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Some Results of QF Rings

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Abstract. Let R be a ring. We give some new characterizations of QF under the special annihilators condition. Some known results are obtained as corollaries.

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary. As usual, we use $J(R)$, $Z(_RR), Z(R_R), Soc(_RR)$ and $Soc(R_R)$ (briefly J, Z_l, Z_r, S_l, S_r)to indicate the Jacobson radical, the left singular ideal , right singular ideal , and the left socle and right socle of the ring R, respectively. The left and right annihilators of a subset X of R are denoted by $l(X)$ and $r(X)$, respectively. We use $N \leq_e M$ to indicate that N is an essential submodule of M.

Recall that a ring R called right mininjective[1] if every right R-homomorphic from any minimal right ideal of R into R is given by left multiplication by an element of R . A ring R is said to be right simple injective if every homomorphism from a right ideal of R to R with simple image can be given by left multiplication by an element of R [2].It is clear that right simple injective rings imply right mininjective rings.

A ring R is called quasi-Frobenius, briefly QF , if R is two-sided Artinian and two-sided self-injective ring, or equivalently, if R has the ACC on right or left annihilators and is right or left self-injective. The class of quasi-Frobenius rings is one of the most important classes of rings, which was introduced as a generalization of group algebras of a finite group over a field(see[3]). There are three open conjectures on QF rings, which have attracted many people to work on them. One of the three conjectures is the Faith-Menal conjecture. Faith's conjecture is an outstanding problem about quasi-Frobenius rings , which has been intensively investigated by various of authors(see[4] and [5] for more). There is a great deal of research devoted to improve this result by weakening the Artinian condition or the injectivity or both. Nicholson and Yousif[1] proved that any right and left mininjective ring, right Artinian ring is QF. In [6], Nicholson and Yousif extended the condition from Artinian ring to a semilocal ring with ACC on right annihilators

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such that $S_r \leq_e R_R$.

Because of these results, it is natural to ask whether these results are also correct when weaker condition of Noetherian ring or ACC on left annihilators. For this purpose, we will consider the following condition for a given ring R:

$(*)$ The ascending chain $r(S_1) \subseteq r(S_2S_1) \subseteq \cdots$ terminates for any sequence $\{S_1, S_2, \cdots\} \subseteq R$.

Obviously, Artinian rings, Noetherian rings and ring having ACC on left annihilators satisfy $(*)$.

First we show that if R is a right mininjective ring and R satisfies $(*)$ such that $S_r \leq_e R_R$, then R is semiprimary . From this, we have (1) if R is a left and right mininjective ring with $J^2 = 0$ and R satisfies $(*)$ such that $S_r \leq_e R_R$, then R is quasi-Frobenius; (2) if R is a right simple injective ring with $J^2 = 0$ and R satisfies $(*)$ such that $S_r \leq_e R_R$, then R is quasi-Frobenius. General background material can be found in Anderson and Fuller[7].

2. Ring of satisfies condition $(*)$

We start with the following lemma.

Lemma 2.1. *Let R be a ring satisfies $(*)$. Then the following hold:*
(1) *For any sequence $a_1, a_2, \cdots \in R$, there is a positive integer n such that $r(a_{n+1}) \cap a_n \cdots a_1R = 0$.*
(2) *If a right ideal K of R is right T-nilpotent, then K is nilpotent.*
(3) *Z_r is right nilpotent.*

Proof. (1) By hypothesis, there is a positive integer n such that $r(a_{n+1} \cdots a_1) = r(a_n \cdots a_1) = \cdots$. For any $t \in r(a_{n+1}) \cap a_n \cdots a_1R$, then there is $r \in R$ such that $t = a_n \cdots a_1 r$ and $r(a_{n+1})t = 0$ and so $a_{n+1} \cdots a_1 r = 0$,i.e., $r \in r(a_{n+1} \cdots a_1) = r(a_n \cdots a_1)$. Thus $t = 0$ and so $r(a_{n+1}) \cap a_n \cdots a_1R = 0$.
(2) Assume that K is not nilpotent, then $K^n \neq 0$ for any positive integer n. By hypotheses there exists a natraul number m such that $r(K^m) = r(K^{m+1}) = \cdots$. So there exists $0 \neq a_1 \in K^m$ such that $K^m a_1 \neq 0$. Therefore, $K^{2m}a_1 \neq 0$. Similarly, there exists $0 \neq a_2 \in K^m$ such that $K^m a_2 a_1 \neq 0$. Continuous this proceeding, there exists $a_1, a_2, \cdots \in K^m$ such that $a_n a_{n-1} \cdots a_1 \neq 0$, a contradiction. Thus, K is nilpotent.
(3) If there is a sequence $a_1, a_2, \cdots \in Z_r$ such that $a_n a_{n-1} \cdots a_1 \neq 0$ for any positive integer n, then $a_n a_{n-1} \cdots a_1 R \neq 0$. Since $r(a_{n+1}) \leq_e R_R$, $r(a_{n+1}) \cap a_n \cdots a_1 R \neq 0$. This is a contradiction by (2). □

Lemma 2.2([8]). *Let $a - aca$ is a regular element for all $a, c \in R$, then a is a regular element.*

Lemma 2.3. *If R is a right mininjective ring and satisfies $(*)$ such that $S_r \leq_e R_R$, then R is semiprimary.*

Proof. Since R is right mininjective, $S_r \subseteq S_l$. So $J \subseteq l(S_r)$. But $S_r \leq_e R_R$, so

$l(S_r) \subseteq Z_r$. Note that R satisfies $(*)$, then Z_r is nilpotent by lemma 2.1. Therefore $Z_r \subseteq J$ and so $J = l(S_r) = Z_r$ is nilpotent. Next we show that $\bar{R} = R/J$ is regular. If $\bar{a} \neq \bar{0}$,then there exists a nonzero right ideal K such that $r(a) \cap K = 0$ by $J = Z_r$. Since $S_r \leq_e R_R$, there exists a minimal right ideal $bR \subseteq K$ and So $r(a) \cap bR = 0$. Define $f : abR \to bR$ by $f(abr) = br$ for all $r \in R$. It is clear that f is well-defined. Since R is right mininjective and abR is a minimal right ideal, there exists $c \in R$ such that $f(abr) = cabr$ for all $r \in R$. Thus $b = cab$ and so $b \in r(a - aca) \backslash r(a)$. If $a - aca \in J$, then $\bar{a}$ is a regular element. If not, let $a_1 = a - aca$. In the same way we get $a_2 = a_1 - a_1c_1a_1$ for some $c_1 \in R$ and $r(a_1) \subset r(a_2)$. Repeating the above-mentioned process, we get astrictly ascending chain

$$r(a_1) \subset r(a_2) \subset r(a_3) \subset \cdots,$$

where $a_{i+1} = a_i - a_ic_ia_i$ for some $c_i \in R, i = 1, 2, \cdots$. Let $b_1 = a_1, b_2 = 1 - a_1c_1, b_3 = 1 - a_2c_2, \cdots, b_{i+1} = 1 - a_ic_i, \cdots$, then $a_1 = b_1, a_2 = b_2b_1, a_3 = b_3b_2b_1, \cdots, a_{i+1} = b_{i+1}b_i \cdots b_2b_1, \cdots$, whence we have the following strictly ascending chain

$$r(b_1) \subset r(b_2b_1) \subset r(b_3b_2b_1) \subset \cdots,$$

which contradicts the hypothesis. So there exists a positive integer n such that $a_{n+1} \in J$, then $\bar{a}$ is a regular element of $\bar{R}$ by lemma 2.2. Therefore $\bar{R}$ is von Neumann regular ring. Finally, we show that R is semilocal. By the proof of [9, Theorem 3.4], $\bar{R}$ is semisimple Artinian. Therefore R is semilocal. From above R is semilocal and J is nilpotent. So R is semiprimary. □

Theorem 2.1. *Let R be a left and right mininjective ring with $J^2 = 0$ and R satisfies $(*)$ in which $S_r \leq_e R_R$. Then R is quasi-Frobenius.*

Proof. By Lemma 2.3, R is semiprimary. Thus, R is QF by [6,Theorem 3.40] □

Remark 1. None of the conditions is superfluous in Theorem 2.1. For example, the ring of integers Z is a two-sided mininjective and noetherian ring with $S_r = 0$. But it is not a QF ring. Example 2.5 ([6]) indicate that a right mininjective and left artinian ring R is not left mininjective. In fact, R has ACC on both right and left annihilators and so S_r, S_l are both essential. It is well known that left PF is not right PF , and so it is not QF. Indeed, Such ring satisfies the conditions except the chain condition in Theorem 2.1.

Moreover$J^2 = 0$ is needed. For example, If R is a field, the ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ is a right and left Artinian ring with $J^2 = 0$, but R is neither right or left mininjective. And Example 2.6([6]) indicate that a commutative, local, mininjective ring with $J^2 = 0$ and satisfies $(*)$ such that $S_r \leq_e R_R$ that is not QF.

Corollary 2.1([6,Theorem 3.31])**.** *Let R be a semilocal, left and right mininjective ring with ACC on right annihilators in which $S_r \leq_e R_R$. Then R is quasi-Frobenius.*

Theorem 2.2. *The following are equivalent for a ring R:*
(1) *R is right and left mininjective , right Noetherian, and $S_r \leq_e R_R$.*
(2) *R is right and left mininjective with $J^2 = 0$, right finitely cogenerated, and R satisfies $(*)$.*
(3) *R is quasi-Frobenius.*

Proof. (1)⇒(2) and (3)⇒(2) are clear.
(2)⇒(3) By lemma 2.1, Z_r is right nilpotent. and so $Z_r \subseteq J$. Thus R is semilocal and $J = Z_r$ by [6, Lemma 8.1]. However, R is semiprimary and so R is QF by Theorem 2.1. □

Since right simple injective ring is right mininjective ring, We can extend Theorem 2.1 to the following theorem:

Theorem 2.3. *Let R be a right simple injective ring and satisfies $(*)$ in which $S_r \leq_e R_R$. Then R is quasi-Frobenius.*

Proof. By Lemma 2.3, R is semiprimary. Then R is left P-injective by [6, Theorem 6.16], which implies that R is left mininjective. Thus R is QF by Theorem 2.1. □

Lemma 2.4. *Let R be right P-injective and satisfies $(*)$. Then the following hold:*
(1) *R is right perfect ring.*
(2) *If J is right T-nilpotent, then J is nilpotent.*
(3) *R is left perfect ring.*

Proof. (1) Assume that for all $a_i \in R$,$i = 1, 2, \cdots$, $Ra_1 \supseteq Ra_2a_1 \supseteq \cdots$ is descending chain of principal left ideals of R. Then$r(a_1) \subseteq r(a_2a_1) \subseteq \cdots$. Since R satisfies $(*)$, then there exists a natraul number n such that $r(a_n \cdots a_2a_1) = r(a_{n+1}a_n \cdots a_2a_1)$. Since R is right P-injective ring, $Ra_n \cdots a_2a_1 = l(r(a_n \cdots a_2a_1)) = l(r(a_{n+1} \cdots a_2a_1)) = Ra_{n+1} \cdots a_2a_1$. Hence R is right perfect ring .
(2)By hypotheses there exists a natraul number n such that $r(J^n) = r(J^{n+1}) = \cdots$. Assume J is not nilpotent, then$J^{n+1} \neq 0$ for any $n \in Z^+$. So there exists $x_1 \in J$ such that $J^n x_1 \neq 0$. Therefore, $J^{n+1}x_1 \neq 0$. Similarly, there exists $x_2 \in J$ such that $J^{n+1}x_2x_1 \neq 0$. Repeaing the process to obtain $x_1, x_2, \cdots \in J$ such that $x_m \cdots x_1 \in J \setminus r(J^n)$, a contradiction. Thus, J is nilpotent.
(3) By(2), R is right perfect ring. Hence, R/J is semisimple and J is right T-nilpotent. By(3), J is nilpotent. So R is left perfect ring. □

Theorem 2.4. *Let R be a right P-injective and satisfies $(*)$ with J is a right R-module which have finite Goldie dimension, then R is Artinian ring.*

Proof. By lemma 2.4, R is left and right perfect ring, J is nilpotent. So R/J is semisimple Artinian. Let $J^n = 0$, $J^{n-1} \neq 0$, then J^{n-1} is semisimple R/J-module. By hypotheses, Goldie dimension of J^{n-1} is finite. Hence J^{n-1} is semisimple Artinian R/J-module. Since $J{\cdot}J^{n-2}/J^{n-1} = \bar{0}$, J^{n-2}/J^{n-1} is semisimple R/J-module and Goldie dimension is finite. Thus, J^{n-2}/J^{n-1} is semisimple Artinian. But R/J-submodule of J^{n-1}, J^{n-2}/J^{n-1} coincide with R-submodule of J^{n-1}, J^{n-2}/J^{n-1}.

Hence J^{n-1} and J^{n-2}/J^{n-1} Artinian R-module. Since short sequence of R-module

$$0 \to J^{n-1} \to J^{n-2} \to J^{n-2}/J^{n-1} \to 0$$

is exact. Thus J^{n-2} is Artinian R-module.

Since $J \cdot J^{n-3}/J^{n-2} = \bar{0}$, J^{n-3}/J^{n-2} is semisimple R/J-module and Goldie dimension is finite. Thus, J^{n-3}/J^{n-2} is R/J-module and so is Artinian R-module. By short sequence of R-module

$$0 \to J^{n-2} \to J^{n-3} \to J^{n-3}/J^{n-2} \to 0$$

is exact. Thus J^{n-2} is Artinian R-module. Continuous this proceeding, we get that J is Artinian R-module. Therefore , By short sequence of R-module

$$0 \to J \to R \to R/J \to 0$$

is exact. Hence R is Artinian ring. □

Theorem 2.5. *The following are equivalent for a ring R:*
(1) *R is right simple injective , right Noether ring with $S_r \leq_e R_R$.*
(2) *R is right simple injective, right finite dimensional, and satisfies $(*)$ with $S_r \leq_e R_R$.*
(3) *R is quasi-Frobenius.*

Proof. (3)⇒(1)⇒(2) are clear.
(2)⇒(3) By lemma 2.3, R is semiprimary. Since R is right simple injective , R is right self-injective. Thus, R is Artinian ring by Theorem 2.4. So R is quasi-Frobenius.
In general, R is a right P-injective and satisfies $(*)$ need not be QF(see [10]). In the next theorem we show that a right 2-injective and satisfies $(*)$ with J is a right R-module which have finite Goldie dimension is quasi-Frobenius. □

Theorem 2.6. *Let R be a right 2-injective and satisfies $(*)$ with J is a right R-module which have finite Goldie dimension. Then R is quasi-Frobenius.*

Proof. By Theorem 2.4, R is left Artinian ring. Then R has ACC on right annihilators. Thus R is QF by [10, Corollary 3]. □

Corollary 2.2. *R is right FP-injective ring satisfies $(*)$ and J is a right R-module which have finite Goldie dimension, then R is QF.*

A ring R is called left min-CS, if every minimal left ideal is essential in a direct summand of R_R.

Theorem 2.7. *Let R be a left min-CS, left P-injective and satisfies $(*)$ with J is a right R-module which have finite Goldie dimension. Then R is quasi-Frobenius.*

Proof. By Theorem 2.4, R is right Artinian, then R is a left GPF ring. So R is left $Kasch$ and $S_r = S_l = S$ by [6, Theorem 5.31], then $Soc(Re)$ is simple for each local idempotent $e \in R$ by [6, Lemma 4.5]. Thus $(eR/eJ) \cong l(J)e = S_re = Soc(Re)$ is simple for every local idempotent $e \in R$. Since R is semiperfect, each simple right R-module is isomorphic to eR/eJ for some local idempotent $e \in R$ by [7, Theorem 27.10]. Hence R is right mininjective by [6, Theorem 2.29]. From above, R is a two-sided mininjective and right Artinian ring, then it is QF by [6, Theorem 3.31]. □

Corollary 2.3([11, Theorem 2.21]). *If R is right Noetherian left CS and left P-injective, then R is QF.* vspace0.2in

References

[1] W. K. Nicholson and M. F. Yousif, *Mininjective rings.*, J. Algebra 187(1997), 548-578.

[2] W. K. Nicholson, J. K. Park and M. F. Yousif, *Extensions of simple-injective rings.*, Comm. Algebra, **28(10)**(2000), 4665-4675.

[3] T. Nakayama, *On Frobenius Rings. II*, Ann. math, **42(1)**(1941), 1-21.

[4] C. Faith and D. V. huyhn, *When self-injective rings are QF: A report on a problem*, Journal of Algebra and Its Application, **1(1)**(2002), 75-105.

[5] W. K. Nicholson, and M. F. Yousif, On Quasi-Frobenius Rings, In: Gray F. Birkenmeier, Jae Keol Park and yong Soo Park, International Symposium on Ring Theory, Birkhäuser, Boston. basel. Berlin, 1999.

[6] W.K. Nicholson, and M.F. Yousif, Quasi-Frobenius Rings. Cambridge Tracts in Mathematics 158, Cambridge University Press, 2003.

[7] F. W. Anderson and K. R. Futter, Rings and Modules Categories. New York, Springer-Verlag, 1974.

[8] K. R. Goodearl, Von Neumann regular rings. London, Pitman, 1979.

[9] Chen J. and Ding N., *On general principally injective rings.*, Comm in Algebra, **27**(1999), 2097-2116.

[10] E. A. Rutter, *Rings with the principal extension property.*, Comm. Algebra, **3(3)**(1975), 203-212.

[11] Chen. J., Li. W., *On artiness of right CF rings.*, Comm. Algebra 32(11),(2004), 4485-4494.