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On a New Hilbert-type Integral Inequality

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Abstract. By introducing a parameter and estimating the weight coefficient, we obtain a new Hilbert-type integral inequality with a composite kernel and a best constant factor. As applications,we also consider its equivalent forms and reverse forms.

1. Introduction

If f, g are non-negative real functions such that $0 < \int_0^\infty f^p(t)dt < \infty$ and $0 < \int_0^\infty g^q(t)dt < \infty$, then we have (see [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < pq\left(\int_0^\infty f^p(t)dt\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(t)dt\right)^{\frac{1}{q}}, \tag{1.1}$$

where the constant factor pq is the best possible. In 2004, by introducing a parameter λ ($\lambda > 2-\min\{p,q\}$), Yang[2] gave a generalization of (1.1) with the best constant factor as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda,y^\lambda\}}dxdy < k_\lambda(p)\left(\int_0^\infty t^{1-\lambda}f^p(t)dt\right)^{\frac{1}{p}}\left(\int_0^\infty t^{1-\lambda}g^q(t)dt\right)^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $k_\lambda(p) = \dfrac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}$ is the best possible.

In 2005, Yang[3] gave a new Hilbert-type integral inequalities as follows: for $2-\min\{p,q\} < \lambda < 1$, we still have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^\lambda}dxdy < \tilde{K}_\lambda(p)\left(\int_0^\infty x^{1-\lambda}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{1-\lambda}g^q(x)dx\right)^{\frac{1}{q}}, \tag{1.3}$$

where the constant factor $\tilde{K}_\lambda(p) = B(\dfrac{p+\lambda-2}{p}, 1-\lambda) + B(\dfrac{q+\lambda-2}{q}, 1-\lambda)$ is the best possible. In 2006, Yang [4] built two new bilinear integral operator inequalities

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with composite as follows:

$$\int_0^\infty \int_0^\infty \frac{\mid x-y\mid^{\lambda-1}}{(\max\{x,y\})^\lambda} f(x)g(y)dxdy < k'_\lambda(p) \parallel f \parallel_p \parallel g \parallel_q, \tag{1.4}$$

$$\int_0^\infty \int_0^\infty \frac{\mid x-y\mid^{\lambda-1}}{(\min\{x,y\})^\lambda} f(x)g(y)dxdy < K'_\lambda(p) \parallel f \parallel_p \parallel g \parallel_q, \tag{1.5}$$

where the constant factor $k'_\lambda(p) = B\left(\lambda, \frac{1}{p}\right) + B\left(\lambda, \frac{1}{q}\right)$ and $K'_\lambda(p) = B\left(\lambda, \frac{1}{p} - \lambda\right) + B\left(\lambda, \frac{1}{q} - \lambda\right)$ are the best possible.

The main objective of this paper is to give a new Hilbert integral inequality as (1.5). As applications, we also consider the equivalent form and its reverse.

2. Some lemmas

First, we need the following formula of the β function (see[5]): for $u, v > 0$,

$$B(u,v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}} dt = \int_0^1 (1-t)^{u-1} t^{v-1} dt = \int_1^\infty \frac{(t-1)^{u-1}}{t^{u+v}} dt. \tag{2.1}$$

Lemma 2.1. *If* $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ r > 1,\ \frac{1}{r} + \frac{1}{s} = 1,\ 0 < \lambda < \frac{1}{2}$, *define the weight function* $\omega_\lambda(s,x)$ *as:*

$$\omega_\lambda(s,x) = \int_0^\infty \frac{(\min\{x,y\})^\lambda \cdot x^{\frac{\lambda}{r}}}{|x-y|^{2\lambda}} y^{-1+\frac{\lambda}{s}} dy, \quad x \in (0,\infty). \tag{2.2}$$

Then we have

$$\omega_\lambda(s,x) = B(1-2\lambda, (1+\frac{1}{r})\lambda) + B(1-2\lambda, (1+\frac{1}{s})\lambda). \tag{2.3}$$

Proof. For fixed x, setting $u = \frac{y}{x}$ in (2.2), we have

$$\begin{aligned}
\omega_\lambda(s,x) &= \int_0^\infty \frac{(\min\{1,u\})^\lambda u^{-1+\frac{\lambda}{s}}}{|u-1|^{2\lambda}} du \\
&= \int_0^1 \frac{u^{-1+(1+\frac{1}{s})\lambda}}{(1-u)^{2\lambda}} du + \int_1^\infty \frac{u^{-1+\frac{\lambda}{s}}}{(u-1)^{2\lambda}} du \\
&= \int_0^1 (1-u)^{(1-2\lambda)-1} u^{-1+(1+\frac{1}{s})\lambda} du + \int_1^\infty \frac{(u-1)^{(1-2\lambda)-1}}{u^{(1-2\lambda)+(1+\frac{1}{r})\lambda}} du \\
&= B(1-2\lambda, (1+\frac{1}{s})\lambda) + B(1-2\lambda, (1+\frac{1}{r})\lambda).
\end{aligned}$$

Hence, (2.3) is valid and the lemma is proved. □

Note. By (2.3), we still have

$$\omega_\lambda(r,y) = B(1-2\lambda,(1+\frac{1}{r})\lambda) + B(1-2\lambda,(1+\frac{1}{s})\lambda). \tag{2.4}$$

Lemma 2.2. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $0 < \lambda < \frac{1}{2}$, *and* $0 < \varepsilon < \frac{q\lambda(s+1)}{2s}$, $0 < \delta < \frac{(s+1)\lambda}{2s}$, *then we have*

$$H := \int_1^\infty x^{-1-\varepsilon}\left(\int_0^{\frac{1}{x}} \frac{u^{-1-\frac{\varepsilon}{q}+(1+\frac{1}{s})\lambda}}{(1-u)^{2\lambda}}du\right)dx = O(1). \tag{2.5}$$

Proof. For $0 < \varepsilon < \frac{q\lambda(s+1)}{2s}$, $0 < \delta < \frac{(s+1)\lambda}{2s}$, by (2.1), we obtain

$$\begin{aligned}
0 < H &\leq \int_1^\infty x^{-1}\left(\int_0^{\frac{1}{x}} \frac{u^{-1-\frac{1}{q}\frac{q\lambda(s+1)}{2s}+(1+\frac{1}{s})\lambda}}{(1-u)^{2\lambda}}du\right)dx \\
&= \int_0^1 \frac{u^{-1+\frac{\lambda(s+1)}{2s}}}{(1-u)^{2\lambda}}du\left(\int_1^{\frac{1}{u}} x^{-1}\right)dx \\
&= \int_0^1 \frac{u^{-1+\frac{\lambda(s+1)}{2s}}}{(1-u)^{2\lambda}}(-\ln u)du \\
&= \int_0^1 (1-u)^{(1-2\lambda)-1}u^{-1+\frac{\lambda(s+1)}{2s}-\delta}(-u^\delta \ln u)du.
\end{aligned}$$

Since $f(u) = -u^\delta \ln u$ is continuous in (0,1] and $f(u) \to 0\,(u \to 0^+)$, there exists $L > 0$, such that $|-u^\delta \ln u| \leq L$, $u \in [0,1]$. Then we have $0 < H \leq LB(1-2\lambda, \frac{\lambda(s+1)}{2s} - \delta)$. Hence, (2.5) is valid and the lemma is proved. □

Lemma 2.3. *Suppose that* $p > 0, (p \neq 1)$, $\frac{1}{p}+\frac{1}{q} = 1$, $r > 1$, $\frac{1}{r}+\frac{1}{s} = 1$, $0 < \lambda < \frac{1}{2}$, *and* $0 < \varepsilon < \frac{|q|\lambda}{2}$.
(1) *If* $p > 1$, *then we have*

$$\begin{aligned}
I &:= \int_1^\infty\int_1^\infty \frac{(\min\{x,y\})^\lambda x^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}}}{|x-y|^{2\lambda}}y^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}}dxdy \\
&= \frac{1}{\varepsilon}\left[B(1-2\lambda,(1+\frac{1}{s})\lambda-\frac{\varepsilon}{q}) + B(1-2\lambda,(1+\frac{1}{r})\lambda+\frac{\varepsilon}{q})\right] - O(1);
\end{aligned} \tag{2.6}$$

(2) *If* $0 < p < 1$*, then we have*

$$
\begin{aligned}
I &:= \int_1^\infty \int_1^\infty \frac{(\min\{x,y\})^\lambda x^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}}}{|x-y|^{2\lambda}} y^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}} dxdy \\
&\leq \frac{1}{\varepsilon}\left[B(1-2\lambda,(1+\frac{1}{s})\lambda-\frac{\varepsilon}{q})+B(1-2\lambda,(1+\frac{1}{r})\lambda+\frac{\varepsilon}{q})\right]. \qquad (2.7)
\end{aligned}
$$

Proof. (1) Setting $u=\frac{y}{x}$, by (2.1) and (2.5), we have

$$
\begin{aligned}
I &= \int_1^\infty x^{-1-\varepsilon}\left(\int_{\frac{1}{x}}^\infty \frac{(\min\{1,u\})^\lambda u^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}}}{|1-u|^{2\lambda}}du\right)dx \\
&= \int_1^\infty x^{-1-\varepsilon}\left(\int_{\frac{1}{x}}^1 \frac{u^{-1-\frac{\varepsilon}{q}+(1+\frac{1}{s})^\lambda}}{(1-u)^{2\lambda}}du\right)dx \\
&\quad +\int_1^\infty x^{-1-\varepsilon}\left(\int_1^\infty \frac{(u-1)^{(1-2\lambda)-1}}{u^{(1-2\lambda)+\frac{\varepsilon}{q}+(1+\frac{1}{r})\lambda}}du\right)dx \\
&= \frac{1}{\varepsilon}\int_0^1 (1-u)^{(1-2\lambda)-1}u^{(1+\frac{1}{s})\lambda-\frac{\varepsilon}{q}-1}du \\
&\quad -\int_1^\infty x^{-1-\varepsilon}\left(\int_0^{\frac{1}{x}} \frac{u^{-1-\frac{\varepsilon}{q}+(1+\frac{1}{s})\lambda}}{(1-u)^{2\lambda}}du\right)dx+\frac{1}{\varepsilon}B(1-2\lambda,\frac{\varepsilon}{q}+(1+\frac{1}{r})\lambda) \\
&= \frac{1}{\varepsilon}\left[B(1-2\lambda,(1+\frac{1}{s})\lambda-\frac{\varepsilon}{q})+B(1-2\lambda,(1+\frac{1}{r})\lambda+\frac{\varepsilon}{q})\right]-O(1).
\end{aligned}
$$

(2)Setting $u=\frac{y}{x}$, by (2.1), we have

$$
\begin{aligned}
I &\leq \int_1^\infty x^{-1-\varepsilon}\left(\int_0^\infty \frac{(\min\{1,u\})^\lambda u^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}}}{|1-u|^{2\lambda}}du\right)dx \\
&= \int_1^\infty x^{-1-\varepsilon}\left(\int_0^1 (1-u)^{(1-2\lambda)-1}u^{(1+\frac{1}{s})\lambda-\frac{\varepsilon}{q}-1}du\right)dx \\
&\quad +\int_1^\infty x^{-1-\varepsilon}\left(\int_1^\infty \frac{(u-1)^{(1-2\lambda)-1}}{u^{(1-2\lambda)+\frac{\varepsilon}{q}+(1+\frac{1}{r})\lambda}}du\right)dx \\
&= \frac{1}{\varepsilon}\left[B(1-2\lambda,(1+\frac{1}{s})\lambda-\frac{\varepsilon}{q})+B(1-2\lambda,(1+\frac{1}{r})\lambda+\frac{\varepsilon}{q})\right].
\end{aligned}
$$

Hence the lemma is proved. □

3. Main results and applications

Theorem 3.1. *If* $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ r>1,\ \frac{1}{r}+\frac{1}{s}=1,\ 0<\lambda<\frac{1}{2},\ f,g\geq 0$ *such* *that* $0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx < \infty$ *and* $0 < \int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x)dx < \infty$, *then we have the following equivalent inequalities:*

$$\int_0^\infty \int_0^\infty \frac{(\min\{x,y\})^\lambda f(x)g(y)}{|x-y|^{2\lambda}} dxdy$$

$$< k_\lambda(r) \left\{ \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x)dx \right\}^{\frac{1}{q}}; \tag{3.1}$$

$$\int_0^\infty y^{\frac{p\lambda}{s}-1} \left(\int_0^\infty \frac{(\min\{x,y\})^\lambda f(x)}{|x-y|^{2\lambda}} dx \right)^p dy < k_\lambda^p(r) \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx, \tag{3.2}$$

where the constant factor $k_\lambda(r) = B(1-2\lambda, (1+\frac{1}{s})\lambda) + B(1-2\lambda, (1+\frac{1}{r})\lambda)$ *and* $k_\lambda^p(r)$ *are the best possible.*

Proof. By Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{(\min\{x,y\})^\lambda f(x)g(y)}{|x-y|^{2\lambda}} dxdy \\
=\ & \int_0^\infty \int_0^\infty \frac{(\min\{x,y\})^\lambda}{|x-y|^{2\lambda}} \cdot \left[\frac{x^{(1-\frac{\lambda}{r})/q}}{y^{(1-\frac{\lambda}{s})/p}} f(x) \right] \cdot \left[\frac{y^{(1-\frac{\lambda}{s})/p}}{x^{(1-\frac{\lambda}{r})/q}} g(y) \right] dxdy \\
\le\ & \left\{ \int_0^\infty \int_0^\infty \left[\frac{(\min\{x,y\})^\lambda}{|x-y|^{2\lambda}} \cdot \frac{x^{\frac{\lambda}{r}}}{y^{(1-\frac{\lambda}{s})}} dy \right] x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx \right\}^{\frac{1}{p}} \\
& \times \left\{ \int_0^\infty \int_0^\infty \left[\frac{(\min\{x,y\})^\lambda}{|x-y|^{2\lambda}} \cdot \frac{y^{\frac{\lambda}{s}}}{x^{(1-\frac{\lambda}{r})}} dx \right] y^{q(1-\frac{\lambda}{s})-1} g^q(y)dy \right\}^{\frac{1}{q}} \\
(3.3)=\ & \left\{ \int_0^\infty \omega_\lambda(s,x) x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \omega_\lambda(r,y) y^{q(1-\frac{\lambda}{s})-1} g^q(y)dy \right\}^{\frac{1}{q}}.
\end{aligned}$$

If (3.3) takes the form of equality, then there exists constants A and B, such that they are not all zero and (see [6])

$$A\frac{x^{(p-1)(1-\frac{\lambda}{r})}}{y^{1-\frac{\lambda}{s}}} f^p(x) = B\frac{y^{(q-1)(1-\frac{\lambda}{s})}}{x^{1-\frac{\lambda}{r}}} g^q(y), \text{ a.e. in } (0,\infty)\times(0,\infty).$$

We find that $Ax \cdot x^{p(1-\frac{\lambda}{r})-1} f^p(x) = By \cdot y^{q(1-\frac{\lambda}{s})-1} g^q(y)$, a.e. in $(0,\infty)\times(0,\infty)$. Hence there exists a constant C, such that

$$Ax \cdot x^{p(1-\frac{\lambda}{r})-1} f^p(x) = C = By \cdot y^{q(1-\frac{\lambda}{s})-1} g^q(y), \text{ a.e. in } (0,\infty).$$

Without loss of generality, suppose that $A \ne 0$, we may get $x^{p(1-\frac{\lambda}{r})-1} f^p(x) = C/(Ax)$, a.e. in $(0,\infty)$, which contradicts $0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx < \infty$. Hence

(3.3) takes a strict inequality. In view of (2.3) and (2.4), we have (3.1). For $\varepsilon > 0$ small enough $\big(\varepsilon < \frac{|q|\lambda}{2}\big)$, setting f_ε and g_ε as:

$$f_\varepsilon(x) = g_\varepsilon(x) = 0,\ x \in (0,1);\ f_\varepsilon = x^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}},\ g_\varepsilon = x^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}},\ x \in [1,\infty).$$

If there exists a positive constant K with $K \le K_\lambda(r)$, such that (3.1) is still valid when we replace $k_\lambda(r)$ by K, then by (2.6), we have

$$\begin{aligned} & B(1-2\lambda,(1+\frac{1}{s})\lambda-\frac{\varepsilon}{q})+B(1-2\lambda,(1+\frac{1}{r})\lambda+\frac{\varepsilon}{q})-o(1) \\ = \ & \varepsilon\int_0^\infty\int_0^\infty \frac{(\min\{x,y\})^\lambda f_\varepsilon(x)g_\varepsilon(y)}{|x-y|^{2\lambda}}dxdy \\ < \ & \varepsilon K\left\{\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f_\varepsilon^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{q(1-\frac{\lambda}{s})-1}g_\varepsilon^q(x)dx\right\}^{\frac{1}{q}} = K. \end{aligned}$$

For $\varepsilon \to 0^+$, it follows that $k_\lambda(r) \le K$. Hence the constant factor $K = k_\lambda(r)$ in (3.1) is the best possible. For $x > 0$, setting a bounded measurable function $[f(x)]_n$ as:

$$[f(x)]_n = \begin{cases} f(x), & f(x) < n; \\ n, & f(x) \ge n. \end{cases}$$

by the condition of $0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx < \infty$, there exists $n_0 \in N$, such that for $n \ge n_0$, $\int_{\frac{1}{n}}^n x^{p(1-\frac{\lambda}{r})-1}[f(x)]_n^p dx > 0$. Setting

$$g_n(y) = y^{\frac{p\lambda}{s}-1}\left(\int_{\frac{1}{n}}^n \frac{(\min\{x,y\})^\lambda[f(x)]_n}{|x-y|^{2\lambda}}dx\right)^{p-1} \quad (\frac{1}{n} < y \le n; n \ge n_0),$$

by (3.1), we have

$$\begin{aligned} 0 \ < \ & \left[\int_{\frac{1}{n}}^n y^{q(1-\frac{\lambda}{s})-1}g_n^q(y)dy\right]^p = \left[\int_{\frac{1}{n}}^n y^{\frac{p\lambda}{s}-1}\left(\frac{(\min\{x,y\})^\lambda[f(x)]_n}{|x-y|^{2\lambda}}dx\right)^p dy\right]^p \\ = \ & \left[\int_{\frac{1}{n}}^n\int_{\frac{1}{n}}^n \frac{(\min\{x,y\})^\lambda[f(x)]_n g_n(y)}{|x-y|^{2\lambda}}dxdy\right]^p \\ (3.4) < \ & k_\lambda^p(r)\left\{\int_{\frac{1}{n}}^n x^{p(1-\frac{\lambda}{r})-1}[f(x)]_n^p dx\right\}\left\{\int_{\frac{1}{n}}^n y^{q(1-\frac{\lambda}{s})-1}g_n^q(y)dy\right\}^{p-1} < \infty. \end{aligned}$$

$$(3.5) \qquad 0 < \int_{\frac{1}{n}}^n y^{q(1-\frac{\lambda}{s})-1}g_n^q(y)dy < k_\lambda^p(r)\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx < \infty.$$

Hence $0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g_\infty^q(y)dy < \infty$, by using (3.1), for $n \to \infty$, both (3.4) and (3.5) still take the forms of strict inequalities, and we have (3.2). On the other hand, suppose (3.2) is valid. By Hölder's inequality, we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{(\min\{x,y\})^\lambda f(x)g(y)}{|x-y|^{2\lambda}}dxdy \\ = & \int_0^\infty \left(y^{\frac{\lambda}{s}-\frac{1}{p}} \int_0^\infty \frac{(\min\{x,y\})^\lambda f(x)}{|x-y|^{2\lambda}}dx\right)\left(y^{-\frac{\lambda}{s}+\frac{1}{p}} g^q(y)\right)dy \\ (3.6)\leq & \left\{\int_0^\infty y^{\frac{p\lambda}{s}-1}\left(\frac{(\min\{x,y\})^\lambda f(x)}{|x-y|^{2\lambda}}dx\right)^p dy\right\}^{\frac{1}{p}} \left\{\int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y)dy\right\}^{\frac{1}{q}}. \end{aligned}$$

Then by(3.2), we have (3.1). Hence (3.1) and (3.2) are equivalent. If the constant factor in (3.2) is not the best possible, then by (3.6), we may get a contradiction that the constant factor in (3.1) is not the best possible. Hence the theorem is proved. □

Theorem 3.2. *If* $0 < p < 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ r > 1,\ \frac{1}{r} + \frac{1}{s} = 1,\ 0 < \lambda < \frac{1}{2},\ f, g \geq 0,$ *such that* $0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx < \infty$, $0 < \int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x)dx < \infty$, *then we have the following equivalent inequalities:*

$$\begin{aligned} I = & \int_0^\infty \int_0^\infty \frac{(\min\{x,y\})^\lambda f(x)g(y)}{|x-y|^{2\lambda}}dxdy \\ (3.7) \qquad & > k_\lambda(r)\left\{\int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x)dx\right\}^{\frac{1}{q}}, \end{aligned}$$

$$(3.8)\quad J := \int_0^\infty y^{\frac{p\lambda}{s}-1}\left(\frac{(\min\{x,y\})^\lambda f(x)}{|x-y|^{2\lambda}}dx\right)^p dy > k_\lambda^p(r)\int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx,$$

$$(3.9)\quad L := \int_0^\infty x^{\frac{q\lambda}{r}-1}\left(\frac{(\min\{x,y\})^\lambda g(y)}{|x-y|^{2\lambda}}dy\right)^q dx < k_\lambda^q(r)\int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y)dy,$$

where the constant factor $k_\lambda(r) = B(1-2\lambda,(1+\frac{1}{s})\lambda) + B(1-2\lambda,(1+\frac{1}{r})\lambda)$ *and* $k_\lambda^p(r), k_\lambda^q(r)$ *are the best possible.*

Proof. By the reverse Hölder's inequality, similar to the proof of (3.1), we have (3.7). We conform that $J > 0$. If $J = \infty$, then (3.8) is naturally valid; if $0 < J < \infty$, setting $g(y) = \left(\frac{(\min\{x,y\})^\lambda f(x)}{|x-y|^{2\lambda}}dx\right)^{p-1}$ $(y > 0)$, then by (3.7), we have

$$\begin{aligned} \infty \ > & \int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y)dy = J = I \\ > & \ k_\lambda(r)\left\{\int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y)dy\right\}^{\frac{1}{q}} > 0 \\ & \int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y)dy = J > \int_0^\infty k_\lambda^p(r) x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx. \end{aligned}$$

Hence we have (3.8). On the other hand, by the reverse Hölder's inequality, we have the reverse of (3.6), and then by (3.8), we obtain (3.7). Hence (3.7) and (3.8) are equivalent.

For $\varepsilon > 0$ small enough $\big(\varepsilon < \dfrac{|q|\lambda}{2}\big)$, setting f_ε and g_ε as:

$$f_\varepsilon(x) = g_\varepsilon(x) = 0,\ x \in (0,1);\ f_\varepsilon = x^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{r}},\ g_\varepsilon = x^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{s}},\ x \in [1,\infty).$$

If there exists a constant K with $K \geq k_\lambda(r)$, such that (3.7) is still valid when we replace $k_\lambda(r)$ by K, then by (2.7), we have

$$\begin{aligned}
& B(1-2\lambda,(1+\frac{1}{s})\lambda-\frac{\varepsilon}{q})+B(1-2\lambda,(1+\frac{1}{r})\lambda+\frac{\varepsilon}{q}) \\
\geq\ & \varepsilon\int_0^\infty\int_0^\infty \frac{(\min\{x,y\})^\lambda f_\varepsilon(x)g_\varepsilon(y)}{|x-y|^{2\lambda}}dxdy \\
>\ & \varepsilon K\left\{\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f_\varepsilon^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{q(1-\frac{\lambda}{s})-1}g_\varepsilon^q(x)dx\right\}^{\frac{1}{q}} = K.
\end{aligned}$$

For $\varepsilon \to 0^+$, it follows that $k_\lambda(r) \geq K$. Hence the constant factor $K = k_\lambda(r)$ in (3.7) is the best possible. By the reverse of (3.6), the constant factor of (3.8) is also the best possible. For $y > 0,\ n \in N$, setting

$$[g(y)]_n \begin{cases} n, & g(y) > n; \\ g(y), & \frac{1}{n} \leq g(y) \leq n; \\ \frac{1}{n}, & g(y) < \frac{1}{n}. \end{cases}$$

by the condition of $0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy < \infty$, there exists $n_0 \in N$, such that for $n \geq n_0,\ 0 < \int_{\frac{1}{n}}^n y^{q(1-\frac{\lambda}{s})-1}[g(y)]_n^q dy > 0$. Setting

$$f_n(x) = x^{\frac{q\lambda}{r}-1}\left(\int_{\frac{1}{n}}^n \frac{(\min\{x,y\})^\lambda[g(y)]_n}{|x-y|^{2\lambda}}dy\right)^{q-1},\ (\frac{1}{n} < x \leq n; n \geq n_0),$$

by (3.7) and in view of $q < 0$, we have

$$\begin{aligned}
\infty\ >\ & \left[\int_{\frac{1}{n}}^n x^{p(1-\frac{\lambda}{r})-1}f_n^p(x)dx\right]^q = \left[\int_{\frac{1}{n}}^n x^{\frac{q\lambda}{r}-1}\left(\frac{(\min\{x,y\})^\lambda[g(y)]_n}{|x-y|^{2\lambda}}dy\right)^q dx\right]^q \\
=\ & \left[\int_{\frac{1}{n}}^n\int_{\frac{1}{n}}^n \frac{(\min\{x,y\})^\lambda f_n(x)[g(y)]_n}{|x-y|^{2\lambda}}dxdy\right]^q >
\end{aligned}$$

$$k_\lambda^q(r)\left\{\int_{\frac{1}{n}}^n x^{p(1-\frac{\lambda}{r})-1}f_n^p(x)dx\right\}^{q-1}\left\{\int_{\frac{1}{n}}^n y^{q(1-\frac{\lambda}{s})-1}[g(y)]_n^q dy\right\} > 0. \tag{3.10}$$

$$0 < \int_{\frac{1}{n}}^n x^{p(1-\frac{\lambda}{r})-1}f_n^p(x)dx < k_\lambda^q(r)\int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy < \infty. \tag{3.11}$$

Hence $0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f_\infty^p(x)dx < \infty$, by using (3.7), for $n \to \infty$, both (3.10) and (3.11) still take the forms of strict inequalities, and we have (3.9).
On the other hand,suppose that (3.9) is valid. By the reverse Hölder's inequality, we have

$$I = \int_0^\infty \left(x^{-\frac{\lambda}{r}+\frac{1}{q}} f(x)\right)\left(x^{\frac{\lambda}{r}-\frac{1}{q}} \int_0^\infty \frac{(\min\{x,y\})^\lambda g(y)}{|x-y|^{2\lambda}} dy\right) dx$$

$$\geq \left\{\int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f_n^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty x^{\frac{q\lambda}{r}-1} \left(\frac{(\min\{x,y\})^\lambda g(y)}{|x-y|^{2\lambda}} dy\right)^q dx\right\}^{\frac{1}{q}}. \tag{3.12}$$

Then by(3.9) and $q < 0$, we have (3.7). Hence (3.7) and (3.9) are equivalent. If the constant factor in (3.9) is not the best possible, then by (3.12), we may get a contradiction that the constant factor in (3.7) is not the best possible. Thus the theorem is proved. □

Remark 3.3. Inequality (3.7), (3.8) and (3.9) are equivalent.

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