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A Note on GQ-injectivity

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Abstract. The purpose of this note is to improve several known results on GQ-injective rings. We investigate in this paper the von Neumann regularity of left GQ-injective rings. We give an answer a question of Ming in the positive. Actually it is proved that if R is a left GQ-injective ring whose simple singular left R-modules are GP-injective then R is a von Neumann regular ring.

Throughout this paper all rings are associative with identity and modules are unitary. We write $J(R)$, $Z_l(R)$ and $Z_r(R)$ for the Jacobson radical, the left singular ideal and the right singular ideal of a ring R, respectively. A left R-module M is called *generalized quasi-injective* (briefly *GQ-injective*) if for any left submodule N which is isomorphic to a complement left submodule of M, every left R-homomorphism from N into M may be extended to an endomorphism of M. R is called a left GQ-injective ring if ${}_RR$ is GQ-injective. Left GQ-injective rings generalize left continuous rings in the sense of Utumi [9]. In [5] Ming proved that if R is left GQ-injective ring then $J(R) = Z_l(R)$ and $R/J(R)$ is von Neumann regular. Recall that a left R-module M is *left principally injective* (briefly *left P-injective*) if, for any principal left ideal Ra of R, every left R-homomorphism from Ra into M extends to one from R into M. A left R-module M is called *generalized left principally injective* (briefly *left* GP*-injective*) if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any left R-homomorphism of Ra^n into M extends to one from R into M. Note that GP-injective modules defined here are also called YJ-injective modules in [6], [8]. R is said to be a left GP-injective ring if ${}_RR$ is GP-injective. Recently it is known that GP-injective rings need not be P-injective [2]. For any element a in R, we denote $l(a)$ and $r(a)$ for the left annihilator and the right annihilator of $\{a\}$, respectively.

Proposition 1. *Let R be a left* GQ*-injective ring. Then the following statements are equivalent.*

(1) *R is left perfect.*

(2) *For every infinite sequence $a_1, a_2, \cdots$ in R, the ascending chain $l(a_1) \subseteq$*

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$l(a_1a_2) \subseteq l(a_1a_2a_3) \subseteq \cdots$ terminates.

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (1) : Since R is left GQ-injective, we have $J(R) = Z_l(R)$ and $R/J(R)$ is von Neumann regular. For a sequence $a_1, a_2, \cdots$ in $J(R)$, there exists a positive integer n such that $l(a_1a_2\cdots a_n) = l(a_1a_2\cdots a_na_{n+1})$ by hypothesis. Hence $Ra_1a_2\cdots a_n \cap l(a_{n+1}) = 0$. Since $a_{n+1} \in J(R) = Z_l(R)$, $a_1a_2\cdots a_n = 0$. Hence $J(R)$ is left T-nilpotent. It remains to show that $R/J(R)$ is orthogonally finite. Let $\{f_1, f_2, \cdots, f_n, \cdots\}$ be a set of countably infinite nonzero orthogonal idempotents in $R/J(R)$. Then there exists nonzero orthogonal idempotents $e_1, e_2, \ldots$ in R such that $f_i = e_i + J(R)$. Let $a_i = 1 - (e_1 + e_2 + \cdots + e_i)$, where $i = 1, 2, \cdots$. Then $a_{i+1} = a_i - a_ie_{i+1}a_i$, $e_{i+1}a_i = e_{i+1} \neq 0$ and $e_{i+1}a_{i+1} = 0$. Hence $l(a_i) \subsetneq l(a_{i+1})$, where $i = 1, 2, \cdots$. Let $b_i = 1 - e_i$ for $i = 1, 2, \cdots$. Then $a_i = b_1b_2\cdots b_i$, and so $l(b_1b_2\cdots b_i) \subsetneq l(b_1b_2\cdots b_{i+1})$. Thus we have the following strictly ascending chain $l(b_1) \subsetneq l(b_1b_2) \subsetneq \cdots$. It is a contradiction. Hence $R/J(R)$ is orthogonally finite. Therefore $R/J(R)$ is semisimple Artinian and so R is left perfect. □

Immediately we have the following.

Corollary 2. *If R is a left* GQ*-injective ring with ACC on left annihilators, then R is semiprimary. In particular, if R is a left continuous ring with ACC on left annihilators, then R is semiprimary.*

Lemma 3. *Let I be a left ideal of R. If $Z_l(R) \cap I$ contains no nonzero nilpotent element, then $Z_l(R) \cap I = 0$.*

Proof. See [5, Lemma 7]. □

In [3], it was shown that if R is a ring whose simple singular left R-modules are GP-injective, then for any nonzero $a \in R$, there exists a positive integer $n = n(a)$ such that $a^n \neq 0$ and $(RaR + l(a^n)) \oplus L = R$ for some left ideal L contained in the left socle of R. Using this result we will give another simple proof of the following proposition which was proved in [6, Proposition 1].

Proposition 4. *If every simple singular left R-module is* GP*-injective, then $Z_l(R) \cap Z_r(R) = 0$.*

Proof. Suppose that $Z_l(R) \cap Z_r(R) \neq 0$. Then by Lemma 3, there exists a nonzero element $a \in Z_l(R) \cap Z_r(R)$ such that $a^2 = 0$. Hence we have $(RaR + l(a)) \oplus L = R$ by [3, Theorem 5]. Since $l(a)$ is essential, $RaR + l(a) = R$. Now $1 = x + y$ where $x \in RaR$ and $y \in l(a)$. This implies $a = xa$. Observe that $r(x) \cap aR \neq 0$ since $x \in Z_r(R)$. Let $z = ar$ be a nonzero element in $r(x) \cap aR$. Then $0 = xz = x(ar) = (xa)r = ar$, a contradiction. □

In [6, Question 1], it is asked whether R is a von Neumann regular ring if R is a left GQ-injective ring whose simple singular left R-modules are GP-injective.

The next theorem gives a positive answer for the question and extends [5, Corollary 1.1].

Theorem 5. *Let R be a left* GQ*-injective ring. Then the following statements are equivalent:*

(1) *R is von Neumann regular.*

(2) *Every cyclic singular left R-module is* GP*-injective.*

(3) *Every simple singular left R-module is* GP*-injective.*

Proof. (1) $\Rightarrow$ (2): See [1, Theorem 2.3].
(2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (1): Since R is left GQ-injective $Z_l(R) = J(R)$ and $R/J(R)$ is von Neumann regular [5, Proposition 1]. Thus it is enough to prove that $Z_l(R) = 0$. Suppose not. Then there exists a nonzero element $a \in Z_l(R)$ such that $a^2 = 0$. Since every simple singular left R-module is GP-injective, we have $R = (RaR + l(a)) \oplus L$ by [3, Theorem 5]. Note that $L = 0$ since $a \in Z_l(R)$. Hence $1 = x + y$, where $x \in RaR \subseteq J(R)$ and $y \in l(a)$. This implies $(1 - x)a = 0$, a contradiction. □

Corollary 6. *The following statements are equivalent for a left GQ-injective ring:*

(1) *R is von Neumann regular.*

(2) *Every simple left R-module is either P-injective or projective.*

Recall that a ring R is *idempotent reflexive ring* [4] if $aRe = 0$ implies $eRa = 0$ for $a, e = e^2 \in R$. A ring R is said to be *abelian* if every idempotent of R is central. Obviously any abelian rings and semiprime rings are idempotent reflexive rings.

The following lemma extends [6, Proposition 2].

Lemma 7. *Let R be an idempotent reflexive ring whose simple singular left modules are* GP*-injective. Then $Z_r(R) = 0$.*

Proof. Suppose that $Z_r(R) \neq 0$. Then by [4, Proposition 7] for each nonzero $a \in Z_r(R)$, we have $a^n = xa^n$ where $x \in RaR$ and $a^n \neq 0$ for some positive integer n. Note that $r(x) \cap a^n R = 0$. Thus $a^n R = 0$, it is a contradiction. □

Consequently, we have the following theorem which extends [6, Corollary 2.1].

Theorem 8. *Let R be a right GQ-injective ring. Then the following statements are equivalent:*

(1) *R is a von Neumann regular ring.*

(2) *R is an idempotent reflexive ring whose simple singular left modules are* GP*-injective.*

Remark 9. In [8, Question 2], it is asked whether R is a von Neumann regular ring if R is a right self-injective ring whose simple singular left R-modules are GP-injective. Theorem 8 is also a partial answer for the question .

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