq 0$. Then by the given condition $ap_{\lambda j}b \in W(q_{\mu i})$. i.e., we get $(i,\, a,\, \lambda)P(j,\, b,\, \mu) = (i,\, ap_{\lambda j}b,\, \mu) \in E_Q$. Again since for $P_1 \in \Gamma$, $0P_1(i, a, \lambda) = (i, a, \lambda)P0 = 0 \in E_{Q_1}$ for all $Q_1 \in \Gamma$ we conclude that S is a right E- Γ-semigroup. □

Definition 3.3. Let S be a Γ-semigroup. A nonempty subset P of S is said to be *partial* Γ*-subsemigroup*, if for $a, b \in P$, there exists $\alpha \in \Gamma$ such that $a\alpha b \in P$.

Theorem 3.4. *Let S be a Γ-semigroup and $E(S) \neq \phi$. Then the regular elements form a partial Γ-subsemigroup if and only if for $e, f \in E(S)$, $e\alpha_1 f$ is regular for some $\alpha_1 \in \Gamma$.*

Proof. Let the regular elements of S form a partial Γ - subsemigroup. Since every idempotent element is regular, the condition holds.

Conversely let the given condition hold. Let a, b be two regular elements of S and $a' \in V_\alpha^\beta(a), b' \in V_\gamma^\delta(b)$. Then $a'\beta a,\ b\gamma b' \in E(S)$. By the given condition there exists $\mu \in \Gamma$ such that $(a'\beta a)\mu(b\gamma b')$ is regular. i.e., there exist $x \in S$ and $\mu_1, \mu_2 \in \Gamma$ such that $(a'\beta a)\mu(b\gamma b') = (a'\beta a\mu b\gamma b')\mu_1 x\mu_2(a'\beta a\mu b\gamma b')$. Now $a\mu b = a\alpha a'\beta a\mu b\gamma b'\delta b = a\alpha((a'\beta a)\mu(b\gamma b'))\delta b = a\alpha((a'\beta a\mu\ b\gamma b')\mu_1 x\mu_2\ (a'\beta a\mu b\gamma b'))\delta b$. Thus we have $a\mu b = (a\mu b)\gamma\ (b'\mu_1 x\mu_2 a')\beta(a\mu b)$ and hence $a\mu b$ is a regular element of S. Hence the proof. □

We now recall Rees congruence on a Γ-semigroup which has been introduced in [3]. Let I be an ideal of a Γ-semigroup S. Let $\rho_I = (I \times I) \cup 1_S$ where 1_S is the equality relation. Thus for $x, y \in S, (x, y) \in \rho_I$ if and only if either $x = y$ or x and y both belong to I. It is clear that ρ_I is an equivalence relation. Now let $(x, y) \in \rho_I, z \in S$ and $\alpha \in \Gamma$. Then there are two possibilities. If $x = y$ then $(x\alpha z, y\alpha z) \in \rho_I$ and $(z\alpha x, z\alpha y) \in \rho_I$ and if x, y both belong to I then also $x\alpha z, y\alpha z \in I$ and $z\alpha x, z\alpha y \in I$ i.e, $(x\alpha z, y\alpha z) \in \rho_I$ and $(z\alpha x, z\alpha y) \in \rho_I$. Hence ρ_I is a Γ-congruence on S. We call this Γ-congruence Rees Γ-congruence on the Γ-semigroup S and denote the Γ-semigroup of all such classes of the elements of Γ-semigroup S by S/ρ_I or simply by S/I and we have $S/I = \{I\} \cup \{\{x\} : x \notin I\}$.

Definition 3.5. If I is a Γ-ideal of a Γ-semigroup S, then S is called an *ideal extension of* I by the Rees quotient Γ-semigroup S/I.

Definition 3.6. Let S be a Γ-semigroup with zero. Then a nonzero element $a \in S$ is said to be *divisor of zero* if there exist an element $\alpha \in \Gamma$ and a nonzero element $b \in S$ such that $a\alpha b = 0$.

Theorem 3.7. *Let S be a Γ-semigroup with $E(S) \neq \phi$. Then S is either E-inversive or an ideal extension of an idempotent free Γ-semigroup by an E^*-inversive*

Γ-semigroup. If S is a right E- Γ-semigroup, then S is either E-inversive or an ideal extension of an idempotent free Γ-semigroup by an E^*-inversive Γ-semigroup which contains no proper zero divisor.

Proof. Let S be not E-inversive Γ-semigroup. Let T be the set of all non E-inversive elements of S. Then by Theorem 2.6, T is a Γ-ideal of S. Since every idempotent element is E-inversive, T is idempotent free. Let A be a nonzero element of S/T, Rees quotient Γ-semigroup. Then $A = \{a\}$ for an E-inversive element $a \in S$. Since a is E-inversive, there exist $x \in S$, $\alpha \in \Gamma$ such that $a\alpha x = e \in E_\beta$ for some $\beta \in \Gamma$. Clearly e is E-inversive. Hence $\{e\} \in S/T$ is different from the zero element $\{T\}$ of S/T. Hence $A\alpha\{x\} = \{e\} \in E_\beta(S/T)$ where $\{e\}$ is nonzero. Thus A is an E^*-inversive element of S/T. □

Let us suppose now that S is a right E-Γ-semigroup and $\{a\}$, $\{b\}$ be two nonzero elements of S/T. Then $a, b \in S$ and they are E-inversive elements. Hence by Theorem 2.7, $a\alpha b$ is E-inversive for some $\alpha \in \Gamma$. This implies that $a\alpha b \notin T$ i.e., $\{a\}\alpha\{b\} \neq \{T\}$ and hence S/T contains no proper zero divisor.

Theorem 3.8. *Let S be a Γ-semigroup and D be a $\mathcal{D}$ class of S. If an element of D is E-inversive then every element of D is E-inversive.*

Proof. Suppose a is an E-inversive element of D. Let $a\mathcal{D}b$. We show that there exist γ, $\delta \in \Gamma$ such that $W_\gamma^\delta(b) \neq \phi$. Since a is E-inversive there exist $a' \in S$ and α, $\beta \in \Gamma$ such that $a' \in W_\alpha^\beta(a)$. Now $a'\mathcal{L}a\alpha a'$ and there exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$. Again since $\mathcal{L}$ is right congruence, we have $c\alpha a'\mathcal{L}a\alpha a'\mathcal{L}a'$. Now since $\mathcal{L} \subseteq \mathcal{D}$ we have $c\alpha a' \in D_{a'}$. Since a' is a regular element, by Theorem 1.12, $c\alpha a'$ is a regular element. Thus there exist $z \in S$ and μ, $\nu \in \Gamma$ such that $z \in V_\mu^\nu(c\alpha a')$. Let $c' = a'\mu z$. Now $c'\nu c\alpha c' = a'\mu z\nu c\alpha a'\mu z = a'\mu z\nu(c\alpha a')\mu z = a'\mu z = c'$. Thus $c' \in W_\alpha^\nu(c)$. Since $\mathcal{R}$ is a left congruence, from $c\mathcal{R}b$ we have $c'\nu b\mathcal{R}c'\nu c\mathcal{R}c'$. Since $\mathcal{R} \subseteq \mathcal{D}$, we have $c'\nu b \in D_{c'}$. Applying Theorem 1.12 we see that $c'\nu b$ is a regular element since c' is a regular element. Thus there exists $w \in V_p^q(c'\nu b)$ for some $p, q \in \Gamma$. Let $b' = wqc'$. Now $b'\nu bpb' = wqc'\nu bpwqc' = wqc' = b'$ and hence $b' \in W_p^\nu(b)$. This completes the proof. □

Theorem 3.9. *Let S be a Γ-semigroup with $E(S) \neq \phi$. Then the following are equivalent:*

(i) *S is right E- Γ-semigroup,*

(ii) *$V_{\beta_1}^{\beta_2}(b)\beta_2 V_{\alpha_1}^{\alpha_2}(a) \subseteq V_{\beta_1}^{\alpha_2}(a\alpha_1 b)$ for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$ and $a, b \in S$,*

(iii) *$W_{\beta_1}^{\beta_2}(b)\beta_2 W_{\alpha_1}^{\alpha_2}(a) \subseteq W_{\beta_1}^{\alpha_2}(a\alpha_1 b)$ for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$ and $a, b \in S$.*

Proof. (i)⇒(ii) Let $a' \in V_{\alpha_1}^{\alpha_2}(a), b' \in V_{\beta_1}^{\beta_2}(b)$. We show that $b'\beta_2 a' \in V_{\beta_1}^{\alpha_2}(a\alpha_1 b)$. Now

$$\begin{aligned} b'\beta_2 a'\alpha_2 a\alpha_1 b\beta_1 b'\beta_2 a' &= b'\beta_2 b\beta_1 b'\beta_2 a'\alpha_2 a\alpha_1 b\beta_1 b'\beta_2 a'\alpha_2 a\alpha_1 a' \\ &= b'\beta_2 (b\beta_1 b'\beta_2 a'\alpha_2 a)\alpha_1 (b\beta_1 b'\beta_2 a'\alpha_2 a)\alpha_1 a' \\ &= b'\beta_2 (b\beta_1 b'\beta_2 a'\alpha_2 a)\alpha_1 a' (\text{ Since } S \text{ is right } E\text{- } \Gamma\text{-semigroup}) \\ &= b'\beta_2 a'. \end{aligned}$$

and

$$\begin{aligned} a\alpha_1 b\beta_1 b'\beta_2 a'\alpha_2 a\alpha_1 b &= a\alpha_1 a'\alpha_2 a\alpha_1 b\beta_1 b'\beta_2 a'\alpha_2 a\alpha_1 b\beta_1 b'\beta_2 b \\ &= a\alpha_1 (a'\alpha_2 a\alpha_1 b\beta_1 b')\beta_2 (a'\alpha_2 a\alpha_1 b\beta_1 b')\beta_2 b \\ &= a\alpha_1 (a'\alpha_2 a\alpha_1 b\beta_1 b')\beta_2 b \text{ (Since } (a'\alpha_2 a)\alpha_1 (b\beta_1 b') \in E_{\beta_2}) \\ &= a\alpha_1 b. \end{aligned}$$

Hence the proof.
(ii)$\Rightarrow$(i) Let e be an α-idempotent and f be a β-idempotent i.e, $e \in V_\alpha^\alpha(e)$ and $f \in V_\beta^\beta(f)$. Then by the given condition $e\alpha f \in V_\alpha^\beta(f\beta e)$ i.e, $e\alpha f\beta f\beta e\alpha e\alpha f = e\alpha f$ which implies $(e\alpha f)\beta(e\alpha f) = e\alpha f$ i.e, $e\alpha f$ is a β-idempotent. Thus (i) holds.
(i) $\Rightarrow$ (iii) is similar to (i) $\Rightarrow$ (ii).
(iii) $\Rightarrow$ (i) Let e be an α-idempotent and f be a β-idempotent. Then $e \in W_\alpha^\alpha(e)$ and $f \in W_\beta^\beta(f)$. Now by (iii) we have $e\alpha f \in W_\alpha^\beta(f\beta e)$ i.e, $(e\alpha f)\beta(f\beta e)\alpha(e\alpha f) = e\alpha f$ which implies $(e\alpha f)\beta(e\alpha f) = e\alpha f$. Thus (i) holds since $e\alpha f$ is a β-idempotent. □

Theorem 3.10. *Let S be a Γ-semigroup. Then $W_{\beta_1}^{\alpha_2}(a\alpha_1 b) \subseteq W_{\beta_1}^{\alpha_1}(b)\alpha_1 W_{\alpha_1}^{\alpha_2}(a)$ for all $a, b \in S$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$.*

Proof. Let $x \in W_{\beta_1}^{\alpha_2}(a\alpha_1 b)$. Then $x = x\alpha_2 a\alpha_1 b\beta_1 x$. So $(x\alpha_2 a)\alpha_1 b\beta_1 (x\alpha_2 a) = x\alpha_2 a$ and $(b\beta_1 x)\alpha_2 a\alpha_1 (b\beta_1 x) = b\beta_1 x$ i.e, $x\alpha_2 a \in W_{\beta_1}^{\alpha_1}(b)$ and $b\beta_1 x \in W_{\alpha_1}^{\alpha_2}(a)$. Again since $x = (x\alpha_2 a)\alpha_1 (b\beta_1 x)$ we have $W_{\beta_1}^{\alpha_2}(a\alpha_1 b) \subseteq W_{\beta_1}^{\alpha_1}(b)\alpha_1 W_{\alpha_1}^{\alpha_2}(a)$. □

From the above two theorems the following corollary follows.

Corollary 3.11. *For any Γ-semigroup S with $E(S) \neq \phi$, S is a right E- Γ-semigroup if and only if $W_\alpha^\beta(a)\beta W_\beta^\gamma(b) = W_\alpha^\gamma(b\beta a)$.*

The following theorem extends the results of Proposition 3.4([9]) in the right E-Γ-semigroup.

Theorem 3.12. *Let S be a right E- Γ-semigroup. Then*
(i) *$V_\alpha^\beta(e) \subseteq W_\alpha^\beta(e) \subseteq E_\beta$ for all $e \in E_\alpha$,*
(ii) *$a'\beta e\gamma a \in E_\alpha$ and $a\alpha e\gamma a' \in E_\beta$ for all $a \in S, a' \in W_\alpha^\beta(a), e \in E_\gamma$,*
(iii) *$V_{\alpha_1}^\beta(a) \cap V_\alpha^\beta(b) \neq \phi$ for some $\alpha_1, \alpha, \beta \in \Gamma$ implies $V_{\alpha_1}^\delta(a) = V_\alpha^\delta(b)$ for all $\delta \in \Gamma$ and for all $a, b \in S$,*
(iv) *$W_\beta^\alpha(e\alpha f) = W_\alpha^\alpha(f\beta e)$ for all $e \in E_\alpha, f \in E_\beta$.*

Proof. (i) It is obvious that $V_\alpha^\beta(e) \subseteq W_\alpha^\beta(e)$. Let $a \in W_\alpha^\beta(e)$. Then $a\beta e\alpha a = a$. Now $a = a\beta e\alpha a = (a\beta e)\alpha(e\alpha a)$. Again $(a\beta e)\alpha(a\beta e) = a\beta e$ and $e\alpha a\beta e\alpha a = e\alpha a$ i.e, $a\beta e \in E_\alpha$ and $e\alpha a \in E_\beta$. Since S is right E- Γ-semigroup, $a = (a\beta e)\alpha(e\alpha a)$ is a β-idempotent.
(ii) Let $a \in S, a' \in W_\alpha^\beta(a), e \in E_\gamma$. Now

$$\begin{aligned}(a'\beta e\gamma a)\alpha(a'\beta e\gamma a) &= (a'\beta a\alpha a'\beta e\gamma a\alpha a'\beta e\gamma a)\\ &= a'\beta(((a\alpha a')\beta e)\gamma((a\alpha a')\beta e))\gamma a\\ &= a'\beta a\alpha a'\beta e\gamma a\\ &= a'\beta e\gamma a.\end{aligned}$$

and

$$\begin{aligned}(a\alpha e\gamma a')\beta(a\alpha e\gamma a') &= a\alpha e\gamma a'\beta a\alpha e\gamma a'\beta a\alpha a'\\ &= a\alpha((e\gamma a'\beta a)\alpha(e\gamma a'\beta a))\alpha a'\\ &= a\alpha e\gamma a'\beta a\alpha a'\\ &= a\alpha e\gamma a'.\end{aligned}$$

(iii) Assume that $a' \in V^{\beta}_{\alpha_1}(a) \cap V^{\beta}_{\alpha}(b)$ and $a^* \in V^{\delta}_{\alpha_1}(a)$. Then we have $a'\beta a\alpha_1 a' = a'$, $a\alpha_1 a'\beta a = a$, $a'\beta b\alpha a' = a'$, $b\alpha a'\beta b = b$, $a^*\delta a\alpha_1 a^* = a^*$ and $a\alpha_1 a^*\delta a = a$. Now proceeding as in the proof of Theorem 3.9 [2] we can show that $b\alpha a^*\delta b = b$ and $a^*\delta b\alpha a^* = a^*$. Thus $a^* \in V^{\delta}_{\alpha}(b)$ i.e., $V^{\delta}_{\alpha_1}(a) \subseteq V^{\delta}_{\alpha}(b)$. Similarly we can show that $V^{\delta}_{\alpha}(b) \subseteq V^{\delta}_{\alpha_1}(a)$. Therefore we have $V^{\delta}_{\alpha_1}(a) = V^{\delta}_{\alpha}(b)$ for all $\delta \in \Gamma$.
(iv) Let $e \in E_\alpha$, $f \in E_\beta$ and $x \in W^{\alpha}_{\beta}(e\alpha f)$ i.e, $x\alpha e\alpha f\beta x = x$. Since $e\alpha f \in E_\beta$, by (i) we have $x \in E_\alpha$. Therefore $x\alpha e\alpha x = x\alpha e\alpha(x\alpha e\alpha f\beta x) = (x\alpha e)\alpha(x\alpha e)\alpha(f\beta x) = x\alpha e\alpha f\beta x = x$ and

$$\begin{aligned}x\alpha f\beta x = (x\alpha e\alpha f\beta x)\alpha(f\beta x) &= x\alpha e\alpha((f\beta x)\alpha(f\beta x))\\ &= (x\alpha e)\alpha(f\beta x) = x\alpha(e\alpha f)\beta x\\ &= x.\end{aligned}$$

Hence

$$\begin{aligned}x\alpha(f\beta e)\alpha x &= (x\alpha e\alpha x)\alpha(f\beta e)\alpha(x\alpha f\beta x)\\ &= x\alpha((e\alpha x\alpha f)\beta(e\alpha x\alpha f))\beta x\\ &= x\alpha(e\alpha x\alpha f)\beta x (\text{ Since } S \text{ is right } E\text{-}\Gamma\text{-semigroup})\\ &= (x\alpha e\alpha x)\alpha f\beta x = x\alpha f\beta x = x.\end{aligned}$$

Hence $x \in W^{\alpha}_{\alpha}(f\beta e)$ i.e, $W^{\alpha}_{\beta}(e\alpha f) \subseteq W^{\alpha}_{\alpha}(f\beta e)$.

Conversely, let $y \in W^{\alpha}_{\alpha}(f\beta e)$. Then $y\alpha f\beta e\alpha y = y$ and by (i) y is an α-idempotent. Now $y\alpha e\alpha y = (y\alpha f\beta e\alpha y)\alpha(e\alpha y) = (y\alpha f)\beta(e\alpha y)\alpha(e\alpha y) = (y\alpha f)\beta(e\alpha y) = y$ and $y\alpha f\beta y = (y\alpha f)\beta(y\alpha f\beta e\alpha y) = (y\alpha f)\beta(y\alpha f)\beta(e\alpha y) = (y\alpha f)\beta(e\alpha y) = y$. Now $y\alpha(e\alpha f)\beta y = (y\alpha f\beta y)\,\alpha(e\alpha f)\beta(y\alpha e\alpha y) = y\alpha((f\beta y\alpha e)\alpha(f\beta y\alpha e))\alpha y$ $= y\alpha(f\beta y\alpha e)\alpha y = (y\alpha f\beta y)\alpha e\alpha y = y\alpha e\alpha y = y$. Hence $y \in W^{\alpha}_{\beta}(e\alpha y)$. Thus (iv) holds. □

Definition 3.13. Let S be a Γ-semigroup, $a \in S$ and $\alpha, \beta \in \Gamma$. The set $I^{\beta}_{\alpha}(a)$ is defined by $I^{\beta}_{\alpha}(a) = \{x \in S : x\beta a \in E_\alpha, a\alpha x \in E_\beta\}$.

Theorem 3.14. *Let S be a Γ-semigroup. Then the following are equivalent:*

(i) $I^{\beta_2}_{\beta_1}(b)\beta_2 I^{\alpha_2}_{\alpha_1}(a) \subseteq I^{\alpha_2}_{\beta_1}(a\alpha_1 b)$ *for* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$ *and* $a, b \in S$,

(ii) $a\alpha e\gamma a' \in E_\beta$ *for all* $a \in S, a' \in I^{\beta}_{\alpha}(a), e \in E_\gamma$,

(iii) $a'\beta e\gamma a \in E_\alpha$ *for all* $a \in S, a' \in I_\alpha^\beta(a), e \in E_\gamma$.

Proof. (i) $\Rightarrow$ (ii): Let $a \in S, a' \in V_\alpha^\beta(a)$ and $e \in E_\gamma$. Since $e \in I_\gamma^\gamma(e)$ and $a' \in I_\alpha^\beta(a)$, from (i) we have $e\gamma a' \in I_\gamma^\beta(a\alpha e)$. Now $a\alpha e\gamma a' = (a\alpha e)\gamma(e\gamma a') \in E_\beta$.

(ii) $\Rightarrow$ (i): Let $a' \in I_{\alpha_1}^{\alpha_2}(a)$ and $b' \in I_{\beta_1}^{\beta_2}(b)$. Then $b\beta_1 b' \in E_{\beta_2}$ and by (ii), $(a\alpha_1 b)\beta_1(b'\beta_2 a') = a\alpha_1(b\beta_1 b')\beta_2 a' \in E_{\alpha_2}$. Again since $a \in I_{\alpha_2}^{\alpha_1}(a')$ and $b \in I_{\beta_2}^{\beta_1}(b')$, similarly we can show that $(b'\beta_2 a')\alpha_2(a\alpha_1 b) = b'\beta_2(a'\alpha_2 a)\alpha_1 b \in E_{\beta_1}$. Hence we have $b'\beta_2 a' \in I_{\beta_1}^{\alpha_2}(a\alpha_1 b)$. Thus (i) is proved. It can be shown (ii) $\Leftrightarrow$ (iii). $\square$

Theorem 3.15. *Let S be a right E- Γ-semigroup and $a \in S$. If $a' \in W_\alpha^\beta(a), e \in E_\gamma, f \in E_\delta$, then $e\gamma a' \in W_\alpha^\beta(a), a'\beta f \in W_\alpha^\delta(a)$ and $e\gamma a'\beta f \in W_\alpha^\delta(a)$.*

Proof. Let $a' \in W_\alpha^\beta(a), e \in E_\gamma, f \in E_\delta$. Now $e\gamma a'\beta a\alpha e\gamma a' = (e\gamma(a'\beta a)\alpha e)\gamma a'\beta a\alpha a'$ $= (e\gamma(a'\beta a))\alpha(e\gamma(a'\beta a))\alpha a' = (e\gamma(a'\beta a))\alpha a'$ (Since $e\gamma(a'\beta a) \in E_\alpha$) $= e\gamma a'$. Hence $e\gamma a' \in W_\alpha^\beta(a)$. Again

$$\begin{aligned}(a'\beta f)\delta a\alpha(a'\beta f) &= a'\beta a\alpha a'\beta f\delta a\alpha a'\beta f \\ &= a'\beta((a\alpha a')\beta f)\delta((a\alpha a')\beta f) \\ &= a'\beta((a\alpha a')\beta f)(\text{ Since } (a\alpha a')\beta f \in E_\delta) \\ &= a'\beta f.\end{aligned}$$

Hence $a'\beta f \in W_\alpha^\delta(a)$. Again from $e\gamma a' \in W_\alpha^\beta(a)$ and $f \in E_\delta$ it follows that $e\gamma a'\beta f \in W_\alpha^\delta(a)$ $\square$.

Theorem 3.16. *Let S be a right E- Γ-semigroup and a be a regular element of S such that $a' \in V_\alpha^\beta(a)$. Then $W_\alpha^\delta(a) = E_\alpha\alpha a'\beta E_\delta$.*

Proof. By the Theorem 3.15 we have $E_\alpha\alpha a'\beta E_\delta \subseteq W_\alpha^\delta(a)$. Now let $a^* \in W_\alpha^\delta(a)$, then $a^* = a^*\delta a\alpha a^* = a^*\delta a\alpha a'\beta a\alpha a^* = (a^*\delta a)\alpha a'\beta(a\alpha a^*) \subseteq E_\alpha\alpha a'\beta E_\delta$. Hence the proof. $\square$

Theorem 3.17. *Let S be a right E- Γ-semigroup. Then the following are equivalent:*

(i) *for $e \in E_\alpha, f \in E_\beta, e\alpha f\beta e = e\alpha f$,*

(ii) *for every $a \in S$, if $a' \in V_{\alpha_1}^{\beta_1}(a)$ and $a'' \in V_{\alpha_2}^{\beta_2}(a)$ then $a\alpha_1 a' = a\alpha_2 a''$,*

(iii) *every $\mathcal{R}$- class contains at most one idempotent,*

(iv) *if for some $\alpha, \beta, \delta \in \Gamma, a' \in W_\alpha^\beta(a)$ and $a^* \in W_\alpha^\delta(a)$ with $a'\mathcal{R}a^*$ then $a' = a^*$,*

(v) *(for all $e \in E_\alpha, e' \in V_\alpha^\beta(e)$) $e'\beta e = e'$.*

Proof. (i) $\Rightarrow$ (v) From Theorem 3.12(i) we have $e' \in E_\beta$. Now from (i) $e' = e'\beta e\alpha e' = e'\beta e$.

(v) $\Rightarrow$ (iv) : Let $a' \in W_\alpha^\beta(a)$ and $a^* \in W_\alpha^\delta(a)$ such that $a'\mathcal{R}a^*$. Then we have $a'\beta a\mathcal{R}a'\mathcal{R}a^*\mathcal{R}a^*\delta a$. Since $(a'\beta a)\alpha x = (a^*\delta a)\alpha x = x$ for all $x \in R_{a'\beta a} = R_{a^*\delta a}$, we have

$$\begin{aligned}(a\alpha a^*)\delta(a\alpha a')\beta(a\alpha a^*) &= a\alpha((a^*\delta a)\alpha(a'\beta a))\alpha a^* \\ &= a\alpha(a'\beta a)\alpha a^* \\ &= a\alpha((a'\beta a)\alpha a^*)= a\alpha a^*.\end{aligned}$$

i.e, $a\alpha a^* \in W_\beta^\delta(a\alpha a')$. Hence by (v) we have

$$a' = (a^*\delta a)\alpha a' = (a^*\delta a)\alpha(a^*\delta a)\alpha a' = a^*\delta((a\alpha a^*)\delta(a\alpha a')) = a^*\delta(a\alpha a^*) = a^*.$$

(iv) $\Rightarrow$ (iii) : Let $e \in E_\alpha$ and $f \in E_\beta$ with $e\mathcal{R}f$, then we have $e\alpha f = f$ and $f\beta e = e$. Now $f\beta e\alpha f = f$ and hence we get $f \in W_\alpha^\beta(e)$. Again $e \in V_\alpha^\alpha(e)$ and by (iv) we have $e = f$.
(iii) $\Rightarrow$ (ii) : Let $a' \in V_{\alpha_1}^{\beta_1}(a)$, $a'' \in V_{\alpha_2}^{\beta_2}(a)$. Then $a\alpha_1 a' \,\mathcal{R}\, a \,\mathcal{R}\, a\alpha_2 a''$. Hence by (iii) we have $a\alpha_1 a' = a\alpha_2 a''$.
(ii) $\Rightarrow$ (i) : Let $e \in E_\alpha$ and $f \in E_\beta$. Now $(e\alpha f)\beta(f\beta e)\alpha(e\alpha f) = e\alpha f\beta e\alpha f = e\alpha f$ and $(f\beta e)\alpha(e\alpha f)\beta(f\beta e) = f\beta e\alpha f\beta e = f\beta e$. Hence $f\beta e \in V_\beta^\alpha(e\alpha f)$. Again $e\alpha f \in V_\beta^\beta(e\alpha f)$. Hence by (ii) we have $(e\alpha f)\beta(f\beta e) = (e\alpha f)\beta(e\alpha f)$. Thus $e\alpha f\beta e = e\alpha f$. □

Theorem 3.18. *Let S be an E-inversive Γ-semigroup. Then the following are equivalent:*

(i) *for $e \in E_\alpha$ and $f \in E_\beta, e\alpha f\beta e = e$,*
(ii) *for $e \in E_\alpha, f \in E_\beta$ and $g \in E_\gamma, e\alpha f\beta g = e\alpha g$,*
(iii) *(for all $e \in E_\alpha$) $W_\alpha^\beta(e) = E_\beta(S)$,*
(iv) *(for all $a, b \in S$) $W_{\alpha_1}^\beta(a) \cap W_{\alpha_2}^\beta(b) \neq \phi$ for some $\alpha_1, \alpha_2, \beta \in \Gamma$ implies $W_{\alpha_1}^\delta(a) = W_{\alpha_2}^\delta(b)$ for all $\delta \in \Gamma$,*
(v) *(for $e \in E_\alpha, f \in E_\beta$) if $W_\alpha^\gamma(e) \cap W_\beta^\gamma(f) \neq \phi$ for some $\gamma \in \Gamma$ then $W_\alpha^\delta(e) = W_\beta^\delta(f)$ for all $\delta \in \Gamma$,*
(vi) *for $e \in E_\alpha$ and $f \in E_\beta$, $e\alpha f \in E_\beta$ and $W_\alpha^\beta(a) = V_\alpha^\beta(a)$ for all $\alpha, \beta \in \Gamma$ and for all regular elements $a \in S$.*

Proof. (i) $\Rightarrow$ (ii) : Let $e \in E_\alpha, f \in E_\beta$ and $g \in E_\gamma$. Then $(e\alpha g)\gamma(e\alpha g) = (e\alpha g\gamma e)\alpha g = e\alpha g$. Thus $e\alpha g$ is γ-idempotent. Now $e\alpha f\beta e = e$ and $g\gamma f\beta g = g$. Thus

$$e\alpha g \quad = (e\alpha f\beta e)\alpha(g\gamma f\beta g) = e\alpha(f\beta(e\alpha g)\gamma f)\beta g = e\alpha f\beta g.$$

Hence (ii) follows.
(ii) $\Rightarrow$ (i) is obvious.
(i) $\Rightarrow$ (iii) : Let $a \in W_\alpha^\beta(e)$. Then $a\beta e \in E_\alpha$ and $e\alpha a \in E_\beta$. Now $a\beta a = (a\beta e\alpha a)\beta(a\beta e\alpha a) = (a\beta e)\alpha(e\alpha a)\beta(a\beta e)\alpha(e\alpha a) = (a\beta e)\alpha(e\alpha a) = a$. Hence $W_\alpha^\beta(e) \subseteq E_\beta$. Again if $f \in E_\beta$ then by (i) $f\beta e\alpha f = f$ i.e, $f \in W_\alpha^\beta(e)$. Hence (iii) holds.
(iii) $\Rightarrow$ (i) is obvious.
(i) $\Rightarrow$ (iv) : Let $x \in W_{\alpha_1}^\beta(a) \cap W_{\alpha_2}^\beta(b)$ and let $a' \in W_{\alpha_1}^\delta(a)$. Then $a'\delta a \in E_{\alpha_1}$ and $a\alpha_1 a' \in E_\delta$. Again by (i) we can show that S is a right E-Γ-semigroup and by Theorem 3.12(ii) $b\alpha_2 a'\delta a\alpha_1 x \in E_\beta$. Now

$$\begin{aligned}
a' &= (a'\delta a)\alpha_1 a' \\
&= ((a'\delta a)\alpha_1(x\beta b)\alpha_2(a'\delta a))\alpha_1 a' \\
&= a'\delta a\alpha_1 x\beta b\alpha_2((a'\delta a)\alpha_1(x\beta a)\alpha_1(a'\delta a))\alpha_1 a' \\
&= a'\delta((a\alpha_1 a')\delta(a\alpha_1 x)\beta(b\alpha_2 a'\delta a\alpha_1 x))\beta(a\alpha_1 a')\delta(a\alpha_1 a') \\
&= a'\delta((a\alpha_1 a')\delta(b\alpha_2 a'\delta a\alpha_1 x))\beta(a\alpha_1 a')\delta(a\alpha_1 a') \text{ (Since (i) } \Rightarrow \text{ (ii))} \\
&= a'\delta b\alpha_2((a'\delta a)\alpha_1(x\beta a)\alpha_1(a'\delta a))\alpha_1 a' \\
&= a'\delta b\alpha_2 a'\delta a\alpha_1 a' = a'\delta b\alpha_2 a'.
\end{aligned}$$

Thus we have $a' \in W^{\delta}_{\alpha_2}(b)$. Hence $W^{\delta}_{\alpha_1}(a) \subseteq W^{\delta}_{\alpha_2}(b)$. Similarly we can show that $W^{\delta}_{\alpha_2}(b) \subseteq W^{\delta}_{\alpha_1}(a)$. Thus $W^{\delta}_{\alpha_2}(b) = W^{\delta}_{\alpha_1}(a)$.
(iv) $\Rightarrow$ (v) is trivial.
(v) $\Rightarrow$ (i) : Let e be an α-idempotent and f be a β-idempotent and since S is E-inversive, we find an $x \in W^{\mu}_{\gamma}(e\alpha f)$ for some $\gamma, \mu \in \Gamma$ such that $x\mu e\alpha f\gamma x = x$. Now $(f\gamma x\mu e)\alpha f\beta(f\gamma x\mu e) = f\gamma x\mu(e\alpha f)\gamma x\mu e = f\gamma x\mu e = f\gamma x\mu(e\alpha f)\gamma x\mu e = (f\gamma x\mu e)\alpha e\alpha(f\gamma x\mu e)$. Which implies that $f\gamma x\mu e \in W^{\alpha}_{\alpha}(e) \cap W^{\alpha}_{\beta}(f)$. Hence by (v), $W^{\delta}_{\alpha}(e) = W^{\delta}_{\beta}(f)$ for all $\delta \in \Gamma$. Since $e \in W^{\alpha}_{\alpha}(e)$ we have $e \in W^{\alpha}_{\beta}(f)$ and hence (i) follows.
(i) $\Rightarrow$ (vi) : Let $e \in E_{\alpha}$ and $f \in E_{\beta}$. Then $(e\alpha f)\beta(e\alpha f) = (e\alpha f\beta e)\alpha f = e\alpha f$. Thus $e\alpha f \in E_{\beta}$. For $a \in S$ we see that $V^{\beta}_{\alpha}(a) \subseteq W^{\beta}_{\alpha}(a)$ for some $\alpha, \beta \in \Gamma$. Now let $a' \in W^{\beta}_{\alpha}(a)$ i.e, $a'\beta a\alpha a' = a'$ and $a\alpha a' \in E_{\beta}$. Since a is regular, there exists $a^* \in V^{\delta}_{\gamma}(a)$ for some $\gamma, \delta \in \Gamma$. i.e, $a\gamma a^* \in E_{\delta}$. Now $a = a\gamma a^*\delta a = (a\gamma a^*)\delta(a\alpha a')\beta(a\gamma a^*)\delta a = a\alpha a'\beta a$. Thus $a' \in V^{\beta}_{\alpha}(a)$ and hence we have $W^{\beta}_{\alpha}(a) = V^{\beta}_{\alpha}(a)$.
(vi) $\Rightarrow$ (i) : Let $e \in E_{\alpha}$ and $f \in E_{\beta}$. Then $(e\alpha f)\beta e\alpha(e\alpha f) = (e\alpha f)\beta(e\alpha f) = e\alpha f$ and hence $e\alpha f \in W^{\beta}_{\alpha}(e) = V^{\beta}_{\alpha}(e)$. Thus we have $e\alpha f\beta e = e\alpha(e\alpha f)\beta e = e$. Hence the proof. □

4. Semidirect product of a semigroup and a Γ-semigroup

Let S be a semigroup and T be a Γ-semigroup. Let $End(T)$ denote the set of all endomorphisms on T i.e., the set of all mappings $f : T \to T$ satisfying $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in T$, $\alpha \in \Gamma$. Clearly $End(T)$ is a semigroup. Let $\phi : S \not\to End(T)$ be a given antimorphism i.e, $\phi(sr) = \phi(r)\phi(s)$ for all $r, s \in S$. If $s \in S$ and $t \in T$, we write t^s for $(\phi(s))(t)$ and $T^s = \{t^s : t \in T\}$. Let $S \times_{\phi} T = \{(s, t) : s \in S, t \in T\}$. We define $(s_1, t_1)\,\alpha(s_2, t_2) = (s_1 s_2, t_1^{s_2}\,\alpha t_2)$ for all $(s_i, t_i) \in S \times_{\phi} T$ and $\alpha \in \Gamma$. Then $S \times_{\phi} T$ is a Γ-semigroup. This Γ-semigroup $S \times_{\phi} T$ is called the semidirect product of the semigroup S and the Γ-semigroup T. In [6] we have studied such type of semidirect product. We recall the following lemmas from [6].

Lemma 4.1. *Let $S \times_{\phi} T$ be a semidirect product of a semigroup S and a Γ-semigroup T. Then*

(i) $(t\alpha u)^s = t^s\alpha u^s$ *for all* $s \in S$, $t, u \in T$ *and* $\alpha \in \Gamma$.

(ii) $(t^s)^r = (t)^{sr}$ *for all* $s, r \in S$ *and* $t \in T$.

Lemma 4.2. *Let $S \times_{\phi} T$ be a semidirect product of a semigroup S and a Γ-semigroup*

T. Then T^x is a Γ-semigroup for all $x \in S$.

We now give the following characterization.

Theorem 4.3. *Let $S \times_\phi T$ be a semidirect product of a semigroup S and a Γ-semigroup T. Then $S \times_\phi T$ is E-inversive if and only if for all $s \in S, t \in T$ there exists $s' \in W(s)$ such that $t^{s's}$ is an E-inversive element of a Γ-semigroup $T^{s's} = \{t^{s's} : t \in T\}$. If S is an E-inversive semigroup and T is an E-inversive Γ-semigroup, then every semidirect product of S and T is E-inversive Γ-semigroup.*

Proof. Let $S \times_\phi T$ be E-inversive Γ-semigroup. Let $s \in S$ and $t \in T$. Then $(s,t) \in S \times_\phi T$. Since $S \times_\phi T$ is E-inversive, $W_\alpha^\beta((s,t)) \neq \phi$ for some α, $\beta \in \Gamma$. Let $(s',t') \in W_\alpha^\beta((s,t))$. Then $(s',t')\beta(s,t)\alpha(s',t') = (s',t')$. i.e, $(s'ss', t'^{ss'}\beta t^{s'}\alpha t') = (s',t')$. Thus $s'ss' = s'$ and $(t')^{ss'}\beta t^{s'}\alpha t' = t'$. Thus $s' \in W(s)$ and hence S is E-inversive semigroup. Now since $(t')^{ss'}\beta t^{s'}\alpha t' = t'$, we have $((t')^{ss'}\beta t^{s'}\alpha t')^{ss's} = (t')^{ss's}$ i.e, $(t')^{ss's}\beta t^{s's}\alpha(t')^{ss's} = (t')^{ss's}$ which implies $(t'^s)^{s's}\beta(t)^{s's}\alpha(t'^s)^{s's} = (t'^s)^{s's}$. Hence $t^{s's}$ is an E-inversive element of the Γ-semigroup $T^{s's} = \{t^{s's} : t \in T\}$.

Conversely let the given condition hold. Let $(s,t) \in S \times_\phi T$. Then by the given condition we have $x \in W(s)$ and $u^{xs} \in W_\alpha^\beta(t^{xs})$ for some $x \in S$ and $u^{xs} \in T^{xs}$. Now $(x,u^x)\beta(s,t)\alpha(x,u^x) = (xsx, u^{xsx}\beta t^x\alpha u^x) = (x, u^{xsx}\beta t^{xsx}\alpha u^{xsx}) = (x, (u^{xs}\beta t^{xs}\alpha u^{xs})^x) = (x, u^{xsx}) = (x, u^x)$ Hence $(x,u^x) \in W_\alpha^\beta((s,t))$. Hence $S \times_\phi T$ is E-inversive Γ-semigroup.

Again let S be an E-inversive semigroup and T be an E-inversive Γ-semigroup. Let $t^x \in T^x$. Since T is E-inversive there exist $u \in T$, α, $\beta \in \Gamma$ such that $u \in W_\alpha^\beta(t)$. i.e., $u\beta t\alpha u = u$ which implies $u^x\beta t^x\alpha u^x = u^x$. Hence T^x is an E-inversive Γ-semigroup for all $x \in S$. Hence if S is an E-inversive semigroup and T be an E-inversive Γ-semigroup then for all $s \in S, t \in T$ there exists $s' \in W(s)$ such that $t^{s's}$ is an E-inversive element of a Γ-semigroup $T^{s's} = \{t^{s's} : t \in T\}$ and hence we conclude that $S \times_\phi T$ is E-inversive Γ-semigroup. □

Theorem 4.4. *Let $S \times_\phi T$ be a semidirect product of a semigroup S and a Γ-semigroup T. Then $S \times_\phi T$ is right E-Γ-semigroup if and only if S is an E-semigroup and $e, f \in E(S)$, $t, u \in T$ and α, $\beta \in \Gamma$ such that $t^e\alpha t = t$, $u^f\beta u = u$ imply that $t^{fef}\alpha u^{ef}\beta t^f\alpha u = t^f\alpha u$.*

Proof. Let $S \times_\phi T$ be a right E-Γ-semigroup and let $(s,t) \in S \times_\phi T$. Again let $e, f \in E(S)$, $t, u \in T$ and α, $\beta \in \Gamma$ such that $t^e\alpha t = t$, $u^f\beta u = u$. Then (e,t) is an α-idempotent and (f,u) is an β-idempotent. Since $S \times_\phi T$ is right E-Γ-semigroup we have $(e,t)\alpha(f,u) \in E_\beta$ and hence $(e,t)\alpha(f,u)\beta(e,t)\alpha(f,u) = (e,t)\alpha(f,u)$ i.e, $(efef, t^{fef}\alpha u^{ef}\beta t^f) = (ef, t^f\alpha u)$. Thus $ef \in E(S)$, i.e., S is an E-semigroup and $t^{fef}\alpha u^{ef}\beta t^f = t^f\alpha u$. Reversing the argument the converse follows. □

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References

[1] F. Catino, and M. M. Miccoli, *On semidirect products of semigroups*, Note di Mat., **9**(1989), 189-194.

[2] S. Chattopadhyay, *Right Orthodox* Γ*-semigroup*, Southeast Asian Bull. of Math., **29**(2005), 23-30.

[3] T. K. Dutta and S. Chattopadhyay, *On Unoformly Strongly Prime* Γ*-Semigroup*, Analale Stiintifice Ale Universitatii "AL. I. CUZA" Tomul LII, S.I, Mathematica, **2**(2006), 325 - 335.

[4] H. Mitsch, M. Petrich, *Basic properties of E-inversive semigroups*, Comm. Algebra, **28**(2000), 5169-5182.

[5] N. K. Saha, *On* Γ*-semigroup II* , Bull. Cal. Math. Soc., **79**(1987), 331-335.

[6] M. K. Sen, and S. Chattopadhyay, *Wreath Product of a semigroup and a* Γ*-semigroup*, Discussiones Mathematicae - General Algebra and Applications, Vol.28(2008), 161 - 178.

[7] M. K. Sen and N. K. Saha, *On* Γ*-semigroup I* , Bull. Cal. Math. Soc., **78**(1986), 181-186.

[8] A. Seth, *Rees's theorem for* Γ*-semigroup* , Bull. Cal. Math. Soc., **81**(1989), 217-226.

[9] Barbara Weipoltshammer, *On classes of E-inversive semigroups and semigroups whose idempotents form a subsemigroup*, Communications in Algebra, **32**(2004), 2929-2948.