KYUNGPOOK Math. J. 49(2009), 349-353

Coefficient Estimates in a Class of Strongly Starlike Functions

JANUSZ SOKÓŁ
Department of Mathematics Rzeszów University of Technology ul. W. Pola 2, 35-959 Rzeszów, Poland
e-mail : jsokol@prz.edu.pl

ABSTRACT. In this paper we consider some coefficient estimates in the subclass $\mathcal{SL}^*$ of strongly starlike functions defined by a certain geometric condition.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathcal{U} = \{z : \; |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. Everywhere in this paper $z \in \mathcal{U}$ unless we make a note. We say that an analytic function f is subordinate to an analytic function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω, analytic in $\mathcal{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$ and $f(z) = g(\omega(z))$. In particular, if g is univalent in $\mathcal{U}$, we have the following equivalence

$$f(z) \prec g(z) \iff f(0) = g(0) \;\text{ and }\; f(\mathcal{U}) \subseteq g(\mathcal{U}).$$

Let us denote $Q(f,z) = \dfrac{zf'(z)}{f(z)}$. The class $\mathcal{SS}^*(\beta)$ of strongly starlike functions of order β

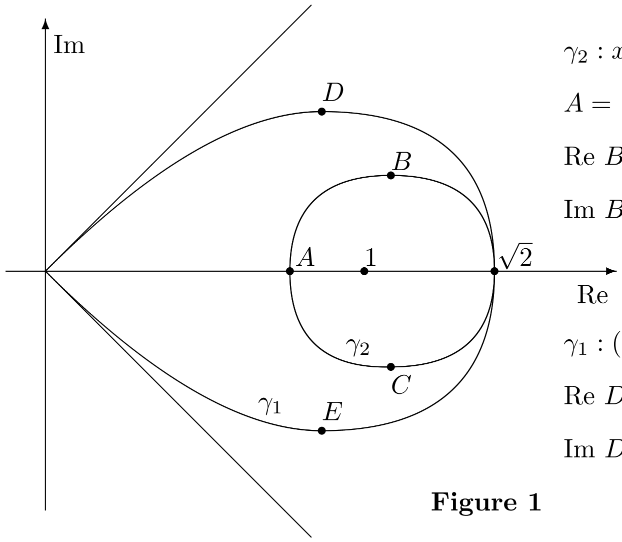
$$\mathcal{SS}^*(\beta) := \{f \in \mathcal{A} : |\text{Arg } Q(f,z)| < \beta\pi/2\}, \quad 0 < \beta \le 1$$

was introduced in [5] and [1]. For $\beta = 1$ this class becomes the well known class S^* of starlike functions. In this paper we consider the class $\mathcal{SL}^*$:

$$\mathcal{SL}^* := \{f \in \mathcal{A} : |Q^2(f,z) - 1| < 1\}. \tag{1}$$

It is easy to see that $f \in \mathcal{SL}^*$ if and only if $Q(f,z) \prec q_0(z) = \sqrt{1+z}$, $q_0(0) = 1$. We observe that $\mathcal{L} := \{w \in \mathbb{C} : \text{Re } w > 0, \;\; |w^2 - 1| < 1\}$ is the interior of the right half of the lemniscate of Bernoulli $\gamma_1 : (x^2+y^2)^2 - 2(x^2-y^2) = 0$, see Figure 1. Moreover $\mathcal{L} \subset \{w : |\text{Arg } w| < \pi/4\}$, thus $\mathcal{SL}^* \subset \mathcal{SS}^*(1/2) \subset \mathcal{S}^*$. The class $\mathcal{SL}^*$ was introduced in [4] and there the authors give also the following representation formula.

Received July 14, 2008; accepted September 12, 2008.
2000 Mathematics Subject Classification: 30C50, 30C45.
Key words and phrases: analytic functions; starlike functions; k–starlike functions; strongly starlike functions.



$\gamma_2 : x^2 = (6+4\sqrt{2})(x-1)^2 + (6+4\sqrt{2})y^2$

$A = \dfrac{4+\sqrt{2}}{7}$

$\mathrm{Re}\ B = \mathrm{Re}\ C = \dfrac{4\sqrt{2}-2}{7}$

$\mathrm{Im}\ B = -\mathrm{Im}\ C = \dfrac{1}{\sqrt{5+4\sqrt{2}}}$

$\gamma_1 : (x^2+y^2)^2 - 2(x^2-y^2) = 0$

$\mathrm{Re}\ D = \mathrm{Re}\ E = \dfrac{\sqrt{3}}{2}$

$\mathrm{Im}\ D = -\mathrm{Im}\ E = \dfrac{1}{2}$

Figure 1

Theorem A([4]). *The function f belongs to the class $\mathcal{SL}^*$ if and only if there exists an analytic function $q \in \mathcal{H}$, $q(0) = 0$, $q(z) \prec q_0(z) = \sqrt{1+z}$, $q_0(0) = 1$ such that*

$$f(z) = z \exp \int_0^z \frac{q(t)-1}{t} dt. \tag{2}$$

Let $q_1(z) = \dfrac{3+2z}{3+z}$, $q_2(z) = \dfrac{5+3z}{5+z}$, $q_3(z) = \dfrac{8+4z}{8+z}$. Because $q_i(z) \prec q_0(z)$, $i = 1,2,3$, then by (2) we obtain that the functions $f_1(z) = z + \dfrac{z^2}{3}$, $f_2(z) = z(1+\dfrac{z}{5})^2$, $f_3(z) = z(1+\dfrac{z}{8})^3$ are in $\mathcal{SL}^*$. If we take $q_0(z) = \sqrt{1+z}$, $q_0(0) = 1$ then we obtain from (2) the function f_0

$$f_0(z) := \frac{4z\exp(2\sqrt{1+z}-2)}{(1+\sqrt{1+z})^2} = z + \frac{1}{2}z^2 + \frac{1}{16}z^3 + \frac{1}{96}z^4 - \frac{1}{128}z^5 + \cdots . \tag{3}$$

Rønning considered in [3] an analogously defined class connected with a parabolic region:

$$\mathcal{S}_p^* := \{f \in \mathcal{A} : \mathrm{Re}[Q(f,z)] > |Q(f,z)-1|\}.$$

Kanas and Wiśniowska introduced in [2] the concept of a k–starlike functions

$$k-\mathcal{ST} := \{f \in \mathcal{A} : \mathrm{Re}[Q(f,z)] > k|Q(f,z)-1|\}, \quad k \geq 0.$$

In this way they obtained a continuous passage from starlike functions ($k = 0$) to the class $\mathcal{S}_p^*$ ($k = 1$). Moreover for $0 < k < 1$ the quantity $Q(f,z)$ takes its values in a convex domain on the right of a hyperbola while for $k > 1$ inside an ellipse. Let us consider the conic region $P(k) = \{w \in \mathbb{C} : \text{Re } w > k|w-1|\}$ connected with the class $k - \mathcal{ST}$ described above. For $k > 1$ the curve $\partial P(k)$ is the ellipse $\gamma_2 : x^2 = k^2(x-1)^2 + k^2y^2$. For $k \geq 2 + \sqrt{2}$ this ellipse lies entirely inside $\overline{\mathcal{L}}$. Therefore $k - \mathcal{ST} \subset \mathcal{SL}^*$, for $k \geq 2 + \sqrt{2}$.

2. Main results

Theorem 1. *If the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belongs to the class $\mathcal{SL}^*$, then*

$$\sum_{k=2}^{\infty}(k^2-2)|a_k|^2 \leq 1. \tag{4}$$

Proof. If $f \in \mathcal{SL}^*$, then $Q(f,z) \prec q_0(z) = \sqrt{1+z}$. Hence $Q(f,z) = \sqrt{1+\omega(z)}$, where ω satisfies $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$. Therefore $f^2(z) = (zf'(z))^2 - f^2(z)\omega(z)$ and using this we can obtain

$$\begin{aligned}
2\pi \sum_{k=1}^{\infty} |a_k|^2 r^{2k} &= \int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta \\
&\geq \int_0^{2\pi} \left|\omega(re^{i\theta})\right| \left|f^2(re^{i\theta})\right| d\theta \\
&= \int_0^{2\pi} \left|(re^{i\theta}f'(re^{i\theta}))^2 - f^2(re^{i\theta})\right| d\theta \\
&\geq \int_0^{2\pi} \left|re^{i\theta}f'(re^{i\theta})\right|^2 - \left|f(re^{i\theta})\right|^2 d\theta \\
&= 2\pi \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2k} - 2\pi \sum_{k=1}^{\infty} |a_k|^2 r^{2k}
\end{aligned}$$

for $0 < r < 1$. The extremes in this sequence of inequalities give

$$2\sum_{k=1}^{\infty} |a_k|^2 r^{2k} \geq \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2k}, \quad 0 < r < 1.$$

Eventually, if we let $r \to 1^-$ then we obtain (4). □

Corollary 1. *If the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belongs to the class $\mathcal{SL}^*$, then $|a_k| \leq \sqrt{\dfrac{1}{k^2-2}}$ for $k \geq 2$.*

Theorem 2. *If the function* $f(z) = \sum_{k=1}^{\infty} a_k z^k$ *belongs to the class* $\mathcal{SL}^*$, *then*

(5) $$|a_2| \le 1/2, \quad |a_3| \le 1/4, \quad |a_4| \le 1/6.$$

Those estimations are sharp.

Proof. If $f(z) = \sum_{k=1}^{\infty} a_k z^k$ belongs to the class $\mathcal{SL}^*$ then $[zf'(z)]^2 = f^2(z)[\omega(z) - 1]$, where ω satisfies $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$. Let us denote

(6) $$[zf'(z)]^2 = \sum_{k=2}^{\infty} A_k z^k, \quad f^2(z) = \sum_{k=2}^{\infty} B_k z^k, \quad \omega(z) = \sum_{k=1}^{\infty} C_k z^k.$$

Then we have

(7) $$A_k = \sum_{l=1}^{k-1} l(k-l) a_l a_{k-l} \ , \quad B_k = \sum_{l=1}^{k-1} a_l a_{k-l}$$

and

(8) $$\sum_{k=2}^{\infty} (A_k - B_k) z^k = \left[\sum_{k=1}^{\infty} C_k z^k\right] \left[\sum_{k=2}^{\infty} B_k z^k\right].$$

Thus

(9) $$A_2 = a_1 = 1, \quad A_3 = 4a_1 a_2 = 4a_2, \quad A_4 = 6a_3 + 4a_2^2, \quad A_5 = 8a_1 a_4 + 12 a_2 a_3$$

and

(10) $$B_2 = a_1 = 1, \quad B_3 = 2a_2, \quad B_4 = 2a_3 + a_2^2, \quad B_5 = 2a_1 a_4 + 2a_2 a_3.$$

Equating the second, third and fourth coefficients of both sides of (8) we obtain

(i) $A_3 - B_3 = C_1 B_2$,

(ii) $A_4 - B_4 = C_1 B_3 + C_2 B_2$,

(iii) $A_5 - B_5 = C_1 B_4 + C_2 B_3 + C_3 B_2$.

So by (9), (10) we have

(j) $a_2 = \frac{1}{2} C_1$,

(jj) $a_3 = \frac{1}{16} C_1^2 + \frac{1}{4} C_2$,

(jjj) $a_4 = \frac{1}{96} C_1^3 + \frac{1}{24} C_1 C_2 + \frac{1}{6} C_3$.

It is well known that $|C_k| \leq 1$, $\sum_{k=1}^{\infty} |C_k|^2 \leq 1$ therefore we obtain (5).
For the proof of sharpness let us consider $q(z) = \sqrt{1+z^n}$. Using the representation formula (2) we obtain the function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ such that $[zf'(z)]^2 = f^2(z)[z^n - 1]$ and with the notation (6) we have

$$\sum_{k=2}^{\infty} (A_k - B_k) z^k = \sum_{k=2}^{\infty} B_k z^{k+n}.$$

So $A_k = B_k$ for $k \leq n+1$. This gives $a_1 = 1$, $a_2 = \cdots = a_n = 0$. While $A_{n+2} - B_{n+2} = B_2$ gives

$$\sum_{l=1}^{n+1} [l(n+2-l) - 1] a_l a_{n+2-l} = 1$$

thus $2na_{n+1} = 1$. Therefore there exists a function f in the class $\mathcal{SL}^*$ such that $f(z) = z + \frac{1}{2n} z^{n+1} + \cdots$. □

Conjecture. *Let $f \in \mathcal{SL}^*$ and $f(z) = \sum_{k=1}^{\infty} a_k z^k$. Then $|a_{n+1}| \leq \frac{1}{2n}$.*

References

[1] D. A. Brannan, W. E. Kirwan, *On some classes of bounded univalent functions*, J. London Math. Soc., **1(2)**(1969) 431-443.

[2] S. Kanas, A. Wiśniowska, *Conic regions and k–uniform convexity*, J. Comput Appl. Math., **105**(1999) 327-336.

[3] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118**(1993) 189-196.

[4] J. Sokół, J. Stankiewicz, *Radius of convexity of some subclasses of strongly starlike functions*, Folia Scient. Univ. Tech. Res., **19**(1996) 101-105.

[5] J. Stankiewicz, *Quelques problèmes extrèmaux dans les classes des fonctions α – angulairement ètoilèes*, Ann. Univ. Mariae Curie–Skłodowska, Sect. A, **20**(1966) 59-75.