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On Compact-covering Images of Locally Separable Metric Spaces

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Abstract. In this paper, we give the internal characterizations of compact-covering s-(resp., π-)images of locally separable metric spaces. As applications of these results, we obtain characterizations of compact-covering quotient s-(resp., π-)images of locally separable metric spaces.

1. Introduction

Finding the internal characterizations of certain images of metric spaces is a considerable interest in general topology. In the past, many nice results have been obtained [6], [11], [12], [17], [18]. Recently, many topologists were engaged in research of internal characterizations of images of locally separable metric spaces, and some noteworthy results were shown. In [12], S. Lin, C. Liu, and M. Dai gave a characterization of quotient s-images of locally separable metric spaces. After that, S. Lin, and P. Yan characterized sequence-covering s-images of locally separable metric spaces in [13]; Y. Ikeda, C. Liu and Y. Tanaka characterized quotient compact images of locally separable metric spaces in [7]; and Y. Ge characterized pseudo-sequence-covering compact images of locally separable metric spaces in [5]. In a personal communication, the first author of [12] and [13] informs that characterizations on compact-covering s-images and compact-covering π-images still have no answer. Thus, it is natural to rise the following question.

Question 1.1. How are compact-covering s-(resp., π-)images of locally separable metric spaces characterized?

In this paper, we give the internal characterizations of compact-covering s-(resp., π-)images of locally separable metric spaces. As applications of these results, we obtain a characterization of compact-covering quotient s-(resp., π-)images of locally separable metric spaces.

Throughout this paper, all spaces are assumed to be regular and T_1, all mappings are assumed continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $f : X \longrightarrow Y$ be a mapping, $x \in X$, and $\mathcal{P}$ be a family of subsets of X, we

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denote $st(x,\mathcal{P}) = \bigcup\{P \in \mathcal{P} : x \in P\}$, $\bigcup\mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, $\bigcap\mathcal{P} = \bigcap\{P : P \in \mathcal{P}\}$, and $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$.

Definition 1.2. Let $\mathcal{P}$ be a family of subsets of a space X, and K be a subset of X.
(1) $\mathcal{P}$ is a *cover for* K *in* X, if $K \subset \bigcup\mathcal{P}$. When $K = X$, a cover for K in X is a *cover of* X [3].
(2) For each $x \in X$, $\mathcal{P}$ is a *network at* x *in* X [15], if $x \in \bigcap\mathcal{P}$ and if $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.
$\mathcal{P}$ is a *network for* X [15], if $\{P \in \mathcal{P} : x \in P\}$ is a network at x in X for every $x \in X$.
(3) $\mathcal{P}$ is a *cfp-cover for* K *in* X, if for each compact subset H of K, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup\{C_F : F \in \mathcal{F}\}$, where C_F is closed and $C_F \subset F$ for every $F \in \mathcal{F}$. Note that such a $\mathcal{F}$ is a *full cover* in the sense of [2]. When $K = X$, a cfp-cover for K in X is a *cfp-cover for* X [20].
(4) $\mathcal{P}$ is a *cfp-network for* K *in* X, if for each compact subset H of K satisfying $H \subset U$ with U open in X, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup\{C_F : F \in \mathcal{F}\} \subset \bigcup\mathcal{F} \subset U$, where C_F is closed and $C_F \subset F$ for every $F \in \mathcal{F}$. Note that a cfp-network $\mathcal{P}$ for K in X is a family to have *property cc for* K [14], and if $K = X$, then $\mathcal{P}$ is a *strong* k*-network* for X in the sense of [2].
It is clear that if $\mathcal{P}$ is a cover (resp., cfp-cover, cfp-network) for X, then $\mathcal{P}$ is a cover (resp., cfp-cover, cfp-network) for K in X.
(5) $\mathcal{P}$ is *point-countable* [6], if every point of X meets at most countably many members of $\mathcal{P}$.

Definition 1.3. Let $f : X \longrightarrow Y$ be a mapping.
(1) f is a *compact-covering* mapping [16], if every compact subset of Y is the image of some compact subset of X.
(2) f is a *pseudo-sequence-covering* mapping [7], if every convergent sequence of Y is the image of some compact subset of X.
(3) f is a *pseudo-open* mapping [1], if $y \in \text{int} f(U)$ whenever $f^{-1}(y) \subset U$ with U open in X.
(4) f is a π*-mapping* [1], if for every $y \in Y$ and for every neighborhood U of y in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$, where X is a metric space with a metric d.
(5) f is an s*-mapping* [1], if $f^{-1}(y)$ is separable for every $y \in Y$.

Definition 1.4. Let X be a space.
(1) X is a *sequential* space [4], if a subset A of X is closed if and only if any convergent sequence in A has a limit point in A.
(2) X is a *Fréchet* space [4], if for each $x \in \overline{A}$, there exists a sequence in A converging to x.

For terms which are not defined here, please refer to [3] and [18].

2. Results

In 1960, V. Ponomarev proved that every first-countable space is precisely an open image of some Baire zero-dimension metric space [3, 4.2 D]. The Ponomarev's method has been generalized [14], and plays a very important role in characterizations of images of metric spaces. We shall use the above method to characterize compact-covering s-images of locally separable metric spaces.

Definition2.1. Let $\mathcal{P}$ be a network for a space X. Assume that there exists a countable network $\mathcal{P}_x \subset \mathcal{P}$ at x in X for every $x \in X$. Put $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$. For every $n \in \mathbb{N}$, put $\Lambda_n = \Lambda$ and endowed Λ_n a discrete topology. Put

$$M = \Big\{a = (\alpha_n) \in \prod_{n \in \mathbb{N}} \Lambda_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x_a \text{ in } X\Big\}.$$

Then M, which is a subspace of the product space $\prod_{n\in\mathbb{N}} \Lambda_n$, is a metric space and each point x_a is unique for every $a \in M$. Define $f : M \longrightarrow X$ by $f(a) = x_a$, then f is a mapping, and $(f, M, X, \mathcal{P})$ is a *Ponomarev-system* [14]. Note that under $\mathcal{P}$ being a point-countable network for X, the Ponomarev-system $(f, M, X, \mathcal{P})$ exists.

It is well known that cfp-networks are preserved by compact-covering mappings. We shall strengthen this result on preservations of cfp-covers and cfp-networks for a compact subset without the assumption of the compact-covering property.

Lemma 2.2. *Let $f : X \longrightarrow Y$ be a mapping.*

(1) *If $\mathcal{P}$ is a cfp-cover for a compact set K in X, then $f(\mathcal{P})$ is a cfp-cover for $f(K)$ in Y.*

(2) *If $\mathcal{P}$ is a cfp-network for a compact set K in X, then $f(\mathcal{P})$ is a cfp-network for $f(K)$ in Y.*

Proof. (1). Let H be a compact subset of $f(K)$. Then $L = f^{-1}(H) \cap K$ is a compact subset of K satisfying $f(L) = H$. Since $\mathcal{P}$ is a cfp-cover for K in X, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $L \subset \bigcup\{C_F : F \in \mathcal{F}\}$, where $C_F \subset F$, and C_F is closed for every $F \in \mathcal{F}$. Because L is compact, every C_F can be chosen compact. It implies that every $f(C_F)$ is closed (in fact, every $f(C_F)$ is compact), and $f(C_F) \subset f(F)$. We get that $H = f(L) \subset \bigcup\{f(C_F) : F \in \mathcal{F}\}$, where $f(\mathcal{F})$ is a finite subfamily of $f(\mathcal{P})$. Then $f(\mathcal{P})$ is a cfp-cover for $f(K)$ in Y.

(2). Similar to the proof of (1). □

Now, we characterize compact-covering s-images of locally separable metric spaces as follows.

Theorem 2.3. *The following are equivalent for a space X.*

(1) *X is a compact-covering s-image of a locally separable metric space,*

(2) *X has a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ satisfying that each X_α has a countable network $\mathcal{P}_\alpha$, and each compact subset K of X has a finite compact cover $\{K_\alpha : \alpha \in \Lambda_K\}$ such that, for each $\alpha \in \Lambda_K$, $\mathcal{P}_\alpha$ is a cfp-network for K_α in X_α.*

Proof. (1) $\Rightarrow$ (2). Let $f : M \longrightarrow X$ be a compact-covering s-mapping from a locally separable metric space M onto X. Since M is a locally separable metric space, $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$, where each M_α is a separable metric space by [3, 4.4.F]. For each $\alpha \in \Lambda$, let $\mathcal{B}_\alpha$ be a countable base of M_α, and put $X_\alpha = f(M_\alpha)$, $\mathcal{P}_\alpha = f(\mathcal{B}_\alpha)$. Then $\{X_\alpha : \alpha \in \Lambda\}$ is a point-countable cover for X, and each $\mathcal{P}_\alpha$ is a countable network for X_α.
Let K be a compact subset of X. Since f is compact-covering, $K = f(L)$ for some compact subset L of M. Because L is a compact subset of M, $\Lambda_K = \{\alpha \in \Lambda : L \cap M_\alpha \neq \emptyset\}$ is finite. For each $\alpha \in \Lambda_K$, put $L_\alpha = L \cap M_\alpha$, then L_α is compact. Denote $K_\alpha = f(L_\alpha)$, we get that $\{K_\alpha : \alpha \in \Lambda_K\}$ is a finite compact cover for K. By [2, Claim 4.2], each $\mathcal{B}_\alpha$ is a cfp-network for M_α. Then $\mathcal{B}_\alpha$ is a cfp-network for L_α in M_α. It follows from Lemma 2.2 that, for each $\alpha \in \Lambda_K$, $\mathcal{P}_\alpha$ is a cfp-network for K_α in X_α.

(2) $\Rightarrow$ (1). For each $\alpha \in \Lambda$ and $n \in \mathbb{N}$, put $\mathcal{P}_\alpha = \{P_\beta : \beta \in \Gamma_\alpha\}$, and denote by $\Gamma_{\alpha,n}$ the countable set Γ_α endowed with the discrete topology. Put

$$M_\alpha = \Big\{b_\alpha = (\beta_{\alpha,n}) \in \prod_{n\in\mathbb{N}} \Gamma_{\alpha,n} : \{P_{\beta_{\alpha,n}} : n \in \mathbb{N}\} \text{ forms a network at some point } x_{b_\alpha} \text{ in } X_\alpha\Big\}.$$

Then M_α, which is a subspace of the product space $\prod_{n\in\mathbb{N}} \Gamma_{\alpha,n}$, is a metric space and x_{b_α} is unique for each $b_\alpha \in M_\alpha$. Define $f_\alpha : M_\alpha \longrightarrow X_\alpha$ by choosing $f_\alpha(b_\alpha) = x_{b_\alpha}$. Then the Ponomarev-system $(f_\alpha, M_\alpha, X_\alpha, \mathcal{P}_\alpha)$ exists. Put $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$. Since every $\mathcal{P}_\alpha$ is countable, M_α is a separable metric space. Then M is a locally separable metric space. Define $f : M \longrightarrow X$ by choosing $f(b_\alpha) = f_\alpha(b_\alpha)$ for every $b_\alpha \in M_\alpha$. It is easy to check that f is continuous and onto.
(a) f is an s-mapping.
For each $x \in X$, since $\{X_\alpha : \alpha \in \Lambda\}$ is a point-countable cover for X, $\Lambda_x = \{\alpha \in \Lambda : x \in X_\alpha\}$ is countable. Note that $\Gamma_{\alpha,n}$ is countable for each $n \in \mathbb{N}$, M_α is a separable metric space. Then $f_\alpha^{-1}(x)$ is a separable subset of M_α for each $\alpha \in \Lambda_x$. Hence $f^{-1}(x) = \bigcup\{f_\alpha^{-1}(x) : \alpha \in \Lambda_x\}$ is a separable subset of M. It implies that f is an s-mapping.
(b) f is compact-covering.
Let K be a compact subset of X. Then K has a finite compact cover $\{K_\alpha : \alpha \in \Lambda_K\}$ such that, for each $\alpha \in \Lambda_K$, $\mathcal{P}_\alpha$ is a cfp-network for K_α in X_α. It follows from [14, Theorem 2] that there exists a compact subset L_α of M_α satisfying $f_\alpha(L_\alpha) = K_\alpha$. Put $L = \bigcup\{L_\alpha : \alpha \in \Lambda_K\}$, then L is a compact subset of M satisfying $f(L) = K$. It implies that f is compact-covering. □

By Theorem 2.3, we get a characterization of compact-covering quotient s-

images of locally separable metric spaces as follows.

Corollary 2.4. *The following are equivalent for a space X.*

(1) *X is a compact-covering quotient (resp., pseudo-open) s-image of a locally separable metric space,*

(2) *X is a sequential (resp., Fréchet) space with a point-countable cover $\{X_\alpha : \alpha \in \Lambda\}$ satisfying that each X_α has a countable network $\mathcal{P}_\alpha$, and each compact subset K of X has a finite compact cover $\{K_\alpha : \alpha \in \Lambda_K\}$ such that, for each $\alpha \in \Lambda_K$, $\mathcal{P}_\alpha$ is a cfp-network for K_α in X_α.*

Proof. (1) $\Rightarrow$ (2). By Theorem 2.3, it is sufficient to prove that X is a sequential (resp., Fréchet) space. This is obvious by [3, 2.4.G].

(2) $\Rightarrow$ (1). It follows from Theorem 2.3 that X is a compact-covering s-image of a locally separable metric space under the mapping f. We get that f is quotient (resp., pseudo-open) by [5, Remark 1.7], and [10, Lemma 2.1]. Then X is a compact-covering quotient (resp., pseudo-open) s-image of a locally separable metric space. □

Definition 2.5. For each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be a cover for X. $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *refinement sequence* for X, if $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$ for each $n \in \mathbb{N}$. A refinement sequence for X is a *refinement* of X in the sense of [5].

Definition 2.6. Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a refinement sequence for X. $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *point-star network* for X, if $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x in X for every $x \in X$. Note that a point-star network is used without the assumption of a refinement sequence in [14], and $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *σ-strong network* for X in the sense of [7].

In Section 2 of [14], S. Lin and P. Yan extended the Ponomarev-system to a sequence of covers for a space as follows.

Definition 2.7. Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a point-star network for a space X. For every $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$, and A_n is endowed with discrete topology. Put

$$M = \Big\{a = (\alpha_n) \in \prod_{n \in \mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x_a \text{ in } X\Big\}.$$

Then M, which is a subspace of the product space $\prod_{n \in \mathbb{N}} A_n$, is a metric space with metric d described as follows.

Let $a = (\alpha_n), b = (\beta_n) \in M$. If $a = b$, then $d(a, b) = 0$. If $a \neq b$, then $d(a, b) = 1/(\min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\})$.

Define $f : M \longrightarrow X$ by choosing $f(a) = x_a$, then f is a mapping, and $(f, M, X, \{\mathcal{P}_n\})$ is a *Ponomarev-system* [19]. Note that without the assumption of a refinement sequence in the notion of point-star networks, then $(f, M, X, \{\mathcal{P}_n\})$ is a *Ponomarev-system* in the sense of [14].

Now, we characterize compact-covering π-images of locally separable metric spaces as follows.

Theorem 2.8. *The following are equivalent for a space X.*

(1) *X is a compact-covering π-image of a locally separable metric space,*

(2) *X has a cover $\{X_\lambda : \lambda \in \Lambda\}$, where each X_λ has a refinement sequence of countable covers $\{\mathcal{P}_{\lambda,n}\}_{n\in\mathbb{N}}$ satisfying the following:*

  (a) *$\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ is a point-star network of X, where $\mathcal{P}_n = \bigcup_{\lambda\in\Lambda} \mathcal{P}_{\lambda,n}$ for each $n \in \mathbb{N}$,*

  (b) *For every compact subset K of X, there exists a finite subset Λ_K of Λ such that K has a finite compact cover $\{K_\lambda : \lambda \in \Lambda_K\}$, and for each $\lambda \in \Lambda_K$ and $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a cfp-cover for K_λ in X_λ.*

Proof. (1) $\Rightarrow$ (2). Let $f : M \longrightarrow X$ be a compact-covering π-mapping from a locally separable metric space M with metric d onto X. Since M is a locally separable metric space, $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, where each M_λ is a separable metric space by [3, 4.4.F]. For each $\lambda \in \Lambda$, denote $f_\lambda = f|_{M_\lambda}$, $X_\lambda = f_\lambda(M_\lambda)$, and $M_\lambda = \overline{D_\lambda}$, where D_λ is a countable dense subset of M_λ.
For each $a \in M_\lambda$ and $n \in \mathbb{N}$, put $B(a, 1/n) = \{b \in M_\lambda : d(a, b) < 1/n\}$, $\mathcal{B}_{\lambda,n} = \{B(a, 1/n) : a \in D_\lambda\}$, and $\mathcal{P}_{\lambda,n} = f_\lambda(\mathcal{B}_{\lambda,n})$. It is clear that $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a refinement sequence of countable covers for X_λ.
(a) $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ is a point-star network for X.
Since $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a refinement sequence for X_λ for each $\lambda \in \Lambda$, $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a refinement sequence for X.
For each $x \in U$ with U open in X. Since f is a π-mapping, $d(f^{-1}(x), M - f^{-1}(U)) > 2/n$ for some $n \in \mathbb{N}$. Then, for each $\lambda \in \Lambda$ with $x \in X_\lambda$, we get $d(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) > 2/n$, where $U_\lambda = U \cap X_\lambda$. Since $\mathcal{P}_{\lambda,n}$ is a cover for X_λ, there exists $f_\lambda(B(a, 1/n)) \in \mathcal{P}_{\lambda,n}$ such that $x \in f(B(a, 1/n))$ for some $a \in D_\lambda$. We shall prove that $B(a, 1/n) \subset f_\lambda^{-1}(U_\lambda)$. In fact, if $B(a, 1/n) \not\subset f_\lambda^{-1}(U_\lambda)$, then there exists $b \in B(a, 1/n) - f_\lambda^{-1}(U_\lambda)$. Since $f_\lambda^{-1}(x) \cap B(a, 1/n) \neq \emptyset$, there exists $c \in f_\lambda^{-1}(x) \cap B(a, 1/n)$. Then $d(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \leq d(c, b) \leq d(c, a) + d(a, b) < 2/n$. It is a contradiction. So $B(a, 1/n) \subset f_\lambda^{-1}(U_\lambda)$, thus $f_\lambda(B(a, 1/n)) \subset U_\lambda$. Then $st(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$, and hence $\bigcup\{st(x, \mathcal{P}_{\lambda,n}) : \lambda \in \Lambda$ with $x \in X_\lambda\} \subset U$. It implies that $st(x, \mathcal{P}_n) \subset U$.
Hence, $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ is a point-star network for X.
(b) For each compact subset K of X, since f is compact-covering, $K = f(L)$ for some compact subset L of M. By compactness of L, $L_\lambda = L \cap M_\lambda$ is compact and $\Lambda_K = \{\lambda \in \Lambda : L_\lambda \neq \emptyset\}$ is finite. For each $\lambda \in \Lambda_K$, put $K_\lambda = f(L_\lambda)$, then $\{K_\lambda : \lambda \in \Lambda_K\}$ is a finite compact cover for K. For each $n \in \mathbb{N}$, since $\mathcal{B}_{\lambda,n}$ is a cfp-cover for L_λ in M_λ, $\mathcal{P}_{\lambda,n}$ is a cfp-cover for K_λ in X_λ by Lemma 2.2.

(2) $\Rightarrow$ (1). For each $\lambda \in \Lambda$, let $x \in U_\lambda$ with U_λ open in X_λ. We get that $U_\lambda = U \cap X_\lambda$ with some U open in X. Since $st(x, \mathcal{P}_n) \subset U$ for some $n \in \mathbb{N}$, $st(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$.

It implies that $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ. Then the Ponomarev-system $(f_\lambda, M_\lambda, X_\lambda, \{\mathcal{P}_{\lambda,n}\})$ exists. Since each $\mathcal{P}_{\lambda,n}$ is countable, M_λ is a separable metric space with metric d_λ described as follows. For $a = (\alpha_n), b = (\beta_n) \in M_\lambda$, if $a = b$, then $d_\lambda(a,b) = 0$, and if $a \neq b$, then $d_\lambda(a,b) = 1/(\min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\})$.
Put $M = \oplus_{\lambda\in\Lambda} M_\lambda$ and define $f : M \longrightarrow X$ by choosing $f(a) = f_\lambda(a)$ for every $a \in M_\lambda$ with some $\lambda \in \Lambda$. Then f is a mapping and M is a locally separable metric space with metric d as follows. For $a, b \in M$, if $a, b \in M_\lambda$ for some $\lambda \in \Lambda$, then $d(a,b) = d_\lambda(a,b)$, and otherwise, $d(a,b) = 1$.
(a) f is a π-mapping.
Let $x \in U$ with U open in X, then $st(x, \mathcal{P}_n) \subset U$ for some $n \in \mathbb{N}$. So, for each $\lambda \in \Lambda$ with $x \in X_\lambda$, we get $st(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$, where $U_\lambda = U \cap X_\lambda$. It is implies that $d_\lambda(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \geq 1/n$. In fact, if $a = (\alpha_k) \in M_\lambda$ such that $d_\lambda(f_\lambda^{-1}(x), a) < 1/n$, then there exists $b = (\beta_k) \in f_\lambda^{-1}(x)$ such that $d_\lambda(a,b) < 1/n$. So $\alpha_k = \beta_k$ if $k \leq n$. Note that $x \in P_{\beta_n} \subset st(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$. Then $f_\lambda(a) \in P_{\alpha_n} = P_{\beta_n} \subset st(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$. Hence $a \in f_\lambda^{-1}(U_\lambda)$. It implies that $d_\lambda(f_\lambda^{-1}(x), a) \geq 1/n$ if $a \in M_\lambda - f_\lambda^{-1}(U_\lambda)$. So $d_\lambda(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \geq 1/n$. Therefore

$$\begin{aligned}
& d(f^{-1}(x), M - f^{-1}(U)) \\
= \; & \inf\{d(a,b) : a \in f^{-1}(x), b \in M - f^{-1}(U)\} \\
= \; & \min\left\{1, \inf\{d_\lambda(a,b) : a \in f_\lambda^{-1}(x), b \in M_\lambda - f_\lambda^{-1}(U_\lambda), \lambda \in \Lambda\}\right\} \\
\geq \; & 1/n > 0.
\end{aligned}$$

It implies that f is a π-mapping.
(b) f is compact-covering.
For each compact subset K of X, there exists a finite subset Λ_K of Λ such that K has a finite compact cover $\{K_\lambda : \lambda \in \Lambda_K\}$, and for each $\lambda \in \Lambda_K$ and $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a cfp-cover for K_λ in X_λ. It follows from [14, Lemma 13] that $K_\lambda = f_\lambda(L_\lambda)$ with some compact subset L_λ of M_λ. Put $L = \bigcup\{L_\lambda : \lambda \in \Lambda_K\}$, then L is a compact subset of M and $f(L) = K$. It implies that f is compact-covering. □

By Theorem 2.8, we get the following.

Corollary 2.9. *The following are equivalent for a space X.*

(1) *X is a compact-covering quotient (resp., pseudo-open) π-image of a locally separable metric space,*

(2) *X is a sequential (resp., Fréchet) space having a cover $\{X_\lambda : \lambda \in \Lambda\}$, where each X_λ has a refinement sequence of countable covers $\{\mathcal{P}_{\lambda,n}\}_{n\in\mathbb{N}}$ satisfying the following:*

   (a) *$\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ is a point-star network of X, where $\mathcal{P}_n = \bigcup_{\lambda\in\Lambda} \mathcal{P}_{\lambda,n}$ for every $n \in \mathbb{N}$,*

   (b) *For every compact subset K of X, there exists a finite subset Λ_K of Λ such that K has a finite compact cover $\{K_\lambda : \lambda \in \Lambda_K\}$, and for each $\lambda \in \Lambda_K$ and $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a cfp-cover for K_λ in X_λ.*

Proof. As in the proof of Corollary 2.4. □

Finally, we give examples to illustrate theorems in the above.

Example 2.10. There exists a compact-covering s-image of a locally separable metric space which is not a compact-covering π-image of any locally separable metric space.

Proof. Let X be a sequential fan S_ω (see [9]). Then X is a Fréchet and $\aleph_0$-space. It follows from [18, Remark 8.(2)] that X is a compact-covering s-image of a locally separable metric space. It is clear that every compact-covering mapping is a pseudo-sequence-covering mapping, and X is not a pseudo-sequence-covering π-image of any metric space [8, Example 2.8]. Then X is not a compact-covering π-image of any locally separable metric space. □

Example 2.11. There exists a compact-covering π-image of a locally separable metric space which is not a compact-covering s-image of any locally separable metric space.

Proof. Let X be a developable space Y in [7, Example 17]. Then X is a compact-covering (quotient) π-image of a locally separable metric space. Moreover, X is not a quotient s-image of any locally separable metric space. It implies that X is not a compact-covering s-image of any locally separable metric space. □

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