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Two-Weighted Intergal Inequalities for Differential Forms

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Abstract. In this paper, we make use of the weight to obtain some two-weight integral inequalities which are generalizations of the Poincaré inequality. These inequalities are extensions of classical results and can be used to study the integrability of differential forms and to estimate the integrals of differential forms. Finally, we give some applications of this results to quasiregular mappings.

1. Introduction

Differential forms have wide applications in many fields, such as tensor analysis, potential theory, partial differential equations and quasiregular mappings. Throughout this paper, we always assume Ω is a connected open subset of R^n. Let $e_1, e_2, \cdots, e_n$ denote the standard unit basis of R^n. For $\ell = 0, 1, \cdots, n$, the linear space of ℓ-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell}$, corresponding to all ordered ℓ-tuples $I = (i_1, \cdots, i_\ell), 1 \leq i_1 < i_2 < \cdots < i_\ell$, is denoted by $\bigwedge^\ell = \bigwedge^\ell(\mathrm{R}^n)$. The Grassmann algebra $\bigwedge = \bigoplus \bigwedge^\ell(\mathrm{R}^n)$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \bigwedge$ and $\beta = \sum \beta^I e_I \in \bigwedge$,the inner product in $\bigwedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all ℓ-tuples $I = (i_1, i_2, \cdots, i_\ell)$ and all integers $\ell = 0, 1, \cdots, n$. We define Hodge star operator $* : \bigwedge \to \bigwedge$ by the rule

$$*1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n \quad \text{and} \quad \alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle * 1$$

for all $\alpha, \beta \in \bigwedge$, the norm of $\alpha \in \bigwedge$ is given by the formula

$$|\alpha|^2 = \langle \alpha, \alpha \rangle = *(\alpha \wedge *\alpha) \in \bigwedge^0 = \mathrm{R}.$$

The Hodge star is an isometric isomorphism on $\bigwedge$ with

$$* : \bigwedge^\ell \to \bigwedge^{n-\ell} \quad \text{and} \quad **(-1)^{\ell(n-\ell)} : \bigwedge^\ell \to \bigwedge^\ell.$$

Let we call $w(x)$ a weight if $w \in L^1_{loc}(\mathrm{R}^n)$ and $w(x) > 0$, a.e. $0 < p < \infty$, we denote

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the weighted L^p-norm of a measurable function f over E by

$$\|f\|_{p,E,w} = \left(\int_E |f(x)|^p w(x)dx\right)^{1/p} = \left(\int_E |f(x)|^p d\mu\right)^{1/p}.$$

A differential ℓ-form on Ω is a Schwartz distribution on Ω with values in $\bigwedge^\ell(\mathrm{R}^n)$. We denote the space of differential ℓ-form by $D'(\Omega, \bigwedge^\ell)$. We write $L^p(\Omega, \bigwedge^\ell)$ for the ℓ-form with

$$\omega(x) = \sum_I \omega_I(x)dx_I = \sum \omega_{i_1 i_2 \cdots i_\ell}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots dx_{i_\ell},$$

where $\omega_i \in L^p(\Omega, \mathrm{R})$ for all ordered ℓ-tuples I. $L^p(\Omega, \mathrm{R})$ is a Banach space with norm

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx\right)^{1/p} = \left(\int_\Omega \left(\sum_I |\omega_I(x)|^2\right)^{p/2} dx\right)^{1/p}.$$

Similarly, $W_p^1(\Omega, \bigwedge^\ell)$ are those differential ℓ-forms on Ω whose coefficients are in $W_p^1(\Omega, \mathrm{R})$. The notations $W_p^1(\Omega, \mathrm{R})$ and $W_p^1(\Omega, \bigwedge^\ell)$ are self-explanatory. We denote the exterior derivative by $d : D'(\Omega, \bigwedge^\ell) \to D'(\Omega, \bigwedge^{\ell+1})$ for $\ell = 0, 1, \cdots, n$. Its formal adjoint operator $d^* : D'(\Omega, \bigwedge^{\ell+1}) \to D'(\Omega, \bigwedge^\ell)$ is given by $d^* = (-1)^{n\ell+1} * d *$ on $D'(\Omega, \bigwedge^{\ell+1})$, $\ell = 0, 1, \cdots, n$.

A differential forms ω is called an $A-$harmonic tensor if ω satisfies the $A-$harmonic equation

$$d^\star A(x, d\omega) = 0, \tag{1.1}$$

where $A : \Omega \times \wedge^l(R^n) \to \wedge^l(R^n)$ satisfies the following conditions:

$$|A(x,\xi)| \le a|\xi|^{p-1} \tag{1.2}$$

and

$$\langle A(x,\xi), \xi\rangle \ge |\xi|^p \tag{1.3}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(R^n)$. Here $a > 0$ is a constant and $1 < p < \propto$ is a fixed exponent associated with (1.1). A solution to (1.1) is an element of the Sobolev space $W^1_{p,loc}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi\rangle = 0$$

for all $\varphi \in W^1_{p,loc}(\Omega, \wedge^{l-1})$ with compact support. We write $\mathrm{R} = \mathrm{R}^1$. Balls or cubes are denote by Q, and σQ is the ball or cubes with the same center as Q and with

$\operatorname{diam}(\sigma Q) = \sigma \operatorname{diam} Q$. The n-dimensional Lebesgue measure of a set $E \subset \mathrm{R}^n$ is denoted by $|E|$. Let $Q \subset \mathrm{R}^n$ be a cube or a ball. To each $y \in Q$ there corresponds a linear operator $K_y : C^\infty(Q, \bigwedge^\ell) \to C^\infty(Q, \bigwedge^{\ell-1})$ defined by

$$K_y(\omega)(x; \xi_1, \cdots, \xi_\ell) = \int_0^1 t^{\ell-1} \omega(tx + y - ty, \xi_1, \cdots, \xi_{\ell-1}) dt$$

and the decomposition $\omega = d(K_y \omega) + K_y(d\omega)$.

Another linear operator $T_Q : C^\infty(Q, \bigwedge^\ell) \to C^\infty(Q, \bigwedge^{\ell-1})$ can be defined by averaging K_y over all points y in Q, $T_Q \omega = \int_Q \phi(y) K_y \omega dy$. Where $\phi \in C_0^\infty(Q)$ is normalized by $\int_Q \phi(y) = 1$. We define the ℓ-form $\omega_Q \in D'(Q, \bigwedge^\ell)$ by $\omega_Q = |Q|^{-1} \int_Q \omega(y) dy$, $\ell = 0, 1, \cdots, n$, and $\omega_Q = d(T_Q \omega)$, for all $\omega \in L^p(Q, \bigwedge^\ell)$, $1 \le p < \infty$.

2. Two-weighted Poincaré inequality

Definition 2.1. We say the weight $\big(w_1(x), w_2(x)\big)$ satisfies the $A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ condition for some $r > 1$ and $0 < \lambda_1, \lambda_2, \lambda_3 < \infty$, and write $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$, if $w_1(x) > 0$, $w_2(x) > 0$ a.e. and

$$\sup_B \left(\frac{1}{|B|} \int_B w_1^{\lambda_1} dx \right) \left(\frac{1}{|B|} \int_B \left(\frac{1}{w_2} \right)^{\lambda_2/(r-1)} dx \right)^{\lambda_3(r-1)} < \infty$$

for any ball $B \subset\subset \Omega$.

If we choose $w_1 = w_2 = w$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in Definition 2.1, we will obtain the usual $A_r(\Omega)$-weight. If we choose $w_1 = w_2 = w$, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda$ in Definition 2.1, we will obtain the $A_r^\lambda(\Omega)$-weight [9]. If we choose $w_1 = w_2 = w$, $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_3 = 1$ in Definition 2.1, we will obtain the $A_r(\lambda, \Omega)$-weight [11].

We will need the following generalized Hölder inequality.

Lemma 2.2. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on R^n, then*

$$\| fg \|_{s,\Omega} \le \| f \|_{\alpha,\Omega} \cdot \| g \|_{\beta,\Omega}$$

for any $\Omega \subset R^n$.

We also need the following two lemmas.

Lemma 2.3. *If $w \in A_r(\Omega)$, then there exist constants $\beta > 1$ and C, independent of w, such that*

$$\| w \|_{\beta,B} \le C |B|^{(1-\beta)/\beta} \| w \|_{1,B}$$

for all balls $B \subset R^n$.

The following weak reverse Hölder inequality appears in [5].

Lemma 2.4. *Let u be an A-harmonic tensor in Ω, $\rho > 1$, and $0 < s,\ t < \infty$. Then there exists a constant C, independent of u, such that*

$$\| u \|_{s,B} \leq C|B|^{(t-s)/st} \| u \|_{t,\rho B}$$

for all balls or cubes B with $\rho B \subset \Omega$.

Different versions of the Poincaré inequality have been established in the study of the Sobolev spaces of the differential forms. The following version of the Poincaré inequality appears in [14].

Lemma 2.5. *If $u \in W_p^1(Q, \bigwedge^\ell)$, $1 \leq p < \infty$, then for any $0 < \sigma \leq 1$*

$$\|u - u_{\sigma Q}\|_{p,Q} \leq \left(\frac{2}{\sigma}\right)^{n/p} \text{diam } (Q) \| \bigtriangledown u(x)\|_{p,Q}$$

the above inequality can be write as

$$\|u - u_{\sigma Q}\|_{p,Q} \leq c(n,p,\sigma,Q) \| \bigtriangledown u(x)\|_{p,Q}$$

for any cubes or balls, $Q, \sigma Q \subset \Omega$.

We now prove the following local two-weight Poincaré inequality for A-harmonic tensors.

Theorem 2.6. *Let $u \in D'(\Omega, \wedge^l)$ be an A-harmonic tensor in a domain $\Omega \subset R^n$ and $\bigtriangledown u \in L^s(\Omega, \wedge^{l+1})$, $l = 0, 1, \cdots, n$. Suppose that $r > 1$, $w_1^{\lambda_1}(x) \in A_r(\Omega)$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$. If $\rho > 1$, $s > \lambda_3(r-1)+1$, then*

$$\left(\int_B |u - u_{\sigma B}|^s w_1^{\lambda_1} dx\right)^{1/s} \leq C \left(\int_{\rho B} | \bigtriangledown u|^s w_2^{\lambda_2 \lambda_3} dx\right)^{1/s} \tag{2.1}$$

for all balls B with $\rho B \subset \Omega$. Here $u_{\sigma B}$ is a closed form and C is a constant independent of u.

Proof. Using Lemma 2.3 with $w_1^{\lambda_1}(x) \in A_r(\Omega)$, there exist constants $\beta > 1$ and $C_1 > 0$, such that

$$\| w_1^{\lambda_1} \|_{\beta,B} \leq C_1|B|^{(1-\beta)/\beta} \| w_1^{\lambda_1} \|_{1,B} \tag{2.2}$$

for any cube or any ball $B \subset R^n$. Choose $t = s\beta/(\beta-1)$, then $1 < s < t$ and

$\beta = t/(t-s)$. Since $1/s = 1/t + (t-s)/st$, by Lemma 2.1 and (2.2), we have

$$
\begin{aligned}
(2.3) \qquad & \Big(\int_B |u-u_{\sigma B}|^s w_1^{\lambda_1} dx\Big)^{1/s} \\
= \ & \Big(\int_B \Big(|u-u_{\sigma B}| w_1^{\lambda_1/s}\Big)^s dx\Big)^{1/s} \\
\le \ & \Big(\int_B |u-u_{\sigma B}|^t dx\Big)^{1/t} \Big(\int_B w_1^{\lambda_1 t/(t-s)} dx\Big)^{(t-s)/st} \\
= \ & \| u-u_{\sigma B} \|_{t,B} \Big(\int_B w_1^{\lambda_1 \beta} dx\Big)^{1/s\beta} \\
= \ & \|u-u_{\sigma B}\|_{t,B} \cdot \|w_1^{\lambda_1}\|_{\beta,B}^{1/s} \\
\le \ & C_1 |B|^{(1-\beta)/s\beta} \|u-u_{\sigma B}\|_{t,B} \cdot \|w_1^{\lambda_1}\|_{1,B}^{1/s}
\end{aligned}
$$

Next, taking $m = s/(\lambda_3(r-1)+1)$, then $1 < m < s$. Since $u_{\sigma B}$ is a closed form, by Lemma 2.4 and Lemma 2.5, we have

$$
\begin{aligned}
(2.4) \qquad \| u-u_{\sigma B} \|_{t,B} & \le C_2 |B|^{(m-t)/mt} \| u-u_{\sigma B} \|_{m,\rho B} \\
& \le C_3 |B|^{(m-t)/mt} \| \nabla u \|_{m,\rho B},
\end{aligned}
$$

where $\rho > 1$. Using Hölder inequality with $1/m = 1/s + (s-m)/sm$, we have

$$
\begin{aligned}
\| \nabla u \|_{m,\rho B} & = \Big(\int_{\rho B} \Big(| \nabla u| w_2^{\lambda_2\lambda_3/s} w_2^{-\lambda_2\lambda_3/s}\Big)^m dx\Big)^{1/m} \\
(2.5) \qquad & \le \Big(\int_{\rho B} | \nabla u|^s w_2^{\lambda_2\lambda_3} dx\Big)^{1/s} \Big(\int_{\rho B} (\frac{1}{w_2})^{\lambda_2\lambda_3 m/(s-m)} dx\Big)^{(s-m)/sm} \\
& = \Big(\int_{\rho B} | \nabla u|^s w_2^{\lambda_2\lambda_3} dx\Big)^{1/s} \Big(\int_{\rho B} (\frac{1}{w_2})^{\lambda_2/(r-1)} dx\Big)^{\lambda_3(r-1)/s} \\
& = \|1/w_2^{\lambda_2}\|_{1/(r-1)\rho B}^{\lambda_3/s} \Big(\int_{\rho B} | \nabla u|^s w_2^{\lambda_2\lambda_3} dx\Big)^{1/s}
\end{aligned}
$$

for all balls B with $\rho B \subset \Omega$. Combining with (2.3), (2.4) and (2.5), we have

$$
\begin{aligned}
(2.6) \qquad & \Big(\int_B |u-u_{\sigma B}|^s w_1^{\lambda_1} dx\Big)^{1/s} \\
\le \ & C_4 |B|^{(1-\beta)/\beta s} |B|^{(m-t)/mt} \Big(\int_B w_1^{\lambda_1} dx\Big)^{1/s} \\
\times \ & \Big(\int_{\rho B} | \nabla u|^s w_2^{\lambda_2\lambda_3} dx\Big)^{1/s} \Big(\int_{\rho B} (\frac{1}{w_2})^{\lambda_2/(r-1)} dx\Big)^{\lambda_3(r-1)/s}.
\end{aligned}
$$

Since $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$, we have

$$
\begin{aligned}
(2.7) \qquad & \|w_1^{\lambda_1}\|_{1,B}^{1/s} \|1/w_2^{\lambda_2}\|_{1/(r-1),\rho B}^{\lambda_3/s} \\
= & \Big(\int_B w_1^{\lambda_1} dx\Big)^{1/s} \Big(\int_{\rho B} (\frac{1}{w_2})^{\lambda_2/(r-1)} dx\Big)^{\lambda_3(r-1)/s} \\
\le & \bigg(\Big(\int_{\rho B} w_1^{\lambda_1} dx\Big)\Big(\int_{\rho B} (\frac{1}{w_2})^{\lambda_2/(r-1)} dx\Big)^{\lambda_3(r-1)}\bigg)^{1/s} \\
= & |\rho B|^{(\lambda_3(r-1)+1)/s} \Big(\frac{1}{|\rho B|}\int_{\rho B} w_1^{\lambda_1} dx\Big)^{1/s} \\
& \times \bigg(\Big(\frac{1}{|\rho B|}\int_{\rho B} (\frac{1}{w_2})^{\lambda_2/(r-1)} dx\Big)^{\lambda_3(r-1)}\bigg)^{1/s} \\
\le & C_5 |\rho B|^{\lambda_3(r-1)/s+1/s} \\
\le & C_6 |B|^{\lambda_3(r-1)/s+1/s}.
\end{aligned}
$$

Substituting (2.7) into (2.6), we obtain

$$
(2.8) \qquad \Big(\int_B |u-u_{\sigma B}|^s w_1^{\lambda_1} dx\Big)^{1/s} \le C\Big(\int_{\rho B} |\bigtriangledown u|^s w_2^{\lambda_2\lambda_3} dx\Big)^{1/s}
$$

for all balls B with $\rho B \in \Omega$. Hence (2.1) holds. The proof of Theorem 2.6 is completed. ☐

If choosing $\lambda_1 = 1$ in Theorem 2.6, we have the following corollary:

Corollary 2.7. *Let $u \in D'(\Omega, \wedge^l)$ be a A-harmonic tensor in a domain $\Omega \subset \mathbf{R}^n$,$\bigtriangledown u \in L^s(\Omega, \wedge^{l+1})$ $l = 0, 1, \cdots, n$. Suppose that $r > 1$, $w_1(x) \in A_r(\Omega)$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(1, \lambda_2, \Omega)$. If $\rho > 1$, $s > \lambda_3(r-1)+1$, then*

$$
(2.9) \qquad \left(\int_B |u-u_{\sigma B}|^s w_1 dx\right)^{1/s} \le C\left(\int_{\rho B} |\bigtriangledown u|^s w_2^{\lambda_2\lambda_3} dx\right)^{1/s}
$$

for all balls B with $\rho B \subset \Omega$. Here $u_{\sigma B}$ is a closed form and C is a constant independent of u.

If choosing $\lambda_1 = 1, \lambda_3 = 1/\lambda_2$ in Corollary 2.6, we have the following symmetric inequality.

Corollary 2.8. *Let $u \in D'(\Omega, \wedge^l)$ be a A-harmonic tensor (1.2) in a domain $\Omega \subset \mathbf{R}^n$,$\bigtriangledown u \in L^s(\Omega, \wedge^{l+1})$, $l = 0, 1, \cdots, n$. Suppose that $r > 1$, $w_1(x) \in A_r(\Omega)$ and $(w_1(x), w_2(x)) \in A_r^{\lambda_3}(1, 1/\lambda_3, \Omega)$. If $\rho > 1$, $s > \lambda_3(r-1)+1$, then*

$$
(2.10) \qquad \left(\int_B |u-u_{\sigma B}|^s w_1 dx\right)^{1/s} \le C\left(\int_{\rho B} |\bigtriangledown u|^s w_2 dx\right)^{1/s}
$$

for all balls B with $\rho B \subset \Omega$. Here $u_{\sigma B}$ is a closed form and C is a constant independent of u.

If choosing $w_1 = w_2 = w$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in Theorem 2.6, we have the following inequality.

Corollary 2.9. *Let $u \in D'(\Omega, \wedge^l)$ be a A-harmonic tensors in a domain $\Omega \subset \mathbf{R}^n$,$\bigtriangledown u \in L^s(\Omega, \wedge^{l+1})$, $l = 0, 1, \cdots, n$. Suppose that $r > 1$ and $w(x) \in A_r(\Omega)$. If $\rho > 1$ and $s > r$, then*

$$\left(\int_B |u - u_{\sigma B}|^s w dx\right)^{1/s} \leq C \left(\int_{\rho B} | \bigtriangledown u|^s w dx\right)^{1/s} \tag{2.11}$$

for all balls B with $\rho B \subset \Omega$. Here $u_{\sigma B}$ is a closed form and C is a constant independent of u.

3. Applications to quasiregular mappings

Quasiregular mappings were first introduced and studied by Yu.G. Reshetnyak in series of articles that began to appear in 1966. There are many details in the monograph [18]. Quasiregular mappings are interesting not only because of the results obtained about them, but also because of many new ideas generated in the course of the development of their theory. It is known that if $f = (f^1, f^2, \cdots, f^n)$ is K-quasiregular in $\mathbf{R}^n$, then

$$u = f^l df^1 \wedge df^2 \cdots \wedge df^{l-1}$$

and

$$v = \star f^{l+1} df^{l+2} \wedge \cdots \wedge df^n$$

$l = 1, 2, \ldots, n$, are conjugate A-harmonic tensors with $p = \dfrac{n}{l}$ and $q = \dfrac{n}{n-l}$. It is also known that u is a solution to (1.1), where A is some operator satisfying (1.2) and (1.3).

By theorem 2.6, we obtain the following local two-weight Poincaré inequality for quasiregular mappings:

$$\begin{aligned} & \Big(\int_B |f^l df^1 \wedge df^2 \ldots \wedge df^{l-1} - (f^l df^1 \wedge df^2 \ldots \wedge df^{l-1})_{\sigma B}|^s w_1^{\lambda_1} dx\Big)^{1/s} \\ \leq \quad & C|B|^{1/n}\Big(\int_{\rho B} |df^1 \wedge df^2 \ldots \wedge df^l|^s w_2^{\lambda_2 \lambda_3} dx\Big)^{1/s}, \end{aligned}$$

where C is a constant independent of u and balls B with $\rho B \subset \Omega$.

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