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Lower Bounds on Boundary Slope Diameters for Montesinos Knots

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Abstract. In this paper, we give two lower bounds on the diameter of the boundary slope set of a Montesinos knot. One is described in terms of the minimal crossing numbers of the knots, and the other is related to the Euler characteristics of essential surfaces with the maximal/minimal boundary slopes.

1. Introduction

Consider a compact possibly non-orientable surface properly embedded in a knot exterior in the 3-sphere S^3. It is called *essential* if it is incompressible and boundary-incompressible. The boundary of an essential surface consists of a parallel family of non-trivial simple closed curves. Thus, on the peripheral torus of the knot, they determine an isotopy class of a non-trivial unoriented simple closed curve, which is called the *boundary slope* of the surface. In [6], Hatcher showed there are only finitely many such boundary slopes. Moreover, Culler and Shalen [2] proved that there always exist at least two such boundary slopes. Therefore, the boundary slope set of a knot is a non-empty, finite set.

The boundary slope set of a knot K in S^3 is a non-empty, finite subset of the rational numbers. See [12] for example. In view of this, Culler and Shalen introduced and studied in [3] the *diameter* of the boundary slope set for K, which is defined as the difference between the maximum and the minimum among non-meridional elements. This will be denoted $\mathrm{Diam}(K)$ in this paper.

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1.1. Actually Culler and Shalen proved in [3] that;

$$\mathrm{Diam}(K) \geq 2$$

holds for a non-trivial knot K in S^3 if K does not have meridional boundary slope.

This can be generalized for an alternating knot K by using the result in [1], [5] as follows:

$$\mathrm{Diam}(K) \geq 2\,\mathrm{cr}(K),$$

where $\mathrm{cr}(K)$ denotes the minimal crossing number of K. See [10].

In this paper, we consider a *Montesinos knot*, which is obtained by composing a number of rational tangles in line, and give the following:

Theorem 1.1. *For a Montesinos knot K, we have the inequality*

$$\mathrm{Diam}(K) \geq 2\,\mathrm{cr}(K) - 6\,.$$

This inequality seems to be sharp in a sense. That is, $\mathrm{Diam}(K) - 2\,\mathrm{cr}(K)$ seems to be able to approach arbitrarily close to -6 by choosing an appropriate Montesinos knot K. See Remark 3.2. This theorem together with the results in [10] and [11] for two-bridge knots gives the following upper and lower bounds.

Corollary 1.2. *For a Montesinos knot K, we have the inequalities*

$$2\,\mathrm{cr}(K) - 6 \leq \mathrm{Diam}(K) \leq 2\,\mathrm{cr}(K)\,.$$

Note that this corollary can be applied to the trivial knot, for which both $\mathrm{Diam}(K)$ and $2\,\mathrm{cr}(K)$ are 0. In fact, as shown in [10], if K is an alternating Montesinos knot, we have the equality

$$\mathrm{Diam}(K) = 2\,\mathrm{cr}(K)\,.$$

1.2. For an alternating knot K, the lower bound for $\mathrm{Diam}(K)$ of $2\mathrm{cr}(K)$ is achieved by considering the checkerboard surfaces F_1, F_2 for its reduced alternating diagram. In fact, these are known to be essential by [1], [5]. Let R_1 and R_2 denote the boundary slopes of F_1 and F_2. By simple calculations, we have $|R_1 - R_2| = 2\,\mathrm{cr}(K)$ and $\chi(F_1) + \chi(F_2) = 2 - \mathrm{cr}(K)$ where $\chi(F)$ denotes the Euler characteristic of a surface F. Hence, the following inequality also holds;

$$\mathrm{Diam}(K) \geq 2\,((-\chi(F_1)) + (-\chi(F_2))) + 4.$$

We also generalize this for Montesinos knots as follows:

Theorem 1.3. *Let K be a non-trivial Montesinos knot. Among its non-meridional boundary slopes, let R_1 and R_2 be the maximum and the minimum respectively. Then there exist two essential surfaces F_1 and F_2 with boundary slopes R_1 and R_2 such that*

$$\mathrm{Diam}(K) = |R_1 - R_2| \quad \geq \quad 2\left(\frac{-\chi}{\sharp s}(F_1) + \frac{-\chi}{\sharp s}(F_2)\right), \tag{1.1}$$

where $\frac{-\chi}{\sharp s}(F_i)$ *denotes the ratio of the negative of the Euler characteristic and the number of sheets for* F_i *for* $i = 1, 2$.

Here, following [8], by the *number of sheets* of an essential surface, we mean the minimal intersection number between the surface and the meridian of the knot.

Remark 1.1. We remark that our lower bound is optimal in a sense. See Remark 4.1. Also it should be compared with an upper bound given in [9, Theorem 1.4]: We showed that

$$|R_1 - R_2| \quad \leq \quad 2\left(\frac{-\chi}{\sharp s}(F_1) + \frac{-\chi}{\sharp s}(F_2)\right) + 4.$$

holds for any pair of essential surfaces F_1, F_2 with boundary slopes R_1, R_2 in a Montesinos knot exterior.

The minimal geometric intersection number of the curves representing two slopes R_1 and R_2 is called the *distance* between R_1 and R_2. This is usually denoted by $\Delta(R_1, R_2)$.

The following immediate corollary to Theorem 1.3 bounds the distance between the maximal and minimal boundary slopes for a Montesinos knot.

Corollary 1.4. *Let* K *be a Montesinos knot. Among its non-meridional boundary slopes, let* R_1 *and* R_2 *be the maximum and the minimum respectively. Then there exist two essential surfaces* F_1 *and* F_2 *with boundary slopes* R_1 *and* R_2 *such that*

$$\Delta(R_1, R_2) \quad \geq \quad 2\left(\frac{-\chi}{\sharp b}(F_1) + \frac{-\chi}{\sharp b}(F_2)\right), \tag{1.2}$$

where $\frac{-\chi}{\sharp b}(F_i)$ *denotes the ratio of the negative of the Euler characteristic and the number of boundary components of* F_i *for* $i = 1, 2$.

Proof. Recall that the distance between the slopes p/q and r/s is given by $|ps - qr|$. Also note that, for an essential surface F with boundary slope R, the number of sheets of F is equal to the product of the denominator of R with the number of boundary components of F. With these facts, the corollary follows from Theorem 1.3. $\square$

Note that this corollary can be applied to the trivial knot. Since an essential surface F of the trivial knot is a disk and the boundary slope R of F is 0, the inequality (1.2) clearly holds.

This can be regarded as a generalization of the following result shown by Culler and Shalen. Suppose that M is a non-exceptional two-surface knot manifold. That is, M is an irreducible, connected, compact, orientable 3 -manifold with single torus boundary such that it has at most two distinct isotopy classes of strict essential surfaces and it is neither Seifert fibered nor an exceptional graph manifold. Let F_1 and F_2 be representatives of the two isotopy classes of connected strict essential surfaces. Let R_i denote the boundary slope of F_i and let $\sharp b_i$ denote the number of boundary components of F_i. Then for $i = 1, 2$ we have

$$\Delta(R_1, R_2) \geq 2\frac{-\chi(F_i)}{\sharp b_1 \cdot \sharp b_2} \ .$$

Please see [4] for details.

This paper is organized as follows. In the next section, we review the algorithm by Hatcher and Oertel given in [7]. In Section 3, we give some formulae to calculate the twist and prove Theorem 1.1. In the last section, we introduce the remainder term, give formulae for calculating its value and show Theorem 1.3.

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2. Montesinos knots and algorithm of Hatcher-Oertel

In this section, we give a brief review of Hatcher and Oertel's work in [7], which is the base of our arguments. We also present terminology used in the rest of the paper. Note that the terms marked with "†" are about notions which do not appear in [7] and are introduced by the authors in accordance with our argument.

2.1. Montesinos knot. Let us start with the definition of Montesinos knots. A *Montesinos knot* is defined as a knot obtained by putting rational tangles together in a row. See Figure 1 for example. A Montesinos knot obtained from rational tangles $T_1, T_2, \cdots, T_N$ will be denoted by $M(T_1, T_2, \cdots, T_N)$. Here and in the sequel, T_i denotes an irreducible fraction or the corresponding rational tangle depending on the context. In the following, we assume that each rational tangle is non-integral, just for the sake of normalization. Furthermore, we will always assume that the number of tangles is at least three. Note that the knots with at most two tangles are two-bridge knots. For two-bridge knots, Theorem 1.1 holds by [11]. Also Theorem 1.3 holds since non-trivial two-bridge knots are all alternating.

2.2. Hatcher-Oertel's algorithm. We here give a very brief review of the

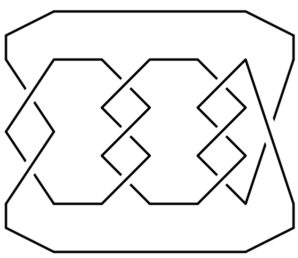

Figure 1: A diagram of $M(1/2, 1/3, -2/3)$

algorithm of Hatcher and Oertel. See [7] or [9] for details.

Given the Montesinos knot $K = M(T_1, T_2, \cdots, T_N)$, divide the 3-sphere S^3 into N 3-balls such that K is decomposed into N rational tangles $(T_1, T_2, \cdots, T_N)$. At the same time, an essential surface F embedded in the exterior of K is divided into surfaces $(F_1, F_2, \cdots, F_N)$. Each of these surfaces F_i can be isotoped into some standard position, and is represented by an "edgepath" γ_i in a "diagram" $\mathcal{D}$. Then the whole of F is represented by an "edgepath system" Γ, which consists of N "edgepaths" as we will describe below.

Remark 2.1. An "edge" corresponds to a piece of the surface called a "saddle". An "edgepath" consists of edges, and so, it corresponds to a surface obtained by combining saddles. Now, in fact, there are two possible choices of saddle for a given edge. Thus, multiple surfaces correspond to the same "edgepath", and there are also multiple surfaces for the same "edgepath system". We here remark that, despite this ambiguity, all the surfaces corresponding to the same "edgepath system" have a common boundary slope, a common value of $-\chi/\sharp s$, a common "twist", and a common "remainder term" as will be defined in a later section. See [9] for details.

2.2.1. Diagram. The graph on the u-v plane defined as follows is called a *diagram* and denoted by $\mathcal{D}$. A vertex of $\mathcal{D}$ is a point $(u, v) = ((q-1)/q, p/q)$ denoted by $\langle p/q \rangle$ or a point $(u, v) = (1, p/q)$ denoted by $\langle p/q \rangle^\circ$ for an irreducible fraction p/q with $q > 0$, or a point $(u, v) = (-1, 0)$ denoted by $\langle 1/0 \rangle$. Two vertices $\langle p/q \rangle$ and $\langle r/s \rangle$ are connected by an edge, which is a straight segment, if $|ps - qr| = 1$. There is another kind of edge called a *horizontal edge*, which connects $\langle p/q \rangle$ and $\langle p/q \rangle^\circ$ for $p/q \neq 1/0$. An important class of non-horizontal edges is the *vertical edges*, which connect the vertices $\langle z \rangle$ and $\langle z+1 \rangle$ for an arbitrary integer z. Another important class of edges is the ∞-*edges*, which connect $\langle 1/0 \rangle$ and $\langle z \rangle$ for each integer z. The diagram $\mathcal{D}$ is illustrated in Figures 2 and 3. By $\mathcal{S}$, we denote the subgraph of $\mathcal{D}$ lying in $0 \leq u \leq 1$.

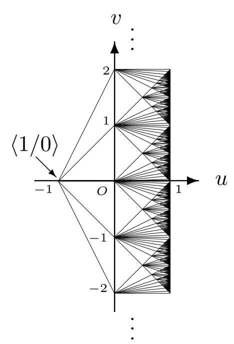

Figure 2: The diagram $\mathcal{D}$

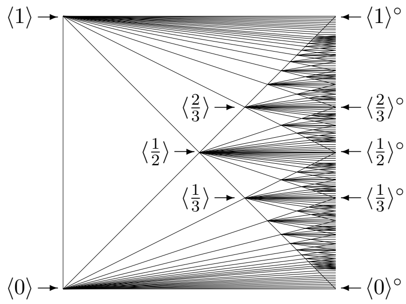

Figure 3: A part of the diagram $\mathcal{D}$ in $[0, 1] \times [0, 1]$

2.2.2. Edgepath and Edgepath system. A path γ on $\mathcal{D}$ is called *minimal* if γ retraces no edges and never goes along two sides of a single triangle in succession. An *edgepath* γ for a rational tangle associated to the irreducible fraction p/q is defined as a minimal path on $\mathcal{D}$. For example, (a) a single point on a horizontal edge $\langle p/q\rangle - \langle p/q\rangle^\circ$, or (b) a path starting at $\langle p/q\rangle$ and proceeding monotonically from right to left or at least vertically. An edgepath is called a *constant edgepath* in the former case, and a *non-constant edgepath* otherwise.

An *edgepath system* Γ for a Montesinos knot $K = M(T_1, T_2, \cdots, T_N)$ is then defined as a tuple $(\gamma_1, \gamma_2, \cdots, \gamma_N)$ of edgepaths where each γ_i is an edgepath for the tangle T_i.

2.2.3. Basic edgepath system and Extended basic edgepath system. We call a non-constant edgepath for a rational tangle T_i whose u-coordinate at the endpoint is 0 and which has no vertical edges a *basic edgepath*†for T_i. A *basic edgepath system*†for a Montesinos knot is then defined as a set of basic edgepaths with the starting points associated to the tangles.

A basic edgepath λ_i for a tangle T_i is extended by combining λ_i and the horizontal edge $\langle T_i\rangle - \langle T_i\rangle^\circ$. We call it the *extended basic edgepath*† $\widetilde{\lambda}_i$ of λ_i. By an *extended basic edgepath system*†, we mean the set of extended basic edgepaths $\widetilde{\Lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \cdots, \widetilde{\lambda}_N)$ obtained from a basic edgepath system $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$.

An extended basic edgepath $\widetilde{\lambda}$ is naturally regarded as a continuous piecewise-affine function $[0, 1] \to \mathbb{R}$ which gives the v coordinate of the point (u, v) on the edgepath for each u. We then regard an extended basic edgepath system $\widetilde{\Lambda}$ as a function defined as the sum $\widetilde{\Lambda}(u) = \sum_{i=1}^{N} \widetilde{\lambda}_i(u)$ of functions $(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \cdots, \widetilde{\lambda}_N)$. Similarly, a basic edgepath system Λ, or an edgepath system in general, can be regarded as a function, which is a restriction of the function $\widetilde{\Lambda}$.

2.2.4. Partial edge. From an extended basic edgepath system $\widetilde{\Lambda} = (\widetilde{\lambda_1}, \widetilde{\lambda_2}, \cdots, \widetilde{\lambda_N})$ for a Montesinos knot $K = M(p_1/q_1,\ p_2/q_2,\ \cdots,\ p_N/q_N)$ and an arbitrary u_0 with $0 < u_0 < 1$, by *cutting* at $u = u_0$, we can make an edgepath system $\Gamma = (\gamma_1, \gamma_2, \cdots, \gamma_N)$ as follows. For each i, if $u_0 \le (q_i - 1)/q_i$, then we set γ_i to be a non-constant edgepath $\lambda_i \cap \{(u, v)|u_0 \le u\} = \widetilde{\lambda}_i \cap \{(u, v)|u_0 \le u \le (q_i - 1)/q_i\}$. If $u_0 > (q_i - 1)/q_i$, then we set γ_i to be a constant edgepath consisting of the point P_i with coordinates $(u, v) = (u_0, p_i/q_i)$ on a horizontal edge $\langle p_i/q_i\rangle - \langle p_i/q_i\rangle^\circ$. In this operation, the edgepath system Γ may include an edge which is a proper subset of an edge of the diagram $\mathcal{D}$. The edge is called a *partial edge*. In contrast, an edge of the diagram $\mathcal{D}$ completely included in an edgepath is called a *complete edge*.

2.2.5. Gluing consistency. As in [7], an edgepath system is associated to a surface embedded in a Montesinos knot exterior. Such an edgepath system must satisfy the following condition: the endpoints of the edgepaths share a common

u-coordinate and their v-coordinates $(v_1, v_2, \cdots, v_N)$ sum to zero:

$$\sum_{i=1}^{N} v_i = 0. \tag{2.1}$$

We refer to these conditions as *gluing consistency*, and call an edgepath system satisfying these conditions a *candidate edgepath system*.

2.2.6. Types of edgepath systems. In order to enumerate all boundary slopes for a Montesinos knot K, we first enumerate all basic edgepath systems for K, which is obviously finitely many, and then list candidate edgepath systems as follows. (I) From each basic edgepath system Λ, by making the extended basic edgepath system $\widetilde{\Lambda}$, solving the equation $\widetilde{\Lambda}(u) = 0$, getting a solution $u_0 > 0$, and cutting $\widetilde{\Lambda}$ at $u = u_0$, we have a candidate edgepath system, which is called a *type I* edgepath system. The endpoints of its edgepaths have a common u-coordinate $u = u_0 > 0$. (II) By adding an appropriate number of vertical edges to some edgepaths of Λ if necessary, we may have a candidate edgepath system, which is called a *type II* edgepath system. The endpoints of its edgepaths have a common u-coordinate 0. (III) By adding suitable (possibly partial) ∞-edges to Λ, we have a candidate edgepath system, which is called a *type III* edgepath system. The endpoints of its edgepaths have a common u-coordinate $u < 0$.

We remark here that our classification is slightly different from that in [7], that is, a basic edgepath system satisfying gluing consistency is regarded as type II in this paper and as type I in [7].

By Corollary 2.4 through Proposition 2.10 of [7], we can determine whether surfaces corresponding to a candidate edgepath system are essential or not. We thus obtain a list of edgepath systems corresponding to some essential surface. The algorithm of Hatcher and Oertel is completed by calculating all the boundary slopes for the surfaces associated to such edgepath systems.

2.2.7. Sign of edge. A non-constant edgepath has a roughly right-to-left direction. With this direction, each non-∞ non-horizontal edge in the edgepath is said to be *increasing* or *decreasing* according to whether v-coordinate increases or decreases as a point moves along the edge according to the direction. For each possibly-partial non-∞-edge e in the edgepath, we assign $+1$ or -1 as the *sign* $\sigma(e)$ if the edge e is increasing or decreasing.

2.2.8. Monotonic edgepath system. At each point $\langle p/q \rangle$ with $q \geq 2$, there exist exactly one increasing leftward edge and exactly one decreasing leftward edge. Hence, for a fixed tangle, there exists a unique basic edgepath consisting of only increasing leftward edges, which is called the *monotonically increasing basic edgepath*†. Similarly, there is a unique *monotonically decreasing basic edgepath*†. Note that they are minimal. Then, for a fixed Montesinos knot K, there exist the unique basic edgepath system consisting of monotonically increasing basic edgepaths and the unique basic edgepath system consisting of monotonically decreasing basic edgepaths, which are called the *monotonically increasing basic edgepath system*†and the *monotonically decreasing basic edgepath system*†. Let Λ_{inc} and Λ_{dec} denote them respectively. When we regard the corresponding extended basic edgepath systems as a function, naturally, $\widetilde{\Lambda_{\mathrm{dec}}}(u) \leq \widetilde{\Lambda}(u) \leq \widetilde{\Lambda_{\mathrm{inc}}}(u)$ holds for any extended basic edgepath system $\widetilde{\Lambda}$ for a fixed K and any $0 \leq u \leq 1$. In addition, Λ_{inc} and Λ_{dec} are convex and concave functions respectively.

2.2.9. Length of edgepath. For each, possibly partial, edge in the edgepath, we assign the length $|e|$, where the length of a complete edge is 1 and the length of a partial edge is less than 1. Precisely, as calculated in [7] or [9], a partial edge e, which is included in a complete edge $\langle p/q\rangle - \langle r/s\rangle$ and has u_0 as the u-coordinate of the endpoint, has the length

$$|e| \quad = \quad \frac{1+s(u_0-1)}{(s-q)(u_0-1)}. \tag{2.2}$$

2.3. Incompressibility. As mentioned in Remark 2.1, a set of surfaces correspond to a candidate edgepath system. In view of this, a candidate edgepath system is called *incompressible*, *compressible* or *indeterminate*, if all the corresponding surfaces are essential, all the corresponding surfaces are inessential, or the set of the corresponding surfaces includes both essential and inessential ones, respectively.

Remark 2.2. The aim of the algorithm of Hatcher and Oertel is to enumerate all the boundary slopes of the orientable essential surfaces. In fact, they determine π_1-injectivity instead of incompressibility. For instance, the difference of these two notions is mentioned in [8]. Note that π_1-injectivity is stronger than incompressibility. This does not matter in [7], since π_1-injectivity is equivalent to incompressibility for orientable surfaces. Though, since we here deal with possibly non-orientable surfaces, we have to be careful about the difference. Fortunately, in what follows, this distinction does not cause any difficulties.

Next, we recall two notions used in [7], and give a lemma about incompressibility of edgepath systems used in later arguments.

Two successive edges $\langle p_3/q_3\rangle - \langle p_2/q_2\rangle - \langle p_1/q_1\rangle$ on the diagram $\mathcal{D}$ are said to be *reversible* if the two edges lie in two triangles of $\mathcal{D}$ sharing a common edge. For example, $\langle 1/0\rangle - \langle 0\rangle - \langle 1/2\rangle$ is reversible since these edges lie in two triangles sharing a common edge $\langle 0\rangle - \langle 1\rangle$. An edgepath is said to be *completely reversible* if all pairs of two successive edges in it are reversible.

For an edgepath, if the last edge of the edgepath is included in $\langle p/q\rangle - \langle r/s\rangle$, then the *final r-value* of the edgepath is defined to be $(s-q)$. In some cases, we may give a positive or negative sign to r-values according to whether the last edge is increasing or decreasing respectively. An edgepath system has a *cycle of final r-values* obtained by collecting the final r-values of N edgepaths in the system.

Here, we summarize some results about incompressibility in propositions in [7] used in this paper.

Lemma 2.1. *For a Montesinos knot $K = M(T_1, T_2, \cdots, T_N)$, where $N \geq 3$, the following hold.*

(1) *A monotonically decreasing type I edgepath system is incompressible.*

(2) *Suppose that a type I or type II edgepath system Γ lying in $\mathcal{S}$ has $(+1, -2, r_3)$ with $r_3 \leq -5$ as its cycle of final r-values. Then Γ is incompressible unless the last edge of γ_3 lies in the same triangle of $\mathcal{S}$, and has the same ending point, as the edge with $r = 2$.*

(3) (a) *A monotonically decreasing type II edgepath system is incompressible.*
(b) *If a basic edgepath system Λ does not satisfy the condition (*):*

> The cycle of final r-values for the γ_i's contains at least one -1, and each -1 in this cycle is separated from the next -1 by r_i's all but possibly one of which are $-\tilde{2}$'s

then there exists an incompressible or indeterminate type II edgepath system obtained from Λ by adding upward vertical edges. Here an edgepath with $-\tilde{2}$ means that the edgepath has final r-value -2 and is completely reversible.

(4) *A type III edgepath system Γ is compressible if and only if $|\Gamma(0)| \leq 1$ holds and at least $N - 2$ edgepaths are completely reversible. On the other hand, a type III edgepath system is incompressible if it is constructed from a basic edgepath system Λ with $|\Lambda(0)| \geq 2$.*

Proof. (1), (3)(a): By Corollary 2.4 and Propositions 2.6, 2.7 and 2.8(a) in [7], if a type I or type II edgepath system is compressible or indeterminate, then its cycle of final r-values must include both positive and negative terms. Since all the signed final r-values of the monotonic edgepath system have a common sign, the edgepath system is incompressible.
(2): Follows from Proposition 2.7(3)(a) in [7].
(3)(b): Follows from Proposition 2.9 in [7], where the condition (*) is stated.
(4): Follows from Proposition 2.5 in [7]. □

3. Calculation and comparison of the twists

This section is devoted to proving Theorem 1.1. The key is the estimation of the maximal and minimal "twists" defined as follows.

The *twist* $\tau(\gamma)$ of an edgepath γ is defined as the sum of $-2\,\sigma(e_i)\,|e_i|$ for non-∞-edges $\{e_i\}$ included in γ. The *twist* $\tau(\Gamma)$ of an edgepath system Γ is then defined as the sum of $\tau(\gamma_i)$ for edgepaths γ_i included in $\Gamma = (\gamma_1, \gamma_2, \cdots, \gamma_N)$. Then, as described in [7], the boundary slope of an essential surface associated to an edgepath system Γ is calculated by $\tau(\Gamma) - \tau(\Gamma_{\text{Seifert}})$, where Γ_{Seifert} denotes the edgepath system Γ to which a Seifert surface for K is associated. Note that ∞-edges and constant edgepaths contribute nothing to the twist.

In the following, we call the maximum among the twists of all the incompressible or indeterminate candidate edgepath systems for a Montesinos knot K the *maximal* *twist* for K. The *minimal twist* for K is defined in the same way.

3.1. Estimation of maximal and minimal twists. We establish the following estimates of the maximal and minimal twists.

Proposition 3.1. *Let $\tau_{\max}$ and $\tau_{\min}$ denote the maximal and minimal twists, respectively, for a Montesinos knot K. Let Λ_{inc} and Λ_{dec} be the monotonically increasing and decreasing basic edgepaths for K with the twists τ_{inc} and τ_{dec}, respectively. Then $\tau_{\max}$ satisfies*

$$\begin{cases} \tau_{\max} = \tau_{\text{dec}} + 2\,\Lambda_{\text{dec}}(0) \geq \tau_{\text{dec}} & \textit{if } \Lambda_{\text{dec}}(0) \geq 0, \\ \tau_{\max} \geq \tau_{\text{dec}} - 6 & \textit{if } \Lambda_{\text{dec}}(0) = -1, \\ \tau_{\max} = \tau_{\text{dec}} & \textit{if } \Lambda_{\text{dec}}(0) \leq -2. \end{cases}$$

Also $\tau_{\min}$ satisfies

$$\begin{cases} \tau_{\min} = \tau_{\text{inc}} + 2\,\Lambda_{\text{inc}}(0) \leq \tau_{\text{inc}} & \textit{if } \Lambda_{\text{inc}}(0) \leq 0, \\ \tau_{\min} \leq \tau_{\text{inc}} + 6 & \textit{if } \Lambda_{\text{inc}}(0) = +1, \\ \tau_{\min} = \tau_{\text{inc}} & \textit{if } \Lambda_{\text{inc}}(0) \geq +2. \end{cases}$$

Once the above proposition is established, Theorem 1.1 is proved as follows.

Proof of Theorem 1.1. As remarked in Subsection 2.1, Theorem 1.1 holds if the number of tangles N in a Montesinos knot K is at most two. Thus we assume that $N \geq 3$. In the following we use the same notations as in Proposition 3.1. If $\Lambda_{\text{dec}}(0) \geq 0$ or $\Lambda_{\text{inc}}(0) \leq 0$ holds, then K is alternating, and we have already shown that $\text{Diam}(K) = 2\,\text{cr}(K)$ in [10]. In other cases, as in [10], $2\,\text{cr}(K) = \tau_{\text{dec}} - \tau_{\text{inc}}$ holds. If $\Lambda_{\text{dec}}(0) = -1$, we have $\tau_{\max} \geq \tau_{\text{dec}} - 6$ and $\tau_{\min} = \tau_{\text{inc}}$. If $\Lambda_{\text{inc}}(0) = +1$, we have $\tau_{\max} = \tau_{\text{dec}}$ and $\tau_{\min} \leq \tau_{\text{inc}} + 6$. Otherwise, we have $\tau_{\max} \geq \tau_{\text{dec}}$ and $\tau_{\min} \leq \tau_{\text{inc}}$. Hence, in all cases, $\text{Diam}(K) = \tau_{\max} - \tau_{\min} \geq \tau_{\text{dec}} - \tau_{\text{inc}} - 6 = 2\,\text{cr}(K) - 6$. Note that $\Lambda_{\text{dec}}(0) = -1$ and $\Lambda_{\text{inc}}(0) = +1$ cannot occur at the same time since the relation $\Lambda_{\text{inc}}(0) = \Lambda_{\text{dec}}(0) + N$ holds and we are assuming $N \geq 3$. □

3.2. Estimation of twists. In order to prove Proposition 3.1, we have to estimate the maximal and minimal twists for a Montesinos knot K. In this subsection, we prepare two lemmas giving formulae for comparing twists of edgepath systems.

3.2.1. Twists of type I edgepath systems. Here we give a lemma used to compare the twists of type I edgepath systems. In the proof, we introduce an integration formula to compute twists of edgepath systems, which is of interest independently.

In the following, we regard an edgepath system as a continuous and piecewise-affine function $[0,1] \to \mathbb{R}$ as explained in 2.2.3. Precisely, for a basic edgepath system or a type I edgepath system Γ, we define a continuous and piecewise-affine function $\varphi : [0,1] \to \mathbb{R}$ as follows. In the case that Γ is a basic edgepath system Λ, we define φ to be $\widetilde{\Lambda}$ regarded as a function. If Γ is of type I, assuming that Γ is constructed by cutting a basic edgepath system Λ at $u = u_0$, we define φ so that $\varphi(u) = 0 = \widetilde{\Lambda}(u_0)$ for $0 \leq u \leq u_0$ and $\varphi(u) = \widetilde{\Lambda}(u)$ for $u_0 \leq u \leq 1$.

Lemma 3.2. *Let $\Lambda_{\rm dec}$ be the monotonically decreasing basic edgepath system for a Montesinos knot K.*

(1) *For any type I edgepath system Γ, its twist $\tau(\Gamma)$ satisfies $\tau(\Gamma) \leq \tau(\Lambda_{\rm dec}) + 2\Lambda_{\rm dec}(0) \leq \tau(\Lambda_{\rm dec}) - 2$.*

(2) *Assume that $\Lambda_{\rm dec}(0) = -1$.*

  (a) *Assume further that there exists a solution $u = u_0$ for the equation $\widetilde{\Lambda_{\rm dec}}(u) = 0$. Let τ be the twist of the type I edgepath system $\Gamma_{\rm I,dec}$ obtained by cutting $\Lambda_{\rm dec}$ at $u = u_0$.*

   (a1) *If $0 < u_0 \leq 1/2$ holds, then τ satisfies $\tau(\Lambda_{\rm dec}) - 4 \leq \tau < \tau(\Lambda_{\rm dec}) - 2$. This twist is maximal among all type I edgepath systems.*

   (a2) *If $1/2 < u_0 \leq 2/3$ holds, then τ satisfies $\tau(\Lambda_{\rm dec}) - 6 \leq \tau < \tau(\Lambda_{\rm dec}) - 4$. This twist is maximal among all type I edgepath systems.*

  (b) *Assume further that $\Lambda_{\rm dec}(1/2) < 0$ holds. Then, for any type I edgepath system Γ, its twist $\tau(\Gamma)$ satisfies $\tau \leq \tau(\Lambda_{\rm dec}) - 4$.*

Proof. We first prepare the following claim.

Claim. Let Γ_a and Γ_b be basic or type I edgepath systems. Assume that φ_a and φ_b are the corresponding functions $[0, 1] \to \mathbb{R}$ for the edgepath systems. If $\varphi_a(u) \geq \varphi_b(u)$ holds for $0 \leq u \leq 1$, then, their twists satisfy $\tau(\Gamma_a) \leq \tau(\Gamma_b)$.

Proof. The following subclaim gives an integration formula to compute twists.

Subclaim. Let Γ be a basic edgepath system or an edgepath system of type I. Then its twist is calculated by the integration of the form

$$\tau(\Gamma) = \int_1^0 -\frac{2}{(u-1)^2}\frac{d\varphi(u)}{du}du. \tag{3.1}$$

Proof. We fix a basic edgepath system Λ and the extended basic edgepath system $\widetilde{\Lambda} = (\widetilde{\lambda_1}, \widetilde{\lambda_2}, \cdots, \widetilde{\lambda_N})$ for Λ. Let $\Gamma_{u_0} = (\gamma_{1,u_0}, \gamma_{2,u_0}, \cdots, \gamma_{N,u_0})$ denote a type I edgepath system obtained from Λ by cutting at $u = u_0$ in our manner. Though Γ_{u_0} may not satisfy the gluing consistency (2.1), we can calculate the twist $\tau(\gamma_{i,u_0})$ and $\tau(\Gamma_{u_0})$ formally. We regard $\tau(\Gamma_u)$ as a function of u from $[0, 1]$ to $\mathbb{R}$.
Assume first that γ_i ends at a point on a decreasing edge $\langle p/q\rangle - \langle r/s\rangle$, where $ps - qr = -1$. Let α denote the twist of complete edges of γ_i included completely in $u \geq u_0$. The derivative of the twist of the edgepath $\gamma_{u_0,i}$ at u_0 is calculated as

$$\frac{d\tau(\gamma_{i,u_0})}{du_0} = \frac{d}{du_0}\left(\alpha + 2\,\frac{1 + s(u_0 - 1)}{(s-q)(u_0 - 1)}\right) = -\frac{2}{(s-q)(u_0-1)^2}$$

using the formula (2.2) of length of a partial edge in Subsubsection 2.2.9. Note that α is constant at $(q-1)/q \le u_0 \le (s-1)/s$. Besides,

$$\frac{d\widetilde{\lambda}_i(u_0)}{du_0} = \frac{d}{du_0}\left(\frac{r}{s}+\left(\frac{p}{q}-\frac{r}{s}\right)\frac{\dfrac{s-1}{s}-u_0}{\dfrac{s-1}{s}-\dfrac{q-1}{q}}\right) = \frac{1}{s-q}.$$

Hence,

$$\frac{d\tau(\gamma_{i,u_0})}{du_0} = -\frac{2}{(u_0-1)^2}\frac{d\widetilde{\lambda}_i(u_0)}{du_0}. \tag{3.2}$$

This identity also holds if γ_i ends at a point on an increasing edge or a horizontal edge. Since $\tau(\Gamma_{u_0})$ is 0 for u_0 close to 1, by summing up (3.2) and performing integration, we have

$$\begin{aligned}
\tau(\Gamma_{u_0}) &= \tau(\Gamma_{u_0})-\tau(\Gamma_1) = \sum_{i=1}^{N}(\tau(\gamma_{i,u_0})-\tau(\gamma_{i,1})) = \sum_{i=1}^{N}\int_1^{u_0}\frac{d\tau(\gamma_{i,u})}{du}du \\
&= \sum_{i=1}^{N}\int_1^{u_0}\left(-\frac{2}{(u-1)^2}\frac{d\widetilde{\lambda}_i(u)}{du}\right)du = \int_1^{u_0}-\frac{2}{(u-1)^2}\frac{d\widetilde{\Lambda}(u)}{du}du. \\
&= \int_1^{0}-\frac{2}{(u-1)^2}\frac{d\varphi(u)}{du}du.
\end{aligned}$$

□

By this subclaim, we have;

$$\begin{aligned}
\tau(\Gamma_a)-\tau(\Gamma_b) &= \int_1^0-\frac{2}{(u-1)^2}\frac{d\varphi_a(u)}{du}du-\int_1^0-\frac{2}{(u-1)^2}\frac{d\varphi_b(u)}{du}du \\
&= \int_1^0-\frac{2}{(u-1)^2}\frac{d(\varphi_a(u)-\varphi_b(u))}{du}du \\
&= \left[-\frac{2}{(u-1)^2}(\varphi_a(u)-\varphi_b(u))\right]_1^0 \\
&\quad -\int_1^0\frac{d}{du}\left(-\frac{2}{(u-1)^2}\right)\cdot(\varphi_a(u)-\varphi_b(u))du \\
&\le 0.
\end{aligned}$$

Note that $\varphi_a(0)=\varphi_b(0)=0$ holds by the gluing consistency. Since Γ_a and Γ_b come from the same Montesinos knot $K=M(T_1,T_2,\cdots,T_N)$, edges of the edgepaths $\gamma_{a,i}$ and $\gamma_{b,i}$ near $u=1$ are both the horizontal edge corresponding to T_i . Hence, there exists $u_1<1$ such that $\varphi_a(u)=\varphi_b(u)$ holds for $u_1<u\le 1$. Eventually, the square bracket has the value zero. □

In the following, let φ_{dec} denote the function for the monotonically decreasing basic edgepath system Λ_{dec}.
(1): Let $\varphi_1 : [0,1] \to \mathbb{R}$ be a type I edgepath system Γ as a function. From Λ_{dec}, by replacing the last segment $(0,z)-(u_1, z+v_1)$ with a polygonal line $(0,0)-(\varepsilon,z)-(u_1,z+v_1)$, we have a function $\varphi_{2,\varepsilon} : [0,1] \to \mathbb{R}$, which is pretty close to Λ_{dec} in $[\varepsilon, 1]$ and is pretty close to the vertical edges connecting $\langle \Lambda_{\mathrm{dec}}(0)\rangle$ and $\langle 0 \rangle$. Since $\varphi_1(u) \geq \varphi_{2,\varepsilon}(u)$ holds for $0 \leq u \leq 1$, by the method in the above claim, we have

$$\begin{aligned}
\tau(\Gamma) &= \int_1^0 -\frac{2}{(u-1)^2}\frac{d\varphi_1(u)}{du}du \\
&\leq \int_1^0 -\frac{2}{(u-1)^2}\frac{d\varphi_{2,\varepsilon}(u)}{du}du \\
&= \int_\varepsilon^0 -\frac{2}{(u-1)^2}\frac{d\varphi_{2,\varepsilon}(u)}{du}du + \int_1^\varepsilon -\frac{2}{(u-1)^2}\frac{d\varphi_{2,\varepsilon}(u)}{du}du
\end{aligned}$$

for each sufficiently small $\varepsilon > 0$. Now, as ε goes to 0, the last expression converges to $2\Lambda_{\mathrm{dec}}(0) + \tau(\Lambda_{\mathrm{dec}})$. Hence,

$$\tau(\Gamma) \leq 2\Lambda_{\mathrm{dec}}(0) + \tau(\Lambda_{\mathrm{dec}}) \leq \tau(\Lambda_{\mathrm{dec}}) - 2.$$

Note that $\Lambda_{\mathrm{dec}}(0) \leq -1$ is necessary to give a type I edgepath system.
(2)(a1): Let $\varphi_{\mathrm{I,dec}}$ denote the function corresponding to the monotonically decreasing type I edgepath system $\Gamma_{\mathrm{I,dec}}$. Then $\varphi_{\mathrm{I,dec}}$ and φ_{dec} coincide for $u \geq u_0$, while $\varphi_{\mathrm{I,dec}}(u) = 0$ and $\varphi_{\mathrm{dec}}(u) = u/u_0 - 1$ hold for $0 \leq u \leq u_0$. By the above subclaim,

$$\begin{aligned}
\tau(\Gamma_{\mathrm{I,dec}}) - \tau(\Lambda_{\mathrm{dec}}) &= -\int_{u_0}^0 -\frac{2}{(u-1)^2}\frac{d\varphi_{\mathrm{dec}}(u)}{du}du \\
&= -\int_{u_0}^0 -\frac{2}{(u-1)^2}\frac{1}{u_0}du = -\frac{2}{1-u_0} \in [-4,-2).
\end{aligned}$$

Suppose that there exists another type I edgepath system Γ whose twist is greater than $\tau(\Gamma_{\mathrm{I,dec}})$ and which is obtained by cutting a basic edgepath system Λ at $u = u_1$. Then we note that Λ must include a part below the horizontal line $v = 0$ by the above claim. On the other hand, Λ satisfies $\Lambda(u_1) = 0$, $\Lambda(u_2) < 0$ and $\Lambda(1/2) \geq 0$ for some u_1 and u_2 satisfying $u_1 < u_2 < u_0 \leq 1/2$. However, this is impossible since Λ is affine in the interval $0 \leq u \leq 1/2$.
(2)(a2): Again, let $\varphi_{\mathrm{I,dec}}$ denote the function corresponding to the monotonically decreasing type I edgepath system $\Gamma_{\mathrm{I,dec}}$. Let φ_1 be a function defined by $\varphi_1(u) = u/u_0 - 1$ for $0 \leq u \leq u_0$ and $\varphi_1(u) = \varphi_{\mathrm{dec}}(u)$ for $u_0 \leq u \leq 1$. The graph of φ_1 includes a segment connecting $(0,-1)$ and $(u_0, 0)$. Thus $\varphi_{\mathrm{dec}}(u) \geq \varphi_1(u)$ holds for $0 \leq u \leq 1$. Note that Λ_{dec} is a concave function. Then, by the above claim and

subclaim we have

$$\begin{aligned}\tau(\Gamma_{\mathrm{I,dec}}) - \tau(\Lambda_{\mathrm{dec}}) &\geq \tau(\Gamma_{\mathrm{I,dec}}) - \tau(\varphi_1) \\ &= -\int_{u_0}^{0} -\frac{2}{(u-1)^2}\frac{d\varphi_1(u)}{du}du = -\int_{u_0}^{0} -\frac{2}{(u-1)^2}\frac{1}{u_0}du \\ &= -\frac{2}{1-u_0} \geq -6.\end{aligned}$$

Also let φ_2 be a function defined by $\varphi_2(u) = 2u - 1$ for $0 \leq u \leq 1/2$, $\varphi_2(u) = 0$ for $1/2 \leq u \leq u_0$, and $\varphi_2(u) = \varphi_{\mathrm{dec}}(u)$ for $u_0 \leq u \leq 1$. The graph of φ_2 includes a piecewise affine path connecting $(0, -1)$, $(1/2, 0)$ and $(u_0, 0)$. Thus $\varphi_{\mathrm{I,dec}}(u) \geq \varphi_2(u)$ holds for $0 \leq u \leq 1$. Again, by the above claim and subclaim, we have

$$\begin{aligned}\tau(\Gamma_{\mathrm{I,dec}}) - \tau(\Lambda_{\mathrm{dec}}) &< \tau(\Gamma_{\mathrm{I,dec}}) - \tau(\varphi_2) \\ &= -\int_{\frac{1}{2}}^{0} -\frac{2}{(u-1)^2}\frac{d\varphi_2(u)}{du}du = -\int_{\frac{1}{2}}^{0} -\frac{2}{(u-1)^2}\cdot 2du \\ &= -4.\end{aligned}$$

Here, $\tau(\varphi)$ denotes the integration value of (3.1) for the function φ.
Suppose that Γ is another type I edgepath system whose twist τ is greater than $\tau(\Gamma_{\mathrm{I,dec}})$. A basic edgepath system Λ including Γ must include a part below the line $v = 0$. Since Λ is affine in the intervals $0 < u < 1/2$ and $1/2 < u < 2/3$, Λ satisfies $\Lambda(0) > 0$, $\Lambda(1/2) < 0$ and $\Lambda(2/3) \geq 0$. However, this is impossible as follows. $\Lambda(1/2) < 0$ means that only one of the edgepaths satisfies $z + 1/2 \leq \lambda_i(1/2) < z + 1$ for some integer z, and thus, at most one of the basic edgepaths can end with an increasing edge. Thus, $\Lambda(0) = 0$ or -1.
(2)(b) Assume that Γ is a type I edgepath system with the function φ. Let φ_1 be a function defined by $\varphi_1(u) = 2 \cdot \Lambda_{\mathrm{dec}}(1/2) \cdot u$ for $0 \leq u \leq 1/2$ and $\varphi_1(u) = \Lambda_{\mathrm{dec}}(u)$ for $1/2 \leq u \leq 1$. The graph of φ_1 includes a segment connecting $(0, 0)$ and $(1/2, \Lambda_{\mathrm{dec}}(1/2))$. Then, $\varphi(u) \geq \varphi_1(u) \geq \varphi_{\mathrm{dec}}(u)$ holds for $0 \leq u \leq 1$. Note that $\varphi_{\mathrm{dec}}(u) = 2 \cdot (\Lambda_{\mathrm{dec}}(1/2) + 1)u - 1$ holds for $0 \leq u \leq 1/2$. By the above claim and subclaim, we have;

$$\begin{aligned}\tau(\Gamma) - \tau(\Lambda_{\mathrm{dec}}) &\leq \tau(\varphi_1) - \tau(\Lambda_{\mathrm{dec}}) \\ &= \int_{1}^{0} -\frac{2}{(u-1)^2}\frac{d(\varphi_1(u) - \varphi_{\mathrm{dec}}(u))}{du}du = \int_{\frac{1}{2}}^{0} -\frac{2}{(u-1)^2}\cdot(-2)du \\ &= -4.\end{aligned}$$

□

3.2.2. Twists of type II or III edgepath systems. The second lemma of this subsection is also for calculating twist and is useful for type II and type III edgepath systems. We first introduce a classification of edgepath systems.

Definition 3.3.

(1) For a basic edgepath λ, we call it: (a) a class A basic edgepath if it is a monotonically decreasing basic edgepath λ_{dec}, (b) a class B basic edgepath if λ and λ_{dec} bound a single triangle and a single vertical edge in the strip $\mathcal{S}$, (c) a class C basic edgepath if λ and λ_{dec} bound two triangles and no vertical edges in the strip $\mathcal{S}$. A class B basic edgepath is obtained from the monotonically decreasing basic edgepath by replacing an edge $\langle z+0\rangle$–$\langle z+1/2\rangle$ by an edge $\langle z+1\rangle$–$\langle z+1/2\rangle$ for some integer z.

(2) For a basic edgepath system Λ, we call it: (a) a class A basic edgepath system if it is a monotonically decreasing basic edgepath system, (b) a class B/C basic edgepath system if exactly one basic edgepath in the basic edgepath system is class B/C and all the other edgepaths are monotonically decreasing (class A),

(3) For a type II edgepath system, it is called class A/B/C if it is obtained from a class A/B/C basic edgepath system by extending by vertical edges.

(4) For a type III edgepath system, it is called class A/B/C if it is obtained from a class A/B/C basic edgepath system by extending by ∞-edges for all edgepaths.

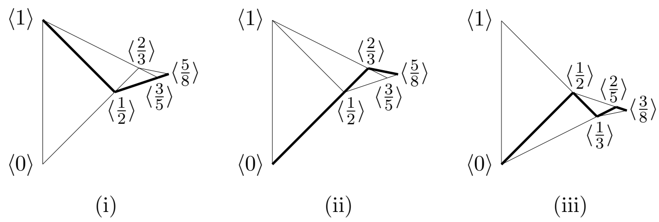

Figure 4: (i) A class B basic edgepath for $\langle 5/8\rangle$, (ii) a class C basic edgepath for $\langle 5/8\rangle$ and (iii) a not-class-C basic edgepath for $\langle 3/8\rangle$. Note that the edgepath in (iii) is not class C since it is not minimal.

Then we obtain the following lemma.

Lemma 3.4. *Let Λ_{dec} be the monotonically decreasing edgepath system with twist $\tau(\Lambda_{\text{dec}})$ for a Montesinos knot K.*

(1) *Let Γ_{II} be a type II edgepath system with twist $\tau(\Gamma_{\text{II}})$. Then,* (a) $\tau(\Gamma_{\text{II}}) = \tau(\Lambda_{\text{dec}}) + 2\,\Lambda_{\text{dec}}(0)$ *if and only if Γ_{II} is of class A,* (b) $\tau(\Gamma_{\text{II}}) = \tau(\Lambda_{\text{dec}}) + 2\,\Lambda_{\text{dec}}(0) - 2$ *if and only if Γ_{II} is of class B,* (c) *otherwise,* $\tau(\Gamma_{\text{II}}) \le \tau(\Lambda_{\text{dec}}) + 2\,\Lambda_{\text{dec}}(0) - 4$.

(2) *Let Γ_{III} be a type III edgepath system with twist $\tau(\Gamma_{\text{III}})$. Then,* (a) $\tau(\Gamma_{\text{III}}) = \tau(\Lambda_{\text{dec}})$ *if and only if Γ_{III} is monotonically decreasing,* (b) $\tau(\Gamma_{\text{III}}) = \tau(\Lambda_{\text{dec}}) - 4$ *if and only if Γ_{III} is of class B or C,* (c) *otherwise,* $\tau(\Gamma_{\text{III}}) \le \tau(\Lambda_{\text{dec}}) - 6$.

Proof. Let $\Lambda_{\rm dec} = (\lambda_{{\rm dec},1}, \lambda_{{\rm dec},2}, \cdots, \lambda_{{\rm dec},N})$ be the monotonically decreasing basic edgepath system.
We first prepare two combinatorial values defined for edgepath systems.
We consider a basic edgepath system $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ for K.
Let L_i and V_i denote the number of triangles and the number of vertical edges in the strip $\mathcal{S}$ bounded by the λ_i and $\lambda_{{\rm dec},i}$ respectively. Set $L = \sum_{i=1}^{N} L_i$ and $V = \sum_{i=1}^{N} V_i$.
Then we have the following claim.

Claim. Let L and V denote the corresponding values for Λ. Then the twist $\tau(\Lambda)$ is calculated by

$$\tau(\Lambda) = \tau(\Lambda_{\rm dec}) - 2\,(L + V).$$

Proof. Let $\{t_1, t_2, \cdots, t_L\}$ be the triangles bounded by the λ_i's and $\lambda_{{\rm dec},i}$'s. Let $\vec{l_i}$ be the directed loop bounding the triangle t_i with anti-clockwise orientation. Let $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_V\}$ be the bounded downward vertical edges. Let $\overrightarrow{\Lambda_{\rm dec}}$ and $\overrightarrow{\Lambda}$ denote the sets of directed edgepaths of $\Lambda_{\rm dec}$ and Λ with the right-to-left orientation. For a leftward edge $\vec{e}$ and $-\vec{e}$ the same edge with opposite orientation, we here define the effect of $-\vec{e}$ on the twist to be the negative of the effect of $\vec{e}$. The effects of $\vec{e}$ and $-\vec{e}$ on the twist cancel out. Then, the directed arcs and loops satisfy the following identity.

$$\sum_{\vec{e} \in \overrightarrow{\Lambda_{\rm dec}} \cup (\cup_{j=1}^{L} \vec{l_j})} \tau(\vec{e}) = \sum_{\vec{e} \in \overrightarrow{\Lambda} \cup (\cup_{j=1}^{V} \vec{v}_j)} \tau(\vec{e}).$$

Now, three vertices of a triangle in the strip $\mathcal{S}$ can be expressed as $P_1 = \langle p/q \rangle = ((q-1)/q, p/q)$, $P_2 = \langle r/s \rangle = ((s-1)/s, r/s)$ and $P_3 = \langle (p+r)/(q+s) \rangle = ((q+s-1)/(q+s), (p+r)/(q+s))$. Note that the u-coordinate of P_3 is greater than those of P_1 and P_2, and the v-coordinate of P_3 lies between those of P_1 and P_2. Thus, the shapes of triangles are somehow restricted. As a result, the contribution of $\vec{l_j}$ is -2. Also, the contribution of a downward vertical edge $\vec{v}_j$ is $+2$. Thus, we have $\tau(\Lambda_{\rm dec}) - 2L = \tau(\Lambda) + 2V$. The twist of Λ is calculated by $\tau(\Lambda) = \tau(\Lambda_{\rm dec}) - 2\,(L+V)$. □

(1) A type II edgepath system $\Gamma_{\rm II}$ is obtained from a basic edgepath system Λ by extending by vertical edges. Let L and V denote the corresponding values for Λ. Now, the effect on the twist of $\Gamma_{\rm II}$ by vertical edges is $2\,\Lambda_{\rm dec}(0) + 2V$. Thus, by the claim above, we have; $\tau(\Gamma_{\rm II}) = \tau(\Lambda) + 2\Lambda_{\rm dec}(0) + 2V = \tau(\Lambda_{\rm dec}) - 2\,(L + V) + 2\Lambda_{\rm dec}(0) + 2V = \tau(\Lambda_{\rm dec}) - 2L + 2\Lambda_{\rm dec}(0)$. Note that if $L \le 1$, then $V \le 1$ holds, and $(L, V) = (1, 0)$ is non-minimal since $(L_i, V_i) = (1, 0)$ must hold for some i. Thus, if $L \le 1$, then the possible values for (L, V) are $(0, 0)$ or $(1, 1)$. These correspond to class A and class B type II edgepath systems.
(2) The twist of a type III edgepath system is the same as its corresponding basic edgepath system Λ. Let L and V denote the corresponding values for Λ. Then, by the claim above, the twist $\tau(\Lambda)$ is calculated as $\tau(\Lambda) = \tau(\Lambda_{\rm dec}) - 2\,(L + V)$. Here we consider the cases satisfying $L + V \le 2$. By definition, V_i is 0 or 1 for each

i. If $V_i = 1$, then $L_i \geq 1$, since some increasing edges must appear in λ_i. Hence, $L_i \geq V_i$ holds for each i, and so does $L \geq V$. If $(L, V) = (1, 0)$, then $(L_i, V_i) = (1, 0)$ holds for some i, and this means that λ_i is not minimal. Eventually, if $L + V \leq 2$, then (L, V) is $(0, 0)$, $(1, 1)$ or $(2, 0)$. Moreover, if (L, V) is $(1, 1)$ or $(2, 0)$, then (L_i, V_i) is $(1, 1)$ or $(2, 0)$ respectively for some unique i, since $(L_i, V_i) = (1, 0)$ is impossible. Thus, at most one basic edgepath of a basic edgepath system can be non-monotonically-decreasing. This implies the assertions (a), (b) and (c) in (2). □

3.3. Proof of Proposition 3.1. This subsection is devoted to proving Proposition 3.1.

In the following proof, we divide the set of Montesinos knots into some classes by the nature of their monotonic basic edgepath systems. For each class, we find an edgepath system which is incompressible or indeterminate, and then calculate or estimate its twist. Since the edgepath system is taken so that its twist gives an appropriate bound on the maximal/minimal twist, this will be enough to prove Proposition 3.1. In fact, our main targets to study are monotonic or nearly monotonic edgepath systems.

Proof of Proposition 3.1. We only prove the assertion about the maximal twist. This is sufficient since the assertion about the minimal twist must hold by symmetry. That is, since $K = M(T_1, T_2, \cdots, T_N)$ has as its mirror image $K' = M(-T_1, -T_2, \cdots, -T_N)$, the assertion about the minimal twist of the knot K is immediately obtained from the assertion about the maximal twist of the knot K'.

Let Λ_{dec} be the monotonically decreasing edgepath system with twist $\tau(\Lambda_{\mathrm{dec}})$ for a Montesinos knot K.

By Lemma 2.1, depending on the value of $\Lambda_{\mathrm{dec}}(0)$, we may be able to obtain an incompressible type II or type III edgepath system. Hence, we first divide the Montesinos knots by the value of $\Lambda_{\mathrm{dec}}(0)$, as follows.

Case 1 : K satisfies $\Lambda_{\mathrm{dec}}(0) \geq 0$.

Case 2 : K satisfies $\Lambda_{\mathrm{dec}}(0) = -1$.

Case 3 : K satisfies $\Lambda_{\mathrm{dec}}(0) \leq -2$.

Remember that $\Lambda_{\mathrm{dec}}(0)$ is an integer.

Claim 1(Case 1)**.** For a Montesinos knot with $\Lambda_{\mathrm{dec}}(0) \geq 0$, there exists a monotonically decreasing type II edgepath system $\Gamma_{\mathrm{II,dec}}$ such that an essential surface is associated to $\Gamma_{\mathrm{II,dec}}$ and its twist $\tau(\Gamma_{\mathrm{II,dec}})$ satisfies that $\tau(\Gamma_{\mathrm{II,dec}}) \geq \tau(\Lambda_{\mathrm{dec}})$.

Proof. From Λ_{dec}, by extending one of its edgepaths by some downward vertical edges if necessary, we have a monotonically decreasing type II edgepath system $\Gamma_{\mathrm{II,dec}}$ in this case. This is incompressible by Lemma 2.1(3)(a). Its twist $\tau(\Gamma_{\mathrm{II,dec}})$ satisfies $\tau(\Gamma_{\mathrm{II,dec}}) = \tau(\Lambda_{\mathrm{dec}}) + 2\,\Lambda_{\mathrm{dec}}(0) \geq \tau(\Lambda_{\mathrm{dec}})$ by Lemma 3.4(1)(a). □

Claim 2(Case 3)**.** For a Montesinos knot with $\Lambda_{\mathrm{dec}}(0) \leq -2$, there is a monotonically decreasing type III edgepath system $\Gamma_{\mathrm{III,dec}}$ such that an essential surface is associated to $\Gamma_{\mathrm{III,dec}}$ and its twist $\tau(\Gamma_{\mathrm{III,dec}})$ satisfies that $\tau(\Gamma_{\mathrm{III,dec}}) = \tau(\Lambda_{\mathrm{dec}})$.

Proof. Since $\Lambda_{\mathrm{dec}}(0) \leq -2$ in this case, $\Gamma_{\mathrm{III,dec}}$ is incompressible by Lemma 2.1(4). Its twist is $\tau(\Gamma_{\mathrm{III,dec}}) = \tau(\Lambda_{\mathrm{dec}})$ by Lemma 3.4(2)(a). □

As we saw in Subsection 2.3, the final r-values of a basic edgepath system are important in showing the existence of a type II edgepath system with an incompressible surface by Lemma 2.1(3)(b). Let $(r_1, r_2, \cdots, r_N)$ denote the final r-values of the monotonically decreasing basic edgepath system. Note that all r_i's are negative naturally. We here divide Case 2 into several subcases according to the numbers of basic edgepaths with $r_i = -1$, $r_i = -2$ or $r_i \leq -3$, as follows.

Case 2-1 : K satisfies $\sharp\{i \,|\, r_i = -1\} = 0$.

Case 2-2 : K satisfies $\sharp\{i \,|\, r_i = -1\} = 1$.

- **Case 2-2-1** : Moreover, K satisfies $\sharp\{i \,|\, r_i \leq -3\} = 0$.
- **Case 2-2-2** : Moreover, K satisfies $\sharp\{i \,|\, r_i \leq -3\} = 1$.
  - **Case 2-2-2-1** : Moreover, K satisfies $\sharp\{i \,|\, r_i = -2\} = 1$. Or equivalently, $N = 3$.
  - **Case 2-2-2-2** : Moreover, K satisfies $\sharp\{i \,|\, r_i = -2\} \geq 2$. Or equivalently, $N \geq 4$.
- **Case 2-2-3** : Moreover, K satisfies $\sharp\{i \,|\, r_i \leq -3\} \geq 2$.

Case 2-3 : K satisfies $\sharp\{i \,|\, r_i = -1\} \geq 2$.

- **Case 2-3-1** : Moreover, Λ_{dec} of K satisfies (*) in Lemma 2.1(3)(b).
- **Case 2-3-2** : Moreover, Λ_{dec} of K does not satisfy (*) in Lemma 2.1(3)(b).

Claim 3(Cases 2-1, 2-2-3, 2-3-2)**.** For a Montesinos knot in these cases, an essential surface is associated to a type II edgepath system $\Gamma_{\mathrm{II,A}}$ obtained by extending the monotonically decreasing basic edgepath system by one upward vertical edge. It has the twist $\tau = \tau(\Lambda_{\mathrm{dec}}) - 2$.

Proof. Λ_{dec} does not satisfy the condition (*) in Lemma 2.1(3)(b) in all these three cases: For there is no $r_i = -1$ in Case 2-1, while there are 2 or more $r_i \leq -3$ despite $\sharp\{i \,|\, r_i = -1\} = 1$ in Case 2-2-3. Thus, by Lemma 2.1(3)(b), we have a class A type II edgepath system $\Gamma_{\mathrm{II,A}}$ to which an incompressible surface is associated. By Lemma 3.4(1)(a), $\Gamma_{\mathrm{II,A}}$ has twist $\tau = \tau(\Lambda_{\mathrm{dec}}) + 2\,\Lambda_{\mathrm{dec}}(0) = \tau(\Lambda_{\mathrm{dec}}) - 2$. □

Claim 4(Cases 2-2-1, 2-2-2-2, 2-3-1)**.** For a Montesinos knot in these cases, the monotonically decreasing type I edgepath system $\Gamma_{\mathrm{I,dec}}$ exists, an essential surface is associated to $\Gamma_{\mathrm{I,dec}}$, and $\Gamma_{\mathrm{I,dec}}$ has twist τ satisfying $\tau(\Lambda_{\mathrm{dec}}) - 4 \leq \tau < \tau(\Lambda_{\mathrm{dec}}) - 2$.

Proof. We consider a part of the graph of the function Λ_{dec} in the region $0 \leq u \leq 1/2$. The slope of the last segment of Λ_{dec} is $R = \sum_{i=1}^{N}(-1/r_i)$. The equation $\widetilde{\Lambda_{\mathrm{dec}}}(u_0) = 0$ is expressed by $R \cdot u_0 - 1 = 0$ in the interval $0 \leq u_0 \leq 1/2$. In these cases, R is 2 or greater since the final r-values include two r_i's equal to -1 or include one r_i equal to -1 and two r_i's equal to -2. Hence, the equation has a solution $u_0 = 1/R \in (0, 1/2]$. Thus, the monotonically decreasing type I edgepath system $\Gamma_{\mathrm{I,dec}}$ exists, and its endpoints have a common u-coordinates $0 < u_0 \leq 1/2$. By Lemma 2.1(1), $\Gamma_{\mathrm{I,dec}}$ is incompressible. As in Lemma 3.2(2)(a1), the twist of $\Gamma_{\mathrm{I,dec}}$ is $\tau(\Lambda_{\mathrm{dec}}) - 4 \leq \tau < \tau(\Lambda_{\mathrm{dec}}) - 2$. □

Next, we prove the claim for the remaining Case 2-2-2-1. In this case, a Montesinos knot K satisfies $N = 3$, $\Lambda_{\mathrm{dec}}(0) = -1$, $\sharp\{i \,|\, r_i = -1\} = 1$, $\sharp\{i \,|\, r_i = -2\} = 1$ and $\sharp\{i \,|\, r_i \leq -3\} = 1$. Without loss of generality, we can assume that $r_1 = -1$, $r_2 = -2$ and $r_3 \leq -3$. Furthermore, we can assume that T_1 satisfies $-1 < T_1 < 0$ and others satisfy $0 < T_i < 1$. That is, any knot K in this case can be isotoped into the normalized one. We here divide further the Montesinos knots in Case 2-2-2-1 into three cases.

Case a : K satisfies $-1/3 \leq T_1 < 0$.

Case b : K satisfies $-1/2 \leq T_1 < -1/3$.

  Case b-a : K satisfies $r_3 = -3$ or -4.

  Case b-b : K satisfies $r_3 \leq -5$.

Claim 5(Cases a, b-a)**.** For a Montesinos knot in these cases, the monotonically decreasing type I edgepath system $\Gamma_{\mathrm{I,dec}}$ exists, an essential surface is associated to $\Gamma_{\mathrm{I,dec}}$, and $\Gamma_{\mathrm{I,dec}}$ has twist τ satisfying $\tau(\Lambda_{\mathrm{dec}}) - 6 \leq \tau < \tau(\Lambda_{\mathrm{dec}}) - 4$.

Proof. $\Lambda_{\mathrm{dec}}(1/2) < 0$ holds since $\lambda_{\mathrm{dec},1}(1/2) = -1/2$, $\lambda_{\mathrm{dec},2}(1/2) = 1/4$, and $\lambda_{\mathrm{dec},3}(1/2) \leq 1/6$ hold. Besides, $\Lambda_{\mathrm{dec}}(2/3) \geq 0$ is obtained from $\lambda_{\mathrm{dec},1}(2/3) = -1/3$, $\lambda_{\mathrm{dec},2}(2/3) = 1/3$ and $\lambda_{\mathrm{dec},3}(2/3) > 0$ in Case a, from $\lambda_{\mathrm{dec},1}(2/3) \geq -1/2$, $\lambda_{\mathrm{dec},2}(2/3) = 1/3$ and $\lambda_{\mathrm{dec},3}(2/3) \geq 1/6$ in Case b-a. Then the solution $1/2 < u_0 \leq 2/3$ exists for the equation $\widetilde{\Lambda_{\mathrm{dec}}}(u) = 0$. Hence there exists a monotonically decreasing type I edgepath system $\Gamma_{\mathrm{I,dec}}$ with $1/2 < u_0 \leq 2/3$. By Lemma 2.1(1), $\Gamma_{\mathrm{I,dec}}$ is incompressible. By Lemma 3.2(2)(a2), $\Gamma_{\mathrm{I,dec}}$ has $\tau(\Lambda_{\mathrm{dec}}) - 6 \leq \tau < \tau(\Lambda_{\mathrm{dec}}) - 4$. □

Claim 6(Case b-b)**.** For a Montesinos knot in this case, from Λ_{dec}, a minimal class B type II edgepath system $\Gamma_{\mathrm{II,B}}$ is obtained by replacing a $\langle -1 \rangle - \langle -1/2 \rangle$ edge of $\lambda_{\mathrm{dec},1}$ by a $\langle 0 \rangle - \langle -1/2 \rangle$ edge. An essential surface is associated to $\Gamma_{\mathrm{II,B}}$, and $\Gamma_{\mathrm{II,B}}$ has twist τ satisfying $\tau = \tau(\Lambda_{\mathrm{dec}}) - 4$.

Proof. Let $\gamma_{\mathrm{II,B},i}$ denote the i-th edgepath of $\Gamma_{\mathrm{II,B}}$. Note that the edgepath $\gamma_{\mathrm{II,B},1}$ is still minimal since $T_1 < -1/3$ holds and $\gamma_{\mathrm{II,B},1}$ does not include the edge $\langle -1/2 \rangle - \langle -1/3 \rangle$. $\Gamma_{\mathrm{II,B}}$ is incompressible by Lemma 2.1(2) since the cycle of final r-values is $(+1, -2, r_3)$ with $r_3 \leq -5$. $\Gamma_{\mathrm{II,B}}$ has twist $\tau = \tau(\Lambda_{\mathrm{dec}}) - 4$, by Lemma 3.4(2)(b). □

Consequently, with these six claims, we have shown Proposition 3.1. □

Remark 3.1. The twists of the maximal and minimal boundary slopes are always non-negative and non-positive respectively, as follows. Since any basic edgepath has at least length 1 and any basic edgepath system has $N \geq 3$ edgepaths, we have $\tau_{\mathrm{dec}} \geq 6$ and $\tau_{\mathrm{inc}} \leq -6$. As described in the argument above, for the maximal or minimal boundary slope, both $\tau_{\max} - \tau_{\mathrm{dec}} \geq -6$ and $\tau_{\min} - \tau_{\mathrm{inc}} \leq 6$ hold. Hence, $\tau_{\min} \leq 0 \leq \tau_{\max}$.

Remark 3.2. Though it is not completely confirmed, the lower bound given in Theorem 1.1 seems to be sharp in a sense. We think about a family of Montesinos knot $K = M(-1/3, 1/3, 1/n)$, which correspond to Claim 5. By calculation, $\mathrm{Diam}(K) - 2\,\mathrm{cr}(K)$ seems to become arbitrary close to -6 as n goes to infinity. However, it seems that no Montesinos knots K can achieve $\mathrm{Diam}(K) = 2\,\mathrm{cr}(K) - 6$.

4. Remainder terms of edgepath systems

In this section, we will prove Theorem 1.3. The keys are the "remainder term" of an edgepath system and Proposition 4.1, which is a technical proposition about the remainder term.

The comparison of the twist and the Euler characteristic: the ratio of the Euler characteristic to the number of sheets for an essential surface with maximal or minimal boundary slope, plays a main role in the proof of Theorem 1.3. Let Γ be an edgepath system with twist $\tau(\Gamma)$. Then, as we saw in [9], the ratio $-\chi/\sharp s(F)$ for a surface F associated to Γ is roughly the sum of lengths of the edgepaths in Γ. Note that this ratio is determined independently of the choice of the surfaces corresponding to Γ. See [9] for details. Thus we also use the notation $-\chi/\sharp s(\Gamma)$ for the value corresponding to a surface associated to Γ. On the other hand, by definition, $\tau(\Gamma)$ is roughly calculated as twice the sum of signed lengths of the edges in Γ. Thus, if Γ is nearly monotonic, $\tau(\Gamma)$ is roughly twice $-\chi/\sharp s(\Gamma)$. In view of this, we define the *remainder term* $\rho(\Gamma)$ of Γ as $\rho(\Gamma) = |\tau(\Gamma)| - 2\,(-\chi/\sharp s(\Gamma))$.

In order to simplify the description, in the following, we use notations $\rho(F)$ and $\tau(F)$ instead of the $\rho(\Gamma)$ and $\tau(\Gamma)$, where Γ is associated to the surface F.

4.1. Proposition for remainder terms. Now let us state a technical proposition. The rest of this paper will be devoted to proving this proposition.

Proposition 4.1. *For a Montesinos knot $K = M(T_1, T_2, \cdots, T_N)$ with $N \geq 3$, there exists an essential surface F such that its twist $\tau(F)$ is maximal and its remainder term $\rho(F)$ is non-negative. This assertion also holds for an essential surface with the minimal twist.*

We can deduce Theorem 1.3 from this proposition.

Proof of Theorem 1.3. For a non-two-bridge Montesinos knot K, by Proposition 4.1, there exist two essential surfaces for K such that their twists τ_1 and τ_2 are maximal and minimal and their remainder terms ρ_1 and ρ_2 are both non-negative. Let R_1 and R_2 be the maximum and the minimum among non-meridional boundary slopes for K, respectively. Then, together with the definition of the twist, we have

$$\begin{aligned} |R_1 - R_2| &= |\tau_1 - \tau_2| = |\tau_1| + |\tau_2| = 2\,(\frac{-\chi_1}{\sharp s_1} + \frac{-\chi_2}{\sharp s_2}) + \rho_1 + \rho_2 \\ &\geq 2\,(\frac{-\chi_1}{\sharp s_1} + \frac{-\chi_2}{\sharp s_2}). \end{aligned}$$

Note that τ_1 and τ_2 can be confirmed to be non-negative and non-positive as in Remark 3.1.

Next, assume that a Montesinos knot K is also a two-bridge knot. The monotonically decreasing and monotonically increasing type III edgepaths correspond to essential surfaces by [8] and give the maximal and minimal slopes by the estimation of the twist of an edgepath in the previous section. We can easily check that the remainder terms are 2 for both essential surfaces. Hence,

$$\mathrm{Diam}(K) = 2\,(\frac{-\chi_1}{\sharp s_1} + \frac{-\chi_2}{\sharp s_2}) + 4.$$

$\square$

4.2. Calculation of the remainder term. Here we include a lemma about calculations of the remainder terms of type I, II and III edgepath systems.

We fix an edgepath system Γ with the twist $\tau(\Gamma)$. We recall one more definition used in [9, Subsection 5.2]. We collect all non-∞-edges $e_{i,j}$ of all non-constant edgepaths γ_i in Γ, divide them into two classes according to the sign $\sigma(e_{i,j})$, and then sum up the lengths of edges for each class. With the total lengths l_+ and l_-, let $\kappa(\Gamma)$ denote $\min(l_+, l_-)$, and call it the *cancel* of the edgepath system. For the cancel, we have the following.

Lemma 4.2. *The remainder term ρ of an edgepath system Γ is calculated from the cancel κ as follows.*

(1) *Assume that Γ is a type I edgepath system. Let Γ_{const} and N_{const} denote the constant edgepaths in Γ and the number of the constant edgepaths, where each constant edgepath γ_i is a point on a horizontal edge $\langle p_i/q_i\rangle$ – $\langle p_i/q_i\rangle^\circ$. Then, the remainder term is calculated by*

$$\rho = -4\kappa + 2\,(N - N_{\mathrm{const}}) - \left(N - 2 - \sum_{\gamma_i \in \Gamma_{\mathrm{const}}} \frac{1}{q_i}\right) \frac{2}{1-u}.$$

(2) *For a type II edgepath system Γ, we have $\rho = 4 - 4\kappa$. Thus, $\rho \geq 0$ holds if $\kappa \leq 1$.*

(3) *For a "non-augmented" type III edgepath system Γ with complete ∞-edges, we have $\rho = -4\kappa$. Thus, $\rho \geq 0$ holds if $\kappa = 0$, that is, the type III edgepath system is monotonic.*

Proof. For non-∞-edges of an edgepath system, let A be the sum of length of the edges, B the sum of signed length of the edges. Then, we have $A = l_+ + l_-$, $B = l_+ - l_-$, $|B| = |l_+ - l_-| = l_+ + l_- - 2\min(l_+, l_-) = A - 2\kappa$, $\tau = -2B$ and $|\tau| = 2|B| = 2\,(A - 2\,\kappa) = 2\,A - 4\,\kappa$.

(1) This is just a calculation of $\rho = |\tau| - 2\,(-\chi/\sharp s)$ using the following formula in [9].

$$\frac{-\chi}{\sharp s} \quad = \quad A + N_{\mathrm{const}} - N + \left(N - 2 - \sum_{\gamma_i \in \Gamma_{\mathrm{const}}} \frac{1}{q_i} \right) \frac{1}{1-u}.$$

(2) By the formula $-\chi/\sharp s = A - 2$ given in [9], we have $\rho = |\tau| - 2\,(-\chi/\sharp s) = 4 - 4\kappa$.
(3) By $-\chi/\sharp s = A$ given in [9], we have $\rho = |\tau| - 2\,(-\chi/\sharp s) = -4\kappa$. $\square$

Note that the term "augmented" is defined in Subsection 4.4.

4.3. Proof of Proposition 4.1. This subsection is devoted to proving Proposition 4.1.

Our strategy to prove the proposition is as follows: In the proof of Proposition 3.1, we divided the set of Montesinos knots into some classes, and found for each class an edgepath system whose twist gives a lower bound on the maximal twist. As in that proof, we will divide the set of Montesinos knots, and for a Montesinos knot in each class, collect edgepath systems with twist equal to or greater than that lower bound, and then prove that their remainder terms are all non-negative.

We first claim that the arguments for some special classes of edgepath systems can be omitted.

Lemma 4.3. *An edgepath system Γ for a Montesinos knot which is*

(a) *an augmented type III edgepath systems,*

(b) *a type III edgepath systems with partial ∞-edges, or*

(c) *a type II edgepath systems with redundant vertical edges*

satisfies one of the following conditions (i), (ii), (iii) *or* (iv).

Assume that F_1 denotes an essential surface corresponding to Γ. Then,

(i) *$\tau(F_1) = \tau(F_2)$ and $(-\chi/\sharp s)(F_1) \geq (-\chi/\sharp s)(F_2)$ hold for some essential surface F_2.*

(ii) *$\tau(F_1) < \tau(F_2)$ holds for some essential surface F_2.*

(iii) *$\rho(F_1) \geq 0$ holds.*

(iv) *$\tau(F_1) = \tau(F_2)$ and $\rho(F_2) \geq 0$ hold for some essential surface F_2.*

By virtue of this lemma, in the following proof of the proposition, we can ignore edgepath systems satisfying; the condition (i), for it is sufficient to check that

$\rho(F_2) \geq 0$; the condition (ii), for $\tau(F_1)$ cannot be the maximum; the conditions (iii) and (iv), for even if F_1 gives the maximal slope, we have $\rho \geq 0$.

We prepare another lemma as follows.

Lemma 4.4. *Let K be a Montesinos knot such that the monotonically decreasing basic edgepath system $\Lambda_{\rm dec}$ satisfies $\Lambda_{\rm dec}(0) = -1$ or 0. Assume that F_1 denotes an essential surface with twist $\tau(F_1)$ associated to a class B or class C type III edgepath system for K. Then there exists an essential surface F_2 with twist $\tau(F_2)$ associated to the monotonically decreasing type III edgepath system such that $\tau(F_1) < \tau(F_2)$ holds.*

Since their proofs are rather technical, we give them in the next subsection separately, in order to make the arguments simpler.

Proof of Proposition 4.1. Following the strategy as we stated above, let us start to prove of the proposition.

Claim 1(Case 1)**.** For a Montesinos knot satisfying $\Lambda_{\rm dec}(0) \geq 0$, there exists an essential surface associated to a monotonically decreasing type II edgepath system such that its twist is maximal and its remainder term is non-negative.

Proof. As in the proof of Proposition 3.1, there exists an incompressible surface associated to a monotonically decreasing type II edgepath system $\Gamma_{\rm II,dec}$. Its twist $\tau(\Gamma_{\rm II,dec})$ is maximal by Lemmas 3.2(1) and 3.4. Its remainder term $\rho(\Gamma_{\rm II,dec})$ is 4 by Lemma 4.2(2). □

Claim 2(Case 3)**.** For a Montesinos knot satisfying $\Lambda_{\rm dec}(0) \leq -2$, there exists an essential surface associated to the monotonically decreasing type III edgepath system such that its twist is maximal and its remainder term is non-negative.

Proof. As in the proof of Proposition 3.1, the monotonically decreasing type III edgepath system $\Gamma_{\rm III,dec}$ is incompressible. Its twist $\tau(\Gamma_{\rm III,dec})$ is maximal by Lemmas 3.2(1) and 3.4. Its remainder term $\rho(\Gamma_{\rm III,dec})$ is 0 by Lemma 4.2(3). □

Claim 3(Cases 2-1, 2-2-3, 2-3-2)**.** For a Montesinos knot in these cases, there exists an essential surface associated to a class A type II edgepath system $\Gamma_{\rm II,A}$ or the monotonically decreasing type III edgepath system such that its twist is maximal and its remainder term is non-negative.

Proof. As in the proof of Proposition 3.1, there exists an incompressible, class A type II edgepath system $\Gamma_{\rm II,A}$. Its twist $\tau(\Gamma_{\rm II,A})$ is maximal among those of type I or type II edgepath systems by Lemmas 3.2(1) and 3.4(1). Its remainder term $\rho(\Gamma_{\rm II,A})$ is 0 by Lemma 4.2(2), since its cancel $\kappa(\Gamma_{\rm II,A})$ is equal to 1.
Only the monotonically decreasing type III edgepath system $\Gamma_{\rm III,dec}$ can have twist greater than $\tau(\Gamma_{\rm II,A})$ by Lemma 3.4(2). Its remainder term $\rho(\Gamma_{\rm III,dec})$ is equal to 0 by Lemma 4.2(3). □

Claim 4(Cases 2-2-1, 2-2-2-2, 2-3-1)**.** For a Montesinos knot in these cases, there exists an essential surface associated to a monotonically decreasing type I edgepath system or a class A type II or III edgepath system such that its twist is maximal and its remainder term is non-negative.

Proof. As in the proof of Proposition 3.1, there exists an incompressible monotonically decreasing type I edgepath system $\Gamma_{\mathrm{I,dec}}$. Its twist $\tau(\Gamma_{\mathrm{I,dec}})$ satisfies $\tau(\Lambda_{\mathrm{dec}})-4 \leq \tau(\Gamma_{\mathrm{I,dec}}) < \tau(\Lambda_{\mathrm{dec}})-2$ and is maximal among those of type I edgepath systems by Lemma 3.2(2)(a1).

Subclaim. Its remainder term $\rho(\Gamma_{\mathrm{I,dec}})$ is non-negative.

Proof. The piecewise linear equation (2.1) has the form $R\cdot u-1=0$ for $0<u<1/2$ where $R=\sum_{i=1}^{N} 1/(-r_i)$. In this case, $\sharp\{i\,|\,r_i=-1\}$ is equal to or greater than $\sharp\{i\,|\,r_i\leq -3\}$. The other r-values are all -2. Thus, the mean value of $(-1/r_1,-1/r_2,\cdots,-1/r_N)$ is $1/2$ or greater. Hence, we have $R\geq N/2$ and a solution $u_0=1/R\leq 2/N$. Note that $\Gamma_{\mathrm{I,dec}}$ does not include any constant edgepath since $u_0<1/2$ holds as shown in the proof of Proposition 3.1. Eventually, by Lemma 4.2(1), $\rho=2N-(N-2)\cdot 2/(1-u)\geq 2N-(N-2)\cdot 2N/(N-2)=0$. □

Only class A type II edgepath systems can have the twist greater than $\tau(\Gamma_{\mathrm{I,dec}})$, which is $\tau(\Lambda_{\mathrm{dec}})-2$, among all type II edgepath systems by Lemma 3.4(1). Its remainder term is equal to 0 by Lemma 4.2(2).
Only class A type III edgepath system can have the twist greater than $\tau(\Gamma_{\mathrm{I,dec}})$, which is $\tau(\Lambda_{\mathrm{dec}})$, among all type III edgepath systems by Lemma 3.4(2). Its remainder term is equal to 0 by Lemma 4.2(3).

Claim 5(Cases a, b-a)**.** For a Montesinos knot in these cases, there exists an essential surface associated to a monotonically decreasing type I edgepath system, a class A or class B type II edgepath system, or the monotonically decreasing type III edgepath system such that its twist is maximal and its remainder term is non-negative.

Proof. As in the proof of Proposition 3.1, there exists an incompressible monotonically decreasing type I edgepath system $\Gamma_{\mathrm{I,dec}}$. Its twist $\tau(\Gamma_{\mathrm{I,dec}})$ satisfies $\tau(\Lambda_{\mathrm{dec}})-6 \leq \tau(\Gamma_{\mathrm{I,dec}}) < \tau(\Lambda_{\mathrm{dec}})-4$ and is maximal among those of type I edgepath systems by Lemma 3.2(2)(a2).

Subclaim. Its remainder term $\rho(\Gamma_{\mathrm{I,dec}})$ is non-negative.

Proof. Recall that in this case $N=3$ holds and the final r-values are $(-1,-2,r_3)$ with $r_3\leq -3$. Since the solution u_0 satisfies $u_0\leq 2/3$, at most one constant edgepath system exists in $\Gamma_{\mathrm{I,dec}}$, and if exists, the constant edgepath is on the edge $\langle -1/2\rangle - \langle -1/2\rangle^{\circ}$. By Lemma 4.2(1), if $N_{\mathrm{const}}=1$, then $\rho=4-1/(1-u_0)\geq 1$. Otherwise, $\rho=6-2/(1-u_0)\geq 0$. □

Only class A or class B type II edgepath systems can have the twist greater than $\tau(\Gamma_{\mathrm{I,dec}})$ among type II edgepath systems by Lemma 3.4(1). Their remainder terms are 0 by Lemma 4.2(2), since their cancels κ are equal to 1.
Only the monotonically decreasing, class B, or class C type III edgepath systems can have the twist greater than $\tau(\Gamma_{\mathrm{I,dec}})$ among all type III edgepath systems by

Lemma 3.4(2). However, in this case, class B or class C type III edgepath systems cannot have the maximal twist by Lemma 4.4. The remainder term of the monotonically decreasing type III edgepath systems is equal to 0 by Lemma 4.2(3). □

Claim 6(Case b-b)**.** For a Montesinos knot in this case, there exists an essential surface associated to a class A or class B type II edgepath system or the monotonically decreasing type III edgepath system such that its twist is maximal and its remainder term is non-negative.

Proof. As in the proof of Proposition 3.1, there exists an incompressible class B type II edgepath system $\Gamma_{\mathrm{II,B}}$. Its twist $\tau(\Gamma_{\mathrm{II,B}})$ is equal to $\tau(\Lambda_{\mathrm{dec}}) - 4$. This is maximal among those of type I edgepath systems by Lemma 3.2(2)(b), for $\lambda_{\mathrm{dec}}(1/2) < 0$ is obtained from $\lambda_{\mathrm{dec},1}(1/2) = -1/2$, $\lambda_{\mathrm{dec},2}(1/2) = 1/4$ and $\lambda_{\mathrm{dec},3}(1/2) = 1/(-2r_3) \leq 1/10$. Its remainder term is 0 by Lemma 4.2(2), since its cancel $\kappa(\Gamma_{\mathrm{II,B}})$ is equal to 1.
Only a class A type II edgepath system can have the twist, which is $\tau(\Lambda_{\mathrm{dec}}) - 2$, greater than $\tau(\Gamma_{\mathrm{II,B}})$ among all type II edgepath systems by Lemma 3.4(1).
Its remainder term is equal to 0 by Lemma 4.2(2).
Only the monotonically decreasing type III edgepath system can have the twist, which is $\tau(\Lambda_{\mathrm{dec}})$, greater than $\tau(\Gamma_{\mathrm{II,B}})$, among all type III edgepath systems by Lemma 3.4(2). Its remainder term is equal to 0 by Lemma 4.2(3). □

Together, these six claims prove the proposition. □

Remark 4.1. We can easily confirm that the lower bound (1.1) of the diameter is best possible. For example, assume that, a Montesinos knot K has 4 tangles, and its monotonically increasing and decreasing basic edgepath systems satisfy $\Lambda_{\mathrm{inc}}(0) = +2$ and $\Lambda_{\mathrm{dec}}(0) = -2$. As in Claim 2 in the proof of Proposition 4.1, the monotonically decreasing type III edgepath system and the monotonically increasing type III edgepath system give the maximal and the minimal twists, and both have $\rho = 0$. Thus, the diameter for K satisfies the equality in the inequality (1.1).

4.4. Special edgepath systems. Though the argument in this subsection is necessary, it is technical and a kind of supplement. The precise definitions of special edgepaths which we have treated separately in the previous subsection are as follows:

- There is an edge $\langle 1/0\rangle^\circ - \langle 1/0\rangle$ called the *augmented edge*. For a type III edgepath system, in some cases, the augmented edge can be attached to some of the edgepaths. The edgepath system thus obtained is called an *augmented type III edgepath system.*
- For a basic edgepath system Λ satisfying $\Lambda(0) = 0$, we can attach partial ∞-edges having a common u-coordinate at their end points to edgepaths in Λ in order to make a type III edgepath system. The edgepath system thus obtained is called a *type III edgepath system with partial ∞-edges.*
- A type II edgepath system is called a *type II edgepath system with redundant*

vertical edges if it includes both upward vertical edges and downward vertical edges.

Proof of Lemma 4.3. We prove the following claims one by one.

Claim 1. An edgepath system in (a) satisfies the condition (i).

Proof. If there is an essential surface F_1 associated to an augmented type III edgepath system, as mentioned in [7], some essential surface F_2 is associated to the corresponding non-augmented type III edgepath system. Thus F_1 and F_2 satisfy (i). □

Claim 2. An edgepath system in (b) satisfies one of the condition (ii), (iii) or (iv).

Proof. Assume that a surface F_1 corresponding to a type III edgepath system with partial ∞-edges is essential. Let Λ be the basic edgepath system corresponding to F_1. If Λ is monotonically decreasing, then $\rho(F_1) \geq 0$ holds. That is, F_1 satisfies (iii). If Λ is of class B or class C, since $\Lambda_{\mathrm{dec}}(0) = -1$ or 0 holds, by Lemma 4.4, an essential surface F_2 is associated to the monotonically decreasing type III edgepath system. Then, F_1 and F_2 satisfy (ii). If $\tau(F_1) = \tau(\Lambda_{\mathrm{dec}}) - 6$, then F_1 may give the maximal twist in Claim 5 of the proof of Proposition 4.1. However, in this case, the monotonically decreasing type I edgepath system also has twist equal to $\tau(\Lambda_{\mathrm{dec}}) - 6$. At the same time, as is shown in Claim 5, $\rho \geq 0$ holds for such a type I edgepath system where ρ is the remainder term. Thus F_1 satisfies (iv). If Λ is any other basic edgepath system, since $\tau(F_1) \leq \tau(\Lambda_{\mathrm{dec}}) - 8$ holds, F_1 cannot give the maximal twist and satisfies (ii). □

Claim 3. An edgepath system in (c) satisfies the condition (i).

Proof. Assume that a type II edgepath system $\Gamma_{\mathrm{II},1}$ has redundant vertical edges and that $\Gamma_{\mathrm{II},1}$ is constructed from some type II edgepath system $\Gamma_{\mathrm{II},2}$ without redundant vertical edges by adding upward and downward vertical edges. Assume further that an essential surface F_1 is associated to $\Gamma_{\mathrm{II},1}$. By combining the latter half of the proof of Proposition 2.9 in [7] and the Exercise just after the proof, we can see that if all surfaces corresponding to $\Gamma_{\mathrm{II},2}$ are not-π_1-injective, then all surfaces corresponding to $\Gamma_{\mathrm{II},1}$ are also not-π_1-injective. As the argument includes a deformed edgepath system with a non-minimal part, the not-π_1-injective surfaces obtained are in fact compressible. We conclude, if no essential surface is associated to $\Gamma_{\mathrm{II},2}$, then no essential surface is associated to $\Gamma_{\mathrm{II},1}$. Now, by assumption, F_1 associated to $\Gamma_{\mathrm{II},1}$ is essential. Hence, there exists an essential surface F_2 associated to $\Gamma_{\mathrm{II},2}$. $\tau(\Gamma_{\mathrm{II},1}) = \tau(\Gamma_{\mathrm{II},2})$ holds since contributions of the redundant vertical edges cancel out each other. Moreover, $(-\chi/\sharp s)(F_1) \geq (-\chi/\sharp s)(F_2)$ holds since F_2 is simpler by virtue of the redundant vertical edges. □

These complete the proof of the lemma. □

Proof of Lemma 4.4. Let $\Gamma_{\mathrm{III},1}$ and $\Gamma_{\mathrm{III},2}$ be a class B or class C type III edgepath system and the monotonically decreasing type III edgepath system respectively. Assume that $\Gamma_{\mathrm{III},2}(0) = -1$ or 0. Assume that an essential surface F_1 is associated

to $\Gamma_{\mathrm{III},1}$.
The compressibility of a type III edgepath system is determined by use of completely reversibility of edgepaths.
As in Lemma 2.1(4), a type III edgepath system Γ is compressible if and only if $|\Gamma(0)| \leq 1$ holds and at least $N-2$ edgepaths are completely reversible. Now, since $\Gamma_{\mathrm{III},2}(0) = -1$ or 0, the first condition holds for both $\Gamma_{\mathrm{III},1}$ and $\Gamma_{\mathrm{III},2}$.
Suppose that $\Gamma_{\mathrm{III},2} = (\gamma_{\mathrm{III},2,1}, \gamma_{\mathrm{III},2,2}, \cdots, \gamma_{\mathrm{III},2,N})$ is compressible and that $\Gamma_{\mathrm{III},1} = (\gamma_{\mathrm{III},1,1}, \gamma_{\mathrm{III},1,2}, \cdots, \gamma_{\mathrm{III},1,N})$ is not compressible. Then, at least $N-2$ edgepaths of $\Gamma_{\mathrm{III},2}$ and at most $N-3$ edgepaths of $\Gamma_{\mathrm{III},1}$ are completely reversible. The difference appears in exactly one pair of edgepaths $\gamma_{\mathrm{III},2,i}$ and $\gamma_{\mathrm{III},1,i}$. Assume that $\gamma_{\mathrm{III},2,i}$ is completely reversible and $\gamma_{\mathrm{III},1,i}$ is not so. Since $\gamma_{\mathrm{III},2,i}$ is a monotonically-decreasing completely-reversible edgepath, the edgepath is limited to an edgepath of the form $\langle 1/0\rangle - \langle z\rangle - \langle z+1/2\rangle - \cdots - \langle z+(p-2)/(p-1)\rangle - \langle z+(p-1)/p\rangle$ for some integer z and $p \geq 2$. The only other possible minimal type III edgepath system for the $(z+(p-1)/p)$-tangle is $\gamma = \langle 1/0\rangle - \langle z+1\rangle - \langle z+(p-1)/p\rangle$. This edgepath γ cannot be a class C type III edgepath. For this edgepath γ to be a class B type III edgepath, p must be 2. However, $\langle 1/0\rangle - \langle z+1\rangle - \langle z+1/2\rangle$ is also completely reversible. Thus, if $\Gamma_{\mathrm{III},2}$ is compressible, then $\Gamma_{\mathrm{III},1}$ is also compressible. Similarly to the argument for (c) of Lemma 4.3, a not-π_1-injective surface obtained by Proposition 2.5 in [7] is moreover compressible. We conclude, if no essential surface is associated to $\Gamma_{\mathrm{III},2}$, then no essential surface is associated to $\Gamma_{\mathrm{III},1}$. Now, by assumption, F_1 associated to $\Gamma_{\mathrm{III},1}$ is essential. Hence, there exists an essential surface F_2 associated to $\Gamma_{\mathrm{III},2}$. Obviously, we have $\tau(\Gamma_{\mathrm{III},1}) < \tau(\Gamma_{\mathrm{III},2})$.
□

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