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Almost Periodic Processes in Ecological Systems with Impulsive Perturbations

Gani Trendafilov Stamov
Department of Mathematics Technical University P.O. Box 348, Sliven, Bulgaria
e-mail : gstamov@abv.bg

Abstract. In the present paper we investigate the existence of almost periodic processes of ecological systems which are presented with nonautonomous N-dimensional impulsive Lotka Volterra competitive systems with dispersions and fixed moments of impulsive perturbations. By using the techniques of piecewise continuous Lyapunov's functions new sufficient conditions for the global exponential stability of the unique almost periodic solutions of these systems are given.

1. Introduction

Impulsive differential equations may be used for mathematical simulation of processes and phenomena, which are subject to short-term perturbations during their evolution. The duration of the perturbations is negligible in comparison with the duration of the process considered, therefore it can be considered instantaneous. Most of the investigations related to impulsive systems are focused on the basic theory of impulsive equations, [8], [11], [13] and seldom produce applications on ecological systems. On the other hand nonautonomous N-dimensional Lotka Volterra competitive systems with dispersions without impulses have attracted the interest of many researchers in the past twenty years, [5]-[7], [12], [18], [21]. Some qualitative characteristics of the solutions of such systems have been investigated by many authors. Stability, periodicity, persistence, permanence are studied by Ahmad and Lazer [1], [2], Ahmad and Stamova [3], Lisena [9], [10], Tineo [19]. There are some perturbations in the real world such as fires and floods, that are not suitable to be considered continually. Competitive system with such sudden perturbations involving impulsive differential equations for mathematical simulations.

In this paper we shall investigate the existence of almost periodic solutions of nonautonomous N-dimensional impulsive Lotka Volterra competitive system with dispersions and fixed moments of impulsive perturbations. The main results related to the study of the existence of almost periodic solutions for impulsive dynamical systems are studied in [13], [14]-[17]. The purpose of this paper is to derive "easily verifiable" sufficient conditions for the existence of almost periodic solutions for a class of nonautonomous N-dimensional impulsive Lotka Volterra competitive sys-

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tem with dispersions and fixed moments of impulsive perturbations. The paper is organized as follows. In Section 2 we give some preliminaries and main definitions. In Section 3 we investigate the existence of almost periodic solutions. By means of piecewise continuous auxiliary functions which are analogues of the classical Lyapunov's functions sufficient conditions are obtained.

2. Preliminaries and basic results

Let $\Re^N$ by the N-dimensional Euclidean space with norm $||u|| = \sum_{i=1}^N |u_i|$, $\Re^+ = [0,\infty)$, $S_\nu = \{u \in \Re^N : \ ||u|| \le \nu\}$, $\nu > 0$, and let: $B = \{\{\tau_k\} : \ \tau_k \in \Re, \ \tau_k < \tau_{k+1}, \ k \in Z\}$ is the set of all sequences $\{\tau_k\}$ which are unbounded and strictly increasing with distance $\rho(\{\tau_k^{(1)}\}, \{\tau_k^{(2)}\})$. $PC = PC[\Re, \Re^N] = \{\varphi : \Re \to \Re^N, \ \varphi$ is a piecewise continuous function with points of discontinuity of the first kind at $\tau_k, \ \{\tau_k\} \in B$ at which $\varphi(\tau_k - 0)$ and $\varphi(\tau_k + 0)$ exist, and $\varphi(\tau_k - 0) = \varphi(\tau_k)\}$. $PC^1[\Re, \Re] = \{\varphi : \Re \to \Re, \ \varphi$ is continuously differentiable everywhere with except of the points $\tau_k, \ \{\tau_k\} \in B$ at which $\dot\varphi(\tau_k - 0)$ and $\dot\varphi(\tau_k + 0)$ exist, and $\dot\varphi(\tau_k - 0) = \dot\varphi(\tau_k)\}$.

We will consider the following nonautonomous N-dimensional impulsive Lotka Volterra competitive system with dispersion and fixed moments of impulsive perturbations

$$
\left\{
\begin{aligned}
&\dot u_i(t) = u_i(t)\big[r_i(t) - a_i(t)u_i(t) - \sum_{j=1, j\neq i}^{N} a_{ij}(t)u_j(t)\big] \\
&\qquad\qquad + \sum_{j=1}^{N} b_{ij}(t)(u_j(t) - u_i(t)), \ t \neq \tau_k, \\
&u_i(\tau_k + 0) = u_i(\tau_k) + d_k u_i(\tau_k), \ k \in Z,
\end{aligned}
\right. \tag{1}
$$

where $t \in \Re, \ i = 1, 2, \cdots, N, \ N \geq 2$ and:

(a) The functions $r_i(t), \ a_i(t) \in C[\Re, \Re], \ 1 \le i \le N$, and $a_{ij}(t) \in C[\Re, \Re], \ i \neq j, \ b_{ij}(t) \in C[\Re, \Re], \ 1 \le i, j \le N$;

(b) The constants $d_k \in \Re, \ \{\tau_k\} \in B, \ k \in Z$.

Let $t_0 \in \Re, \ u_{i0} \in \Re$. Denote by $u(t) = u(t; t_0, u_0), \ u(t) = col(u_1(t), u_2(t), \cdots, u_N(t))$, $u_0 = col(u_{10}, u_{20}, \cdots, u_{N0})$ the solution of (1) with the initial condition

$$u(t_0 + 0; t_0, u_0) = u_0. \tag{2}$$

The solution $u(t) = u(t; t_0, u_0)$ of problem (1), (2) is a piecewise continuous function with points of discontinuity of the first kind at the points $\tau_k, \ k \in Z$. At these points the solutions $u(t)$ are continuous from the left, that is, at the moments of impulse effects $\tau_k^\prime s$ the following relations are valid:

$$u_i(\tau_k - 0) = u_i(\tau_k), \ u_i(\tau_k + 0) = u_i(\tau_k) + d_k u_i(\tau_k), \ k \in Z.$$

We will use piecewise continuous auxiliary functions which are analogues of the classical Lyapunov's functions and consider the following sets:

$G_k = (\tau_{k-1}, \tau_k) \times \Re^N,\ k \in Z;\ G = \cup_{k=-\infty}^{\infty} G_k;$
$V_0 = \{V \in C[G, \Re^+]$, there exist the limits $V(\tau_k - 0, u_0),\ V(\tau_k + 0, u_0),\ V(\tau_k - 0, u_0) = V(\tau_k, u_0),\ u_0 \in S_\nu$, V is locally Lipschitz continuous for $u \in S_\nu\}$.

Let $V \in V_0$. For any $(t, u) \in [\tau_{k-1}, \tau_k) \times S_\nu$ the right-hand derivative $D^+V(t, u(t))$ along the solution $u(t; t_0, u_0)$ of (1) is defined by

$$D^+V(t, u(t)) = \lim_{\delta \to 0^+} inf \delta^{-1}\{V(t + \delta, u(t + \delta)) - V(t, u(t))\}.$$

Given a nonnegative continuous function $g(t)$ which is defined on $\Re$, we set

$$g_M = \sup_{t \in \Re} g(t),\ g_L = \inf_{t \in \Re} g(t).$$

Remark 1. The problems of existence, uniqueness, and continuability of the solutions of impulsive differential equations have been investigated by many authors. Sufficient conditions for the existence of the solutions of such systems are given in [8], [13].

Since the solutions of (1) are piecewise continuous functions we adopt the following definitions for almost periodicity. Let for $T, P \in B$, $s(T \cup P)\ :\ B \to B$ is a map such that the set $s(T \cup P)$ forms a strictly increasing sequence and if $D \subset \Re$ then $\theta_\varepsilon(D) = \{t + \varepsilon,\ t \in D\},\ F_\varepsilon(D) = \cap\{\theta_\varepsilon(D),\ \varepsilon > 0\}$. By $\phi = (\varphi(t), T)$ we denote the element from the space $PC \times B$, and for every sequence of real numbers $\{\alpha_n\},\ n = 1, 2, \cdots$, with $\theta_{\alpha_n}\phi$ denote the sets $\{\varphi(t+\alpha_n), T-\alpha_n\} \subset PC \times B$, where $T - \alpha_n = \{\tau_k - \alpha_n,\ k \in Z\},\ n = 1, 2, \cdots$.

Definition 1([13]). The set of sequences $\{\tau_k^j\},\ \tau_k^j = \tau_{k+j} - \tau_k,\ k \in Z,\ j \in Z$ is said to be *uniformly almost periodic* with respect to $k \in Z$ if for any $\varepsilon > 0$ there exists a relatively dense set in $\Re$ of ε-almost periods common for all the sequences $\{\tau_k^j\}$.

Lemma 1([13]). *The set of sequences $\{\tau_k^j\}$ is uniformly almost periodic, if and only if from each infinite sequences of shifts $\{\tau_k - \alpha_n\},\ k \in Z,\ n = 1, 2, \cdots,\ \alpha_n \in \Re$ we can choose a subsequence, convergent in B.*

Definition 2. The sequence $\{\phi_n\},\ \phi_n = (\varphi_n(t), T_n) \in PC \times B$ is *uniformly convergent* to ϕ, $\phi = (\varphi(t), T) \in PC \times B$ if and only if for any $\varepsilon > 0$ there exists $n_0 > 0$ such that

$$\rho(T, T_n) < \varepsilon,\ ||\varphi_n(t) - \varphi(t)|| < \varepsilon$$

hold uniformly for $n \geq n_0$ and $t \in \Re \setminus F_\varepsilon(s(T_n \cup T))$.

Definition 3. The function $\varphi \in PC$ is said to be *almost periodic piecewise continuous function* with points of discontinuity of the first kind from the set T if for every sequence of real numbers $\{\alpha'_m\}$ there exists a subsequence $\{\alpha_n\}$, $\alpha_n = \alpha'_{m_n}$ such that $\theta_{\alpha_n}\phi$ is compact in $PC \times B$.

Introduce the following assumptions:
H1. The functions $r_i(t),\ a_i(t),\ 1 \leq i \leq N,\ a_{ij}(t),\ i \neq j,\ b_{ij}(t),\ 1 \leq i, j \leq N$, are

almost periodic, nonnegative, continuous and $r_{iL} > 0,\ r_{iM} < \infty,\ a_{iL} > 0,\ a_{iM} < \infty,\ a_{ijL} \geq 0,\ a_{ijM} < \infty,\ i \neq j,\ b_{ijL} \geq 0,\ b_{ijM} < \infty$ for $1 \leq i, j \leq N$.
H2. The sequence $\{d_k\}$, is almost periodic and $-1 < d_k \leq 0$, $k \in Z$.
H3. The set of sequences $\{\tau_k^j\}$, $k \in Z$, $j \in Z$ is uniformly almost periodic and $inf\{\tau_{k+1} - \tau_k,\ k \in Z\} > 0$.

Let the assumptions **H1-H3** hold and let $\{\alpha_m{}'\}$ be an arbitrary sequence of real numbers. Then there exists a subsequence $\{\alpha_n\}$, $\alpha_n = \alpha_{m_n}{}'$ such that the sequences $\{r_i(t+\alpha_n)\}$, $\{a_i(t+\alpha_n)\}$ and $\{a_{ij}(t+\alpha_n)\}$, $i \neq j$, $\{b_{ij}(t+\alpha_n)\}$ are convergent uniformly on $1 \leq i, j \leq N$, to the functions $\{r_i^\alpha(t)\}$, $\{a_i^\alpha(t)\}$, $\{a_{ij}^\alpha(t)\}$, $\{b_{ij}^\alpha(t)\}$ and the sequences $\{\tau_k - \alpha_n\}$, $k \in Z$ are convergent to the sequence $\{\tau_k^\alpha\}$ uniformly with respect to $k \in Z$ as $n \to \infty$. By $\{k_{n_i}\}$ we denote the sequence of integers such that the subsequence $\{\tau_{k_{n_i}}\}$ is convergent to the sequence $\{\tau_k^\alpha\}$ uniformly with respect to $k \in Z$ as $i \to \infty$.
From **H2** it follows that there exists a subsequence of the sequence $\{k_{n_i}\}$ such that the sequences $\{d_{k_{n_i}}\}$ are convergent uniformly to the limits denoted by d_k^α.
Then for every sequence $\{\alpha'_m\}$ the system (1) is moving to the system

$$(3)\qquad \begin{cases} \dot{u}_i(t) = u_i(t)[r_i^\alpha(t) - a_i^\alpha(t)u_i(t) - \displaystyle\sum_{j=1, j\neq i}^{N} a_{ij}^\alpha(t)u_j(t)] \\ \qquad\qquad + \displaystyle\sum_{j=1}^{N} b_{ij}^\alpha(t)(u_j(t) - u_i(t)),\ t \neq \tau_k^\alpha, \\ u_i(\tau_k^\alpha + 0) = u_i(\tau_k^\alpha) + d_k^\alpha u_i(\tau_k^\alpha)),\ k \in Z. \end{cases}$$

Remark 2. In many papers the limiting systems (3) are called hull of the system (1). These systems could be used effectively in the translation technique and in the stability analysis of population models.

In the proof of the main results we shall use the following definitions and lemmas for the system (1). Let $u_0 = col(u_{10}, u_{20}, \cdots, u_{N0})$ and $v_0 = col(v_{10}, v_{20}, \cdots, v_{N0})$, $u_{i0} \in \Re$, $v_{i0} \in \Re$ and

$$u(t) = col(u_1(t), u_2(t), \cdots, u_N(t)),\ v(t) = col(v_1(t), v_2(t), \cdots, v_N(t))$$

are two solutions of (1) with initial conditions

$$u(t_0 + 0; t_0, u_0) = u_0, \quad v(t_0 + 0; t_0, v_0) = v_0.$$

Definition 4([4]). The system (1) is said to be *globally exponentially stable* if for all $\delta > 0$, there exist $\gamma = \gamma(\delta) > 0$ and $c = c(\delta) > 0$ such that if $u_0,\ v_0 \in S_\nu$, with $||u_0 - v_0|| \leq \delta$, then for all $t \geq t_0$,

$$||u(t; t_0, u_0) - v(t; t_0, v_0)|| < \gamma ||u_0 - v_0|| \exp[-c(t - t_0)].$$

Definition 5. Suppose $u(t) = (u_1(t), u_2(t), \cdots, u_n(t))$ is any one solution of system (1) then $u(t)$ is said to be a *strictly positive solution*, if for $1 \le i \le N$,

$$0 < \inf_{t \in \Re} u_i(t) \le \sup_{t \in \Re} u_i(t) < \infty.$$

Lemma 2. *Let the following conditions hold:*
1. *The conditions* **H1-H3** *are satisfied.*
2. *There exist functions* $P_i,\ Q_i \in PC^1[\Re, \Re]$ *such that*

$$P_i(t_0 + 0) \le u_i(t_0 + 0) \le Q_i(t_0 + 0),$$

where $t_0 \in \Re,\ i = 1, 2, \cdots, N$. *Then we have*

$$P_i(t) \le u_i(t) \le Q_i(t) \tag{4}$$

for all $t \ge t_0$ *and* $i = 1, 2, \cdots, N$.

Proof. First we will proof that

$$u_i(t) \le Q_i(t) \tag{5}$$

for all $t \ge t_0$ and $i = 1, 2, \cdots, N$, where $Q_i(t)$ is the maximal solution of the logistic system

$$\begin{cases} \dot{q}_i(t) = q_i(t)\left[r_i(t) - a_i(t)q_i(t)\right],\ t \ne \tau_k, \\ q_i(t_0 + 0) = q_{i0} > 0, \\ q_i(\tau_k + 0) = q_i(\tau_k) + d^M q_i(\tau_k),\ k \in Z, \end{cases} \tag{6}$$

where $d^M = \sup_{k \in Z}\{d_k\}$. The maximal solution $Q_i(t) = Q_i(t; t_0, q_0),\ q_0 = col(q_{10},\ q_{20}, \cdots,\ q_{N0})$ of (6) is defined by the equality

$$Q_i(t; t_0, q_0) = \begin{cases} Q_i^0(t; t_0, Q_i^0 + 0),\ t_0 < t \le \tau_1, \\ Q_i^1(t; \tau_1, Q_i^1 + 0),\ \tau_1 < t \le \tau_2, \\ \cdots \\ Q_i^k(t; \tau_k, Q_i^k + 0),\ \tau_k < t \le \tau_{k+1}, \\ \cdots, \end{cases}$$

where $Q_i^k(t; \tau_k, Q_i^k + 0)$ is the solution of the equation without impulses

$$\dot{q}_i(t) = q_i(t)\left[r_i(t) - a_i(t)q_i(t)\right],$$

in the interval $(\tau_k, \tau_{k+1}],\ k = 0, 1, 2, \cdots$, for which $Q_i^k + 0 = (1 + d^M)Q_i^k(\tau_k; \tau_{k-1}, Q_i^{k-1} + 0),\ k = 1, 2, \cdots,\ 1 \le i \le N$ and $Q_i^0 + 0 = q_{i0}$. By [20] , it follows for (1) that

$$\dot{u}_i(t) \le u_i(t)\left[r_i(t) - a_i(t)u_i(t))\right],\ t \ne \tau_k. \tag{7}$$

Now, let $t \in (t_0, \tau_1]$. If $0 < u_{i0} \le Q_i(t_0+0)$, then elementary differential inequality [8] yields that

$$u_i(t) \le Q_i(t)$$

for all $t \in (t_0, \tau_1]$, i.e., the inequality (5) is valid for $t \in (t_0, \tau_1]$. Suppose that (5) is satisfied for $t \in (\tau_{k-1}, \tau_k]$. Then from $H3$ and the fact that (5) is satisfied for $t = \tau_k$ we obtain

$$\begin{aligned} u_i(\tau_k+0) &= u_i(\tau_k) + d_k u_i(\tau_k) \le u_i(\tau_k) + d^M u_i(\tau_k) \\ &\le Q_i(\tau_k) + d^M Q_i(\tau_k) = Q_i(\tau_k+0). \end{aligned}$$

We apply again the comparison result (7) in the interval $(\tau_k, \tau_{k+1}]$ and obtain

$$u_i(t; t_0, u_o) \le Q_i^k(t; \tau_k, Q_i^k + 0) = Q_i(t; t_0, q_0)$$

i.e., the inequality (5) is valid for $(\tau_k, \tau_{k+1}]$. The proof of (5) is completed by induction.

Further, by analogous arguments, using [20] we obtain from (1) and (7) that

$$\begin{cases} \dot{u}_i(t) \ge u_i(t)\Big[r_i(t) - a_i(t)u_i(t)) - \sum\limits_{j=1, \mathrm{j}\neq i}^{N} a_{ij}(t) \sup\limits_{t\in\Re} Q_i(t)\Big] \\ \qquad - \sum\limits_{j=1}^{N} b_{ij}(t) \sup\limits_{t\in\Re} Q_i(t),\ t \ne \tau_k, \\ u_i(\tau_k+0) \ge u_i(\tau_k) + g^L u_i(\tau_k),\ k \in Z, \end{cases}$$

$i = 1, \cdots, N$, $N \ge 2$, and hence $u_{io} \ge P_i(t_0+0)$ implies that

$$u_i(t) \ge P_i(t) \tag{8}$$

for all $t \in \Re$ and $i = 1, 2, \cdots, N$, where $P_i(t)$ is the minimal solution of the logistic system

$$\begin{cases} \dot{p}_i(t) = p_i(t)\Big[r_i(t) - a_i(t)p_i(t)) - \sum\limits_{\substack{j=1 \\ j\neq i}}^{N} a_{ij}(t) \sup\limits_{t\in\Re} Q_i(t)\Big] \\ \qquad - \sum\limits_{j=1}^{N} b_{ij}(t) \sup\limits_{t\in\Re} Q_i(t),\ t \ne \tau_k, \\ p_i(t_0+0) = p_{i0} > 0, \\ p_i(\tau_k+0) = p_i(\tau_k) + d^L p_i(\tau_k),\ k \in Z, \end{cases} \tag{9}$$

$i = 1, \cdots, N$ and $d^L = \min\limits_{k\in Z}\{d_k\}$ for $1 \le i \le N$. Thus, the proof follows from and (5) and (9). □

Lemma 3. *Let the following conditions hold*:

1. *The conditions* **H1-H3** *are satisfied.*
2. *Let* $u(t) = col(u_1(t), u_2(t), \cdots, u_N(t))$ *is a solution of* (1) *such that* $u_i(t_0+0) > 0$, $1 \le i \le N$.
Then :
1. $u_i(t) > 0$, $1 \le i \le N$, $t \in \Re$.
2. *For* $t \in \Re$ *and* $1 \le i \le N$ *there exist a constants* $A > 0$, $B > 0$, *such that*

$$A \le u_i(t) \le B.$$

Proof of assertion 1. Under hypotheses **H1-H3**, consider the nonimulsive Lotka-Volterra system

$$(10)\qquad \begin{cases} \dot{y}_i(t) = y_i(t)\left[r_i(t) - A_i(t)y_i(t) - \displaystyle\sum_{j=1, j\neq i}^{N} A_{ij}(t)y_j(t)\right] \\ \qquad + \displaystyle\sum_{j=1}^{N} B_{ij}[y_j(t) - y_i(t)],\ t \neq \tau_k,\ t > t_0, \end{cases}$$

where

$$\begin{aligned} A_i(t) &= a_i(t) \prod_{0<\tau_k<t} (1 + d_k), \\ A_{ij}(t) &= a_{ij}(t) \prod_{0<\tau_k<t} (1 + d_k), \\ B_{ij}(t) &= b_{ij}(t) \prod_{0<\tau_k<t} (1 + d_k). \end{aligned}$$

If $y_i(t)$ is solution of (7) then $u_i = y_i \displaystyle\prod_{0<\tau_k<t} (1 + d_k)$ is solution of (1) $1 \le i \le N$.
In fact, for $t \neq \tau_k$ it follows

$$(11)\qquad \dot{u}_i(t) - u_i(t)\left[r_i(t) - a_i(t)u_i(t) - \sum_{j=1, j\neq i}^{N} a_{ij}(t)u_j(t)\right] - \sum_{j=1}^{N} b_{ij}[u_j(t) - u_i(t)]$$

$$\begin{aligned} =\ & \dot{y}_i \prod_{0<\tau_k<t} (1 + d_k) - y_i \prod_{0<\tau_k<t} (1 + d_k)\Big[r_i(t) - a_i(t)y_i \prod_{0<\tau_k<t} (1 + d_k) \\ & - \sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij}(t)y_j \prod_{0<\tau_k<t} (1 + d_k)\Big] - \sum_{j=1}^{N} b_{ij}[y_j \prod_{0<\tau_k<t} (1 + d_k) - y_i \prod_{0<\tau_k<t} (1 + d_k)] \end{aligned}$$

$$= \prod_{0<\tau_k<t}(1+d_k)\Big[\dot{y}_i(t) - y_i(t)\Big[r_i(t) - A_i(t)y_i(t) - \sum_{j=1,j\neq i}^{N} A_{ij}(t)y_j(t)\Big] - \sum_{j=1}^{N} B_{ij}[y_j(t) - y_i(t)]\Big] \equiv 0.$$

For $t = \tau_k$ we have

$$u_i(\tau_k + 0) = \lim_{t\to\tau_k^+} \prod_{0<\tau_k<t}(1+d_k)y_i(t) = \prod_{0<\tau_k<t}(1+d_k)y_i(\tau_k),$$

and

$$u_i(\tau_k) = \prod_{0<\tau_k<t}(1+d_k)y_i(\tau_k).$$

Thus, for every $k \in Z$

$$u_i(\tau_k + 0) = \prod_{0<\tau_k<t}(1+d_k)y_i(\tau_k). \tag{12}$$

From (11) and (12) it follows that $u_i(t)$ is the solution of (1).
The proof that if $u_i = y_i \prod_{0<\tau_k<t}(1+d_k)$ is solution of (1) it follows that $y_i(t),\ 1 \leq i \leq N$ is solution of (10) is analogously. From [20] it follows that for the system without impulses (10) exists positive solution on $t \in \Re$. Then from (11) and (12) it follows that $u_i(t) > 0, 1 \leq i \leq N,\ t \in \Re$. ☐

Proof of assertion 2. From [20] under the conditions of Lemma 3 for the solutions of (6) and (9) it is valid that

$$\alpha_i \leq P_i(t),\ Q_i(t) \leq \beta_i,$$

where $\alpha_i > 0,\ o < \beta_i < \infty$ for all $t \neq \tau_k,\ 1 \leq i \leq N$ and then

$$\alpha_i \leq u_i(t) \leq \beta_i.$$

Also since the solution $u_i(t)$ is left continues at $t = \tau_k$ we have for $t = \tau_1$ that

$$\alpha_i \leq u_i(\tau_1) \leq \beta_i.$$

On the other hand

$$(1+d_1)a_i \leq u_i(\tau_1 + 0) \leq (1+d_k)\beta_i \leq \beta_i.$$

By analogous arguments for the $t \in (\tau_{k-1}, \tau_k]$ it follows

$$0 < \prod_{l=1}^{k}(1+d_l)\alpha_i \leq u_i(\tau_k + 0) \leq \beta_i.$$

Then for all $t \in \Re$ we have

$$A \le u_i(t) \le B,$$

where

$$A = \min_i\{\alpha_i \prod_{k\in Z}(1+d_k)\}, \quad B = \min_i \beta_i.$$

The proof of Lemma 3 is complete. □

3. Main results

For the proof of the main results we consider systems (3) and then discuss the almost periodic solutions of the system (1).

Lemma 4. *Let the following conditions hold:*
1. *The conditions* **H1-H3** *are satisfied.*
2. $\{\alpha_m{}'\}$ *be an arbitrary sequence of real numbers.*
3. *For the systems* (3) *there exist strictly positive solutions.*

Then the system (1) *has a unique strictly positive almost periodic solution.*

Proof. For simplification, we write (1) in the form

$$\begin{cases} \dot u = f(t,u), \ t \ne \tau_k, \\ u(\tau_k+0) = u(\tau_k) + d_k u(\tau_k), \ k \in Z. \end{cases} \tag{13}$$

Let $\phi(t)$ be a strictly positive solution of (13) and let the sequences of real numbers α' and β' are such that for their subsequences $\alpha \subset \alpha'$, $\beta \subset \beta'$, we have $\theta_{\alpha+\beta} f(t,u) = \theta_\alpha \theta_\beta f(t,u)$. Then $\theta_{\alpha+\beta}\phi(t)$, $\theta_\alpha\theta_\beta\phi(t)$ exist uniformly on the compact set $\Re \times B$ and are solutions of the following equation

$$\begin{cases} \dot u = f^{\alpha+\beta}(t,u), \ t \ne \tau_k^{\alpha+\beta}, \\ u(\tau_k+0) = u(\tau_k) + d_k^{\alpha+\beta} u(\tau_k), \ k \in Z. \end{cases}$$

Therefore, $\theta_{\alpha+\beta}\phi(t) = \theta_\alpha\theta_\beta\phi(t)$, and thus according to Lemma 2, [16], it follows that $\phi(t)$ is an almost periodic solution of system (13). The proof is complete. □

Theorem 1. *Let the following conditions hold:*
1. *The conditions* **H1-H3** *are satisfied.*
2. *There exist nonnegative almost periodic functions* $\delta_l(t)$, $1 \le l \le N$ *such that*

$$a_l(t) - \sum_{i=1, i\ne l}^{N} a_{il}(t) - \frac{1}{A}\sum_{i=1}^{N} b_{il}(t) \ge \delta_l(t), \ t \ne \tau_k, \tag{14}$$

for $t \in \Re$, $A > 0$, $k \in Z$. *Then:*
1. *For the system* (1) *there exists a unique strictly positive almost periodic solution.*
2. *If there exists a constant* $c \ge 0$ *such that*

$$\int_{t_0}^{t} \delta(t)ds = c(t - t_0),$$

where $\delta(t) = \min(\delta_1(t), \delta_2(t), \cdots, \delta_N(t))$*, then the almost periodic solution is globally exponentially stable.*

Proof. From construction of the system (3) it follows that there exists a time sequence $\{\sigma_n\}$, $\sigma_n < \sigma_{n+1}$ and $\sigma_n \to \infty$ for $n \to \infty$ such that

$$r_i(t+\sigma_n) \to r_i^\sigma(t),\ a_i(t+\sigma_n) \to a_i^\sigma(t),\ a_{ij}(t+\sigma_n) \to a_{ij}^\sigma(t),$$

$$b_{ij}^\sigma(t+\sigma_n) \to b_{ij}^\sigma(t),\ n \to \infty$$

uniformly on $t \in \Re$, $t \neq \tau_k$, and there exists a subsequence $\{k_n\}$ of $\{n\}$ $k_n \to \infty, n \to \infty$ such that

$$\tau_{k_n} \to \tau_k^\sigma,\ d_{k_n} \to d_k^\sigma.$$

Let $u(t)$ is positive solution of the system (3). From Lemma 3 it follows that there exist two positive constants A and B, such that

$$A \leq \lim_{t\to\infty} \inf u_i(t) \leq \lim_{t\to\infty} \sup u_i(t) \leq B, i = 1, 2, \cdots, N$$

and consequently

$$0 < \inf_{t\in\Re} u_i(t) \leq \sup_{t\in\Re} u_i(t) < \infty,\ i = 1, 2, \cdots, N. \tag{15}$$

Let $u_n(t) = u(t+\sigma_n)$ and for all $t \geq -\sigma_n,\ n = 1, 2, \cdots$, we obtain

$$\begin{cases} \dot{u}_i(t) = u_i(t)[r_i^\sigma(t+\sigma_n) - a_i^\sigma(t+\sigma_n)u_i(t) - \displaystyle\sum_{j=1, j\neq i}^{N} a_{ij}^\sigma(t+\sigma_n)u_j(t)] \\ \qquad + \displaystyle\sum_{j=1}^{N} b_{ij}^\sigma(t+\sigma_n)(u_j(t) - u_i(t)),\ t \neq \tau_{k_n} - \sigma_n, \\ \Delta u_i(\tau_{k_n}) = d_{k_n}^\sigma u_i(\tau_{k_n}),\ k_n \in Z. \end{cases} \tag{16}$$

From **H1-H3** and (14) it follows that there exists a positive constant K such that $|\dfrac{du_n(t)}{dt}| \leq K$ for $n = 1, 2, \cdots$ and for any whole nomber $r > 0$ sequence $\{u_n(t)\}$, $n \leq r$ is uniformly bounded and contionuous for $t \in [-\sigma_n, \infty) \setminus \{\tau_{k_n}\}$. Then from Ascoli-Arzela Theorem it follows that there exists subsequence $\{\sigma_{n_m}\}$ such that the sequence $u_m(t)$ is convergent uniformly on any compact set in $\Re$ for $m \to \infty$.
Set

$$\lim_{m\to\infty} u_m(t) = u^\sigma(t) = (u_1^\sigma(t), u_2^\sigma(t), \cdots, u_n^\sigma(t)).$$

Lemma 3 implies that

$$0 < \inf_{t\in[0,\infty)} u_i^\sigma(t) \leq \sup_{t\in[0,\infty)} u_i^\sigma(t) < \infty,\ i = 1, 2, \cdots, N. \tag{17}$$

From (16) and **H1-H3** it follows that $u^\sigma(t)$ is a solution of the system (3) and consequently for every system in the form (3) there exists at least one strictly positive solution. Now suppose that the system (3) has two arbitrary strictly positive solutions

$$u^\sigma = (u_1^\sigma(t), u_2^\sigma(t), \cdots, u_N^\sigma(t)),\ v^\sigma = (v_1^\sigma(t), v_2^\sigma(t), \cdots, v_N^\sigma(t)).$$

Consider the Lyapunov function

$$V^\sigma(t, u^\sigma(t), v^\sigma(t)) = \sum_{i=1}^{N} |ln \frac{u_i^\sigma(t)}{v_i^\sigma(t)}|, \quad t \in \Re.$$

Then for $t \in \Re,\ t \neq \tau_k^\sigma$ we have that

$$\begin{aligned} D^+V^\sigma(t, u^\sigma(t), v^\sigma(t)) &= \sum_{i=1}^{N} (\frac{\dot{u}_i^\sigma(t)}{u_i^\sigma(t)} - \frac{\dot{v}_i^\sigma(t)}{v_i^\sigma(t)}) \mathrm{sgn}(u_i^\sigma(t) - v_i^\sigma(t)) \\ &\leq \sum_{i=1}^{N} \Big\{ - a_i^\sigma(t)|u_i^\sigma(t) - v_i^\sigma(t)| + \sum_{j=1, j\neq i}^{N} a_{ij}^\sigma(t)|u_j^\sigma(t) - v_j^\sigma(t)| \\ &\quad + \frac{1}{A} \sum_{j=1}^{N} b_{ij}^\sigma(t))|u_j^\sigma(t) - v_j^\sigma(t)|\big\}. \end{aligned}$$

Thus in view of hypothesis (14) we obtain

$$D^+V^\sigma(t, u^\sigma(t), v^\sigma(t)) \leq -\delta^\sigma(t) m^\sigma(t),\ t \in \Re,\ t \neq \tau_k^\sigma, \tag{18}$$

where $\delta_i^\sigma(t + \sigma_n) \to \delta_i^\sigma(t),\ n \to \infty,\ i = 1, 2, \cdots, N,$

$$\delta^\sigma(t) = \min(\delta_1^\sigma(t), \delta_2^\sigma(t), \cdots, \delta_N^\sigma(t)),\ m^\sigma(t) = \sum_{i=1}^{n} |u_i^\sigma - v_i^\sigma|.$$

On the other hand for $t = \tau_k^\sigma$ we have

$$\begin{aligned} V^\sigma(\tau_k^\sigma{+}0, u^\sigma(\tau_k^\sigma + 0), v^\sigma(\tau_k^\sigma + 0)) &= \sum_{i=1}^{N} |ln \frac{u_i^\sigma(\tau_k^\sigma + 0)}{v_i^\sigma(\tau_k^\sigma + 0)}| \\ = \sum_{i=1}^{N} |ln \frac{(1 + d_k^\sigma) u_i^\sigma(\tau_k^\sigma)}{(1 + d_k^\sigma) v_i^\sigma(\tau_k^\sigma)}| &= V^\sigma(\tau_k^\sigma, u^\sigma(\tau_k^\sigma), v^\sigma(\tau_k^\sigma)). \end{aligned} \tag{19}$$

From (18) and (19) it follows

$$D^+V^\sigma(t, u^\sigma(t), v^\sigma(t)) \leq 0,\ t \in \Re,\ t \neq \tau_k^\sigma,$$

and

$$V^\sigma(\tau_k^\sigma + 0, u^\sigma(\tau_k^\sigma + 0), v^\sigma(\tau_k^\sigma + 0)) - V^\sigma(\tau_k^\sigma, u^\sigma(\tau_k^\sigma), v^\sigma(\tau_k^\sigma)) = 0,$$

and hence

$$V^\sigma(t,u^\sigma(t),v^\sigma(t)) \leq V^\sigma(t_0,u^\sigma(t_0),v^\sigma(t_0))$$

for all $t \geq t_0$, $t_0 \in \Re$. From the above inequality, (18) and (19) we get

$$\int_{t_0}^{t} \delta^\sigma(s)m^\sigma(s)ds \leq V^\sigma(t_0) - V^\sigma(t), \quad t \geq t_0.$$

Therefore,

$$\int_{t_0}^{\infty} |u_i^\sigma(s) - v_i^\sigma(s)|ds < \infty, \quad i = 1,2,\cdots,N,$$

and $u_i^\sigma(t) - v_i^\sigma(t) \to \ 0$ as $t \to \infty$. Let $\mu^\sigma = \inf\limits_{t\in\Re}\{u_i^\sigma, v_i^\sigma,\ i = 1,2,\cdots,N\}$. From the definition of $V^\sigma(t)$ we have

$$\begin{aligned} V^\sigma(t,u^\sigma(t),v^\sigma(t)) &= \sum_{i=1}^{N}[lnu_i^\sigma - lnv_i^\sigma] \\ &\leq \frac{1}{\mu^\sigma}\sum_{i=1}^{N}|u_i^\sigma - v_i^\sigma|. \end{aligned}$$

Hence $V^\sigma(t) \to 0,\ t \to \infty$. We have that $V^\sigma(t)$ is nonincreasing nonnegative function on $\Re$ and from (19) we obtain that

$$V^\sigma(t) = 0,\ t \in \Re. \tag{20}$$

From (20) it follows that $u_i^\sigma \equiv v_i^\sigma$ for all $t \in \Re$ and $i = 1,2,\cdots,N$. Therefore for an arbitrary sequence of real numbers $\{\alpha_m{}'\}$ the system (3) has a unique strictly positive solution. From Lemma 4 analogously it follows that system (1) has an unique strictly positive almost periodic solution $u(t)$. Now consider again the Lyapunov function

$$V(t) = V(t,u(t),v(t)) = \sum_{i=1}^{N}|ln\frac{u_i(t)}{v_i(t)}|,$$

where $v(t) = (v_1(t),v_2(t),\cdots,v_N(t))$ is an arbitrary solution of (1) with initial condition $v(t_0+0) = v_0$.

By Mean Value Theorem it follows that for any closed interval contained in $t \in (\tau_{k-1},\tau_k]$, $k \in Z$ there exist positive numbers r and R such that for $1 \leq i \leq N$, $r \leq u_i(t),\ v_i(t) \leq R$ and

$$\frac{1}{R}|u_i(t) - v_i(t)| \leq |lnu_i(t) - lnv_i(t)| \leq \frac{1}{r}|u_i(t) - v_i(t)|. \tag{21}$$

Hence we obtain

$$V(t_0+0,u_0,v_0) = \sum_{i=1}^{N}|lnu_i(t_0) - lnv_i(t_0)| \leq \frac{1}{r}||u_0 - v_0||. \tag{22}$$

On the other hand

$$D^+V(t,u(t),v(t)) \leq -\delta(t)m(t) \leq -\delta(t)rV(t,u(t),v(t)), \tag{23}$$

where $t \in \Re,\ t \neq \tau_k$ and

$$m(t) = \sum_{i=1}^{N} |u_i - v_i|,\ \ \delta(t) = \min(\delta_1(t), \delta_2(t), \cdots, \delta_N(t)).$$

For $t \in \Re,\ t = \tau_k$

$$V(\tau_k + 0, u(\tau_k + 0), v(\tau_k + 0)) = \sum_{i=1}^{N} |ln \frac{u_i(\tau_k + 0)}{v_i(\tau_k + 0)}| \tag{24}$$

$$= \sum_{i=1}^{N} |ln \frac{(1 + d_k)u_i(\tau_k)}{(1 + d_k)v_i(\tau_k)}| = V(\tau_k, u(\tau_k), v(\tau_k)).$$

From (21), (22) and (23) it follows

$$V(t,u(t),v(t)) \leq V(t_0 + 0, u_0, v_0) \exp\left\{-r\int_{t_0}^{t} \delta(s)ds\right\}. \tag{25}$$

Therefore, from (21), (25) and (23) we deduce the inequality

$$\sum_{i=1}^{N} |u_i(t) - v_i(t)| \leq \frac{R}{r}||u_0 - v_0||e^{-rc(t-t_0)},$$

$t \geq t_0$. This shows that the unique almost periodic solution $u(t)$ of the system (1) is globally exponentially stable. The proof of Theorem 1 is complete. □

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