KYUNGPOOK Math. J. 49(2009), 411-418

Weak Strictly Persistence Homeomorphisms and Weak Inverse Shadowing Property and Genericity

Bahman Honary
Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran
e-mail : honary@math.um.ac.ir

Alireza Zamani Bahabadi*
Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran;
Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, Iran
e-mail : bahabadi@wali.um.ac.ir

Abstract. In this paper we introduce the notions of strict persistence and weakly strict persistence which are stronger than those of persistence and weak persistence, respectively, and study their relations with shadowing property. In particular, we show that the weakly strict persistence and the weak inverse shadowing property are locally generic in $Z(M)$.

1. Introduction

Let (M, d) be a compact metric space and let $f : M \to M$ be a homeomorphism (a discrete dynamical system on M). A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is called an *orbit* of f, denote by $o(x, f)$, if for each $n \in \mathbb{Z}$, $x_{n+1} = f(x_n)$ and is called a *δ-pseudo-orbit* of f if

$$d(f(x_n), x_{n+1}) \leq \delta, \forall n \in \mathbb{Z}.$$

We denote the set of all homeomorphisms of M by $Z(M)$. Introduce in $Z(M)$ the complete metric

$$d_0(f, g) = \max\{\max_{x\in M} d(f(x), g(x)), \max_{x\in M} d(f^{-1}(x), g^{-1}(x))\},$$

which generates the C^0 topology.

$f \in Z(M)$ is called *persistence* [1] if for every $\epsilon > 0$, there is $\delta > 0$ such that for every $x \in M$ and $g \in Z(M)$ with $d_0(f, g) < \delta$ there is $y \in X$ satisfying $d(f^n(x), g^n(y)) < \epsilon$, for all $n \in \mathbb{Z}$.We say that $f \in Z(M)$ is weak persistence if for $\epsilon > 0$, there is $\delta > 0$ such that for every $x \in M$ and $g \in Z(M)$ with $d_0(f, g) < \delta$,

---

* Corresponding author.

Received February 3, 2008; accepted March 10, 2009.
2000 Mathematics Subject Classification: Primary 37C50; Secondary 54H20.
Key words and phrases: inverse shadowing property, persistence δ-pseudo-orbit, shadowing property.

there is $y \in M$ satisfying $o(y, g) \subset N_\epsilon(o(x, f))$, where $N_\epsilon(S)$ is the ϵ-neighborhood of the set $S \subset M$.

Inverse shadowing was introduced by Corless and Pilyugin [5] and also as a part of the concept of bishadowing by Diamond et al [6]. Kloeden and Ombach [7] refined this property using the concept of δ-method. One can also see [8], [9] for more information about the concept of δ-method. Let $X^{\mathbb{Z}}$ be the space of all two sided sequences $\xi = \{x_n : n \in \mathbb{Z}\}$ with elements $x_n \in X$, endowed with the product topology. For $\delta > 0$, let $\Phi_f(\delta)$ denote the set of all δ-pseudo orbits of f. A mapping $\varphi : X \longrightarrow \Phi_f(\delta) \subset X^{\mathbb{Z}}$ is said to be a δ-method for f if $\varphi(x)_0 = x$, where $\varphi(x)_0$ is the 0-component of $\varphi(x)$. If φ is a δ-method which is continuous then it is called a continuous δ-method. The set of all δ-methods(resp. continuous δ-methods) for f will be denoted by $\pi_0(f, \delta)$ (resp. $\pi_c(f, \delta)$). If $g : X \longrightarrow X$ is a homeomorphism with $d_0(f, g) < \delta$, then g induces a continuous δ-method φ_g for f defined by

$$\varphi_g(x) = \{g^n(x) : n \in \mathbb{Z}\}.$$

Let $\pi_h(f, \delta)$ denote the set of all continuous δ-methods φ_g for f which are induced by $g \in Z(M)$ with $d_0(f, g) < \delta$. A homeomorphiosm f is said to have the *inverse shadowing property* with respect to the class π_α, α=0,c,h, if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ-method φ in $\pi_\alpha(f, \delta)$ and any point $x \in X$ there exists a point $y \in X$ for which

$$d(f^n(x), \varphi(y)_n) < \epsilon, n \in \mathbb{Z}.$$

A homeomorphiosm f is said to have the *weak inverse shadowing property* with respect to the class π_α, α=0,c,h, if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ-method φ in $\pi_\alpha(f, \delta)$ and any point $x \in X$ there exists a point $y \in X$ for which

$$\varphi(y) \subset N_\epsilon(o(x, f)).$$

In this paper we introduce the notions of strict persistence and weakly strict persistence which are stronger than those of persistence and weak persistence, respectively, and study their relations with shadowing property (see Theorems 1 and 2). Choi, Kim and Lee [2] showed that weak inverse shadowing property with respect to the class π_h(weak persistence property) is generic in $Z(M)$. As an improvement of main result in [2], we show the C^0 genericity of weak strictly persistence in an open subset of $Z(M)$ (Theorem 3) and C^0 genericity of weak inverse shadowing property with respect to the class π_0 in an open subset of $Z(M)$ (Theorem 4).

We say that $x \in M$ is a *weak stable point* for f if for any $\epsilon > 0$ there is $\delta > 0$ and positive integer N such that

$$o(z, f) \subset N_\epsilon(\{f^i(z) : i = -N, \cdots, N\})$$

for every $z \in M$ with $d(x, z) < \delta$. We say that f is weak stable if every point of M is a weak stable point for f. The authors in [3] showed that for a homeomorphism f on compact metric space M, the set of weak stable points is residual in M. Moreover if f is minimal, then f is a weak stable homepmorphism. We say that f

is *strictly persistence* if for each $\epsilon > 0$ there is $\delta > 0$ such that if $d_0(f,g) < \delta$ for $g \in Z(M)$ then $d(f^n(x), g^n(x)) < \epsilon$ for every $x \in M$ and $n \in \mathbb{Z}$. We say that f is *weak strictly persistence* if for each $\epsilon > 0$ there exist $\delta > 0$ such that if $d_0(f,g) < \delta$ then $d_H(\overline{o(x,f)}, \overline{o(x,g)}) < \epsilon$ for every $x \in M$, where d_H is the Hausdorff metric on the set of closed subsets of M. A property P is said to be *generic* for elements of a topological space M if the set of all $x \in M$ satisfying P is residual, i.e., it includes a countable intersection of open and dense subsets of M.

2. Results

A homeomorphism f on M is called minimal if $f(A) = A$, A closed, imply either $A = M$ or $A = \phi$. It is easy to see that f is minimal iff $\overline{O(x,f)} = M$ for each $x \in M$. The following proposition shows that every minimal homeomorphism on a compact metric space is weak strictly persistence.

Proposition 1. *Let f be a homeomorphism on a compact metric space M. If f is minimal, then f is weak strictly persistence.*

Proof. Let $\epsilon > 0$ be arbitrary and $U = \{U_i : i = 1, \cdots, k\}$ be a finite covering of M by open sets with diameter less than or equal $\frac{\epsilon}{2}$. As in the proof of Proposition 2 in [3], for each $x \in M$ there exits $\delta_x > 0$ and $N_x \in \mathbb{N}$ such that for each $y \in N_{\delta_x}(x)$ and $i \in \{1, 2, \cdots, k\}$, there exits $j \in \{-N_x, -N_x+1, \cdots, N_x\}$, such that $f^j(y) \in U_j$ and $o(y,f) \subset N_\epsilon(f^i(y))_{i=-N_x}^{N_x}$. Since M is compact, there exist $\{x_1, x_2, \cdots, x_n\} \subset M$ such that $\{N_{\delta_{x_i}}(x_i)\}_{i=1}^n$ covers M. Put

$$N_f = \max\{N_{x_i} : 1 \leq i \leq n\}.$$

Choose $\delta > 0$ such that $d(f^i(x), g^i(x)) < \frac{\epsilon}{4}$, $|i| \leq N_f$, for each $x \in M$, $g \in Z(M)$ with $d_0(f,g) < \delta$. For each $x \in M$, $g \in Z(M)$ with $d_0(f,g) < \delta$ and $l \in \mathbb{Z}$ there exits $j \in \{1, 2, \cdots, k\}$, such that $g^l(x) \in U_j$ and so $d(f^{k_j}(x), g^l(x)) < \frac{\epsilon}{2}$, for some $|k_j| \leq N_f$. Hence

$$d(g^l(x), g^{k_j}(x)) \leq d(f^{k_j}(x), g^l(x)) + d(f^{k_j}(x), g^{k_j}(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon.$$

This shows that $o(x,g) \subset N_\epsilon(g^i(x))_{i=-N_f}^{N_f}$ for each $x \in M$, $g \in Z(M)$ with $d_0(f,g) < \delta$. Hence

$$o(x,g) \subset N_{3\epsilon}(o(x,f)) \quad \text{and} \quad o(x,f) \subset N_{3\epsilon}(o(x,g)),$$

for each $x \in M$, $g \in Z(M)$ with $d_0(f,g) < \delta$ and so f is weak strictly persistence. □

A homeomorphism f is called *positively* (*negatively*) *recurrent* if $x \in \omega_f(x)$ ($x \in \alpha_f(x)$) for each x in M.

Theorem 1. *Let f be a weak strictly persistence homeomorphism of a compact connected metric space M which is not minimal. If M is not one point, and if f is positively recurrent, then there exists a neighborhood W_f of f such that for any $g \in W_f$, g does not have the shadowing property.*

Proof. Since f is not minimal there is $y \in M$ with $\overline{o(y, f)} \neq M$. Let x be in M such that $x \notin \overline{o(y, f)}$ and $d(x, \overline{o(y, f)}) = 4\epsilon$. Let W_f be a neighborhood of f such that $d_H(o(x, f), o(x, g)) < \frac{\epsilon}{2}$ for each x in M and g in W_f. Suppose that there exists $g \in W_f$ with the shadowing property. As $d_H(\overline{o(x, f)}, \overline{o(x, g)}) < \frac{\epsilon}{2}$ then $d(x, \overline{o(y, g)}) > 3\epsilon$. Let $\delta \in (0, \epsilon)$ be such that every δ-pseudo-orbit of g can be ϵ-shadowed by some point. By connectedness of M there is a $\frac{1}{2}\delta$-chain $\{z_0 = x, z_1, z_2, \cdots, z_{n+1} = y\}$ from x to y. Consider δ-pseudo-orbit $\{a_i\}_{i=0}^{\infty} = \{z_0, z_1, z_2, \cdots, z_{n+1} = y, f(y), \cdots\}$. Since g has the shadowing property there is $a \in M$ such that $d(g^i(a), a_i) < \epsilon$ for all $i \geq 0$. Hence $d(a, a_0) = d(a, x) < \epsilon$ and $d(g^{i+k}(a), g^i(y)) < \epsilon$, $i \geq 0$, where $k = n + 1$. So we have

$$o^+(a, g^k) \subset N_\epsilon(o^+(y, g)) \subset N_\epsilon(\overline{o^+(y, g)}) \subset N_\epsilon(\overline{o(y, g)})$$

and hence

$$\omega_g(a) = \omega_g(g^k(a)) \subset \overline{o^+(a, g^k)} \subset \overline{N_\epsilon(\overline{o(y, g)})}.$$

Since $a \in \omega_f(a)$ and $d_H(o(a, f), o(a, g)) < \frac{\epsilon}{2}$, we have $d(a, \omega_g(a)) < \epsilon$ and $d(a, \overline{o(y, g)}) \leq 2\epsilon$. Therefore

$$d(x, \overline{o(y, g)}) \leq d(x, a) + d(a, \overline{o(y, g)}) \leq 3\epsilon$$

with contradicts $d(x, \overline{o(y, g)}) > 3\epsilon$. Thus completes the proof. □

Theorem 2. *Let f be chain transitive and has the shadowing property. If f is a stable homeomorphism then f is weak strictly persistence.*

Proof. Let $\epsilon > 0$ be arbitrary and $U = \{U_i : i = 1, \cdots, k\}$ be a finite covering of M by open sets with diameter less than or equal $\frac{\epsilon}{2}$. Let x be arbitrary and choose $0 < \delta' < \frac{\epsilon}{2}$ and $\{x_1 = x, x_2, \cdots, x_k\}$ such that $N_{\delta'}(x_i) \subset U_i$ for $i \in \{1, 2, \cdots, k\}$. Since M is compact and f is stable, there is $0 < \delta < \delta'$ such that for each x and y in M with $d(x, y) < \delta$, $d(f^i(x), f^i(y)) < \frac{\delta'}{2}$ for $i \geq 0$. Since f has the shadowing property, there is $\gamma > 0$ such that each γ-pseudo-orbit, δ shadowed by some point of M. Since f is chain transitive there are γ-chains from x_i to x_i+1 for $0 \leq i \leq k$ where $x_{k+1} = x$. Constructing a γ-pseudo-orbit from x to x containing $x_2, x_3, \cdots, x_k$, by composing these γ-chains. There is $z \in M$ such that δ-shadows this γ-pseudo-orbit and hence there are $l_1 < l_2 < \cdots < l_k$, such that

$$d(f^{l_i}(z), x_i) < \delta \quad for \quad i \in \{1, 2, \cdots, k\},$$

hence

$$d(f^{l_1+j}(z), f^j(x)) < \frac{\delta'}{2} \quad for \quad j \geq 0$$

and so

$$d(f^{l_i-l_1}(x), x_i) \leq d(f^{l_i}(z), x_i) + d(f^{l_i}(z), f^{l_i-l_1}(x)) < \delta + \frac{\delta'}{2} < \frac{\delta'}{2} + \frac{\delta'}{2} = \delta'$$

for $i \in \{1, 2, \cdots, k\}$. Hence $f^{l_i-l_1}(x) \in N_{\delta'}(x_i) \subset U_i$ and so $f^{l_i-l_1}(x) \in U_i$ for $i \in \{1, 2, \cdots, k\}$. Choose $\delta_x > 0$ such that $d(f^i(x), f^i(y)) < \beta$ for $0 \leq i \leq l_k - l_1$, with x, y with $d(x, y) < \delta_x$, where $\beta > 0$ is such that $N_\beta(f^{l_i-l_1}(x)) \in U_i$ for $i \in \{1, 2, \cdots, k\}$. This shows that for each $y \in N_{\delta_x}(x)$ and $i \in \{1, 2, \cdots, k\}$, there exits $j \in \{0, 1, \cdots, N_x\}$, such that $f^j(y) \in U_j$ and $o(y, f) \subset N_\epsilon(f^i(y))_{i=-N_x}^{N_x}$ where $N_x = l_k - l_1$. Since x is arbitrary hence the above property satisfy for each $x \in M$. Since M is compact, there exist $\{x_1, x_2, \cdots, x_n\} \subset M$ such that $\{N_{\delta_{x_i}}(x_i)\}_{i=1}^n$ covers M. Put

$$N_f = \max\{N_{x_i} : 1 \leq i \leq n\}.$$

Choose $\delta > 0$ such that

$$d(f^i(x), g^i(x)) < \frac{\epsilon}{4}, \quad |i| \leq N_f,$$

for each $x \in M$, $g \in Z(M)$ with $d_0(f, g) < \delta$. For each $x \in M$, $g \in Z(M)$ with $d_0(f, g) < \delta$ and $l \in \mathbb{Z}$ there exits $j \in \{1, 2, \cdots, k\}$, such that $g^l(x) \in U_j$ and so $d(f^{k_j}(x), g^l(x)) < \dfrac{\epsilon}{2}$, for some $|k_j| \leq N_f$. Hence

$$d(g^l(x), g^{k_j}(x)) \leq d(f^{k_j}(x), g^l(x)) + d(f^{k_j}(x), g^{k_j}(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon.$$

This shows that $o(x, g) \subset N_\epsilon(g^i(x))_{i=-N_f}^{N_f}$ for each $x \in M$, $g \in Z(M)$ with $d_0(f, g) < \delta$. Hence

$$o(x, g) \subset N_{3\epsilon}(o(x, f)) \quad \text{and} \quad o(x, f) \subset N_{3\epsilon}(o(x, g)),$$

for each $x \in M$, $g \in Z(M)$ with $d_0(f, g) < \delta$ and so f is weak strictly persistence.

□

We denote the set of all weak stable homeomorphisms by WSH. The following theorem has been shown In [3, theorem 1]. We give the proof of following theorem again.

Theorem *A. Assume that int(WSH)* $\neq \phi$ *and let* W *be an open subset of* $Z(M)$ *consisting of weak stable homeomorphisms. Then there is a residual subset* R *of* W *such that for each* $f \in R$ *and* $\epsilon > 0$*, there is a neighborhood* U_f *of* f *and positive integer* N_f *such that* $o(x, g) \subset \bigcup_{i=-N_f}^{N_f} \overline{N_\epsilon(g^i(x))}$*, for each* $g \in U_f$ *and* $x \in M$*.*

To prove this theorem we need the following two lemmas.

Lemma 1. *Let $\epsilon > 0$ be arbitrary. Then the function $\psi_\epsilon : W \longrightarrow \mathbb{N}$; defined by*

$$\psi_\epsilon(f) = N_f,$$

is lower semi-continuous, where

$$N_f = \min\{N \in \mathbb{N} : o(x,f) \subset \bigcup_{i=-N}^{N} \overline{N_\epsilon(f^i(x))} \forall x \in M\}.$$

Proof. For $f \in W$ there is $x_0 \in M$ such that $o(x_0, f) \not\subseteq \bigcup_{i=-N_f+1}^{N_f-1} \overline{N_\epsilon(f^i(x_0))}$. So $f^k(x_0) \notin \bigcup_{i=-N_f+1}^{N_f-1} \overline{N_\epsilon(f^i(x_0))}$ for some $k \in \mathbb{Z}$ with $|k| \geq N_f$. Choose $\epsilon' > 0$ such that $d(f^{N_f}(x_0), f^l(x_0)) \geq \epsilon + \epsilon', -N_f + 1 \leq l \leq N_f - 1$. $(*)$.

Choose a neighborhood U_f of f such that $d(f^i(x), g^i(x)) < \frac{\epsilon'}{2}$, $|i| \leq k+1$, for each $x \in M$ and $g \in U_f$. If $N_g < N_f$ then $d(g^k(x_0), g^l(x_0)) < \epsilon$ for some $-N_f + 1 \leq l \leq N_f - 1$.So

$$d(f^k(x_0), f^l(x_0)) \leq d(f^k(x_0), g^k(x_0)) +$$
$$d(g^k(x_0), g^l(x_0)) + d(g^l(x_0), f^l(x_0)) <$$
$$\frac{\epsilon'}{2} + \epsilon + \frac{\epsilon'}{2} = \epsilon + \epsilon',$$

which contradicts $(*)$. Hence $N_g \geq N_f$. This complete the proof of the lemma. □

The following lemma is well known, and see [10] for the proof.

Lemma 2. *Let X be a Bair topological space and $\Gamma : X \longrightarrow \mathbb{N}$ be a lower semi-continuous map. Then there exists a residual subset R of X such that $\Gamma |_R$ is locally constant on each point of R.*

Proof of theorem A. For any $\epsilon > 0$, let R_ϵ be a residual subset of W such that ψ_ϵ is locally constant on R_ϵ. Then $R = \cap\{R_{\frac{1}{n}} : n = 1, 2, \cdots\}$ is a required residual set. □

Theorem 3. *Let M be a compact metric space. Then either $int(WSH) = \phi$ in $Z(M)$, or for every $f \in int(WSH)$ and every open neighborhood W of f, in $int(WSH)$, the weak strictly persistence is generic in W.*

Proof. Assuming $int(WSH) \neq \phi$, $h \in int(WSH)$, and let W be an open neighborhood of h in $int(WSH)$. Let R be in as Theorem A and $f \in R$ and $\epsilon > 0$ be arbitrary. By Theorem A there is a neighborhood U_f of f and positive integer N_f such that $o(x,g) \subset \bigcup_{i=-N_f}^{N_f} \overline{N_\epsilon(g^i(x))}$, for each $g \in U_f$ and $x \in M$. Choose $\delta > 0$ such that $N_\delta(f) \subset U_f$ and $d(f^i(x), g^i(x)) < \frac{\epsilon}{4}$, $|i| \leq N_f$, for each $x \in M$, $g \in Z(M)$ with $d_0(f,g) < \delta$. This show that

$$o(x,g) \subset N_{4\epsilon}(o(x,f)) \quad \text{and} \quad o(x,f) \subset N_{4\epsilon}(o(x,g)),$$

for each $x \in M$, $g \in Z(M)$ with $d_0(f,g) < \delta$, and hence f is weak strictly persistence.This completes the proof of theorem. □

A space M is said to be *generalized homogeneous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\{x_1, \cdots, x_n\}, \{y_1, \cdots, y_n\} \subset M$ is a pair of sets of mutually disjoint elements satisfying $d(x_i, y_i) \leqslant \delta$, $i \in \{1, \cdots, n\}$, then there exists $h \in Z(M)$ satisfying $d_0(h.id_M) \leqslant \epsilon$ and $h(x_i) = y_i, i \in \{1, \cdots, n\}$. Such a δ is called an ϵ-modulus of homogeneity of M.

The proof of the following lemma is similar to that proof of lemma 9 in [4].

Lemma 3. *Let $f : M \longrightarrow M$ be a homeomorphism of a compact manifold. Let $k \geq 0$ be an integer and let $\tau > 0$ and $\eta > 0$ be given. Then for any set of points $\{x_{-k}, \cdots, x_{-2}, x_{-1}, x_0, x_1, \cdots, x_k\}$ with $d(f(x_i), x_{i+1}) < \tau$, $-k \leq i \leq k-1$, there exists a set of points $\{x'_{-k}, \cdots, x'_{-2}, x'_{-1}, x'_0, x'_1, \cdots, x'_k\}$ with $x'_0 = x_0$ such that*
(a) $d(x_i, x'_i) < \eta$, $-k \leq i \leq k$,
(b) $d(f(x'_i), x'_{i+1}) < 2\tau$, $-k \leq i \leq k-1$,
(c) $x'_i \neq x'_j$ *if* $i \neq j$, $-k \leq i \leq k, -k \leq j \leq k$.

Theorem 4. *Let M be a generalized homogeneus space with no isolated point. Then either $int(WSH) = \phi$, or for every $f \in int(WSH)$ and every open neighborhood W of f in $int(WSH)$ the weak inverse shadowing property with respect to the class π_0 is generic in W.*

Proof. Assuming $int(WSH) \neq \phi$ and $h \in int(WSH)$, and let W be an open neighborhood of h in $int(WSH)$. Let R be as in Theorem A and $f \in R$ and $\epsilon > 0$ be arbitrary. By Theorem A there is a neighborhood U_f of f and positive integer N_f such that

$$o(x, g) \subset \bigcup_{i=-N_f}^{N_f} \overline{N_\epsilon(g^i(x))},$$

for each $g \in U_f$ and $x \in M$. Choose $\beta > 0$ such that

$$N_\delta(f) \subset U_f \quad (\quad N_\beta(f) = \{g \in Z(M) : d_0(f,g) < \beta\})$$

and

$$d(f^i(x), g^i(x)) < \frac{\epsilon}{4} |i| \leq N_f,$$

for each $x \in M$, $g \in Z(M)$ with $d_0(f,g) < \delta$. Let $\gamma > 0$ be a β-modulus of homogeneity of M, and put $0 < \delta < \min\{\frac{\gamma}{2}, \frac{\epsilon}{2}\}$.Let φ be a δ-method and $x \in M$. We show that there is $y \in M$ such that $\varphi(y) \subset N_\epsilon(o(x, f))$. $\varphi(x)$ is contained in an ϵ-neighborhood of its "finite part", i.e., there exist $l \in \mathbb{N}$ such that $\varphi(x) \subset N_\epsilon(y_t)$, where $y_t = \{\varphi(x)_n\}_{n=-l}^{l}$ and $l \geq N_f$. By Lemma 3 and proof of Lemma 9 in [4]. Since M has no isolated point we can easily find a finite 2δ-pseudo-orbit $y'_t = \{y'_n\}_{n=-l}^{l}$ such that $y'_0 = x$ and $y_t \subseteq N_\epsilon(y'_t)$ and $y'_i \neq y'_j$ for $i, j \in \{-l, \cdots, l\}, i \neq j$. Let $h \in Z(M), d_0(h, id_M) \leq \beta$, be a homeomorphism connecting $f(y'_i)$ to y'_{i+1} for

$i \in \{-l, \cdots, l\}$. Set $g = hof$. Then the sequence

$$o(x,g) = \{\cdots, g^{-2}(y'_{-l}), g^{-1}(y'_{-l}), y'_{-l}, y'_{-l+1}, \cdots, y'_l, g(y'_{-l}), g^2(y'_{-l}), \cdots\}$$

is an orbit of g. Since $g \in N_\beta(f)$, $o(x,g) \subset \bigcup_{i=-N_f}^{N_f} \overline{N_\epsilon(\{y'_i\})}$, since $d(f^i(x), y'_i) < \dfrac{\epsilon}{4}$, $|i| \leq N_f$, we have $o(x,g) \subset N_{3\epsilon}(o(x,f))$ and therefore $\varphi(x) \subset N_{5\epsilon}(o(x,f))$, which completes the proof of theorem. □

Acknowledgment. We would like to thank the referee for valuable suggestions.

References

[1] J. Lewowicz, *Persistence in expansive systems*, Ergodic Theory Dynam. Systems, **3**(1983), 567-578.

[2] T. Choi, S. Kim, K. Lee, *Weak inverse shadowing and genericity*, Bull. Korean Math. Soc., **43(1)**(2006), 43-52.

[3] B. Honary, A. Zamani Bahabadi, *Orbital shadowing property*, Bull. Korean Math. Soc., **45(4)**(2008), 645-650.

[4] P. Walters, On the pseudo orbit tracing property and its relationship to stability, Lecture Notes in Math., Vol. 668, Springer, Berlin, 1978, 231-244.

[5] R. Corless and S. *Plyugin, Approximate and real trajectories for generic dynamical systems*, J. Math. Anal. Appl., **189**(1995), 409-423.

[6] P. Diamond, P. Kloeden, V. Korzyakin and A. Pokrovskii, *Computer robustness of semihypebolic mappings*, Random and computational Dynamics, **3**(1995), 53-70.

[7] P. Kloeden, J. Ombach and A. Pokroskii, *Continuous and inverse shadowing*, Functional Differential Equations, **6**(1999), 137-153.

[8] K. Lee, *Continuous inverse shadowing and hyperbolicity*, Bull. Austral. Math. Soc., **67**(2003), 15-26.

[9] P. Diamond, Y. Han and K. Lee, *Bishadowing and hyperbolicity*, International Journal of Bifuractions and chaos, **12**(2002), 1779-1788.

[10] K. Kuratowski, "Topology2", AcademicPress-PWN-PolishSciencePublishers, Warszawa, 1968. MR0259835(41 4467).