KYUNGPOOK Math. J. 49(2009), 289-297

Stability of Quartic Mappings in Non-Archimedean Normed Spaces

Alireza Kamel Mirmostafaee
Department of Mathematics, Centre of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, Mashhad 91775, Iran
e-mail : `mirmostafaei@ferdowsi.um.ac.ir`

Abstract. We establish a new method to prove Hyers-Ulam-Rassias stability of the quartic functional equation

$$f(2x+y)+f(2x-y)+6f(y)=4[f(x+y)+f(x-y)+6f(x)]$$

in non-Archimedean normed linear spaces.

1. Introduction

In 1940, Ulam [20] at the University of Wiscosin proposed the following stability problem:
Let $(G_1,*)$ be a group and let $(G_2,\diamond,d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon>0$, does there exists $\delta(\epsilon)>0$ such that if $h:G_1\to G_2$ satisfies the inequality

$$d\Big(h(x*y),h(x)\diamond h(y)\Big)<\delta \quad x,y\in G_1,$$

then there is a homomorphism $H:G_1\to G_2$ with $d\Big(h(x),H(x)\Big)<\epsilon$ for all $x\in G_1$?

In 1941, Hyers [5] gave a partial answer to this question. Hyers' theorem was generalized by T. Aoki [1] for additive mappings and by Th. M. Rassias [18] for linear mappings by considering an unbounded Cauchy difference. The concept of the Hyers–Ulam–Rassias stability was originated from Th. M. Rassias' paper [18] for the stability of the linear mappings and its importance in the proof of further results in functional equations.
During the last decades several stability problems for various functional equations have been investigated by many mathematicians; we refer the reader to [3], [6], [7], [11], [12], [19] and references therein.

The functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{1.1}$$

Received May 20, 2008; accepted September 3, 2008.
2000 Mathematics Subject Classification: Primary 39B82; Secondary 46S10.
Key words and phrases: quartic functional equation, Hyers-Ulam-Rassias stability, non-Archimedean normed spaces.

is called the *quartic functional equation*, since the function $f(x) = x^4$ is a solution of (1.1). Note that f is called quartic because of the identity

$$(2x+y)^4 + (2x-y)^4 = 4(x+y)^4 + 4(x-y)^4 + 24x^4 - 6y^4. \tag{1.2}$$

Every solution of the quartic functional equation is said to be a *quartic mapping*. It is proved in [10] that a function $f : X \to Y$ between real normed spaces is quartic if and only if there exists a symmetric biquadratic function $F : X \times X \to Y$ such that $f(x) = F(x,x)$ for all $x \in X$. The first result on the stability of the quartic functional equation was obtained by J. M. Rassias [17]. Also L. Cădariu [2], H. -M. Kim [9], S. H. Lee, S. M. Im and I. S. Hwang [10], Najati [15] and C. Park [16] investigated the stability of quartic functional equation.

Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\ | : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have

(i) $|a| \geq 0$ and equality holds if and only if $a = 0$,

(ii) $|ab| = |a||b|$,

(iii) $|a+b| \leq \max\{|a|, |b|\}$.

The condition (iii) is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii) that $|n| \leq 1$ for each integer n. We always assume in addition that $|\ |$ is non trivial, i.e., that there is an $a_0 \in \mathbb{K}$ such that $|a_0| \neq 0, 1$.

Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\ .\ |$. A function $||\ .\ || : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $||x|| = 0$ if and only if $x = 0$;

(ii) $||rx|| = |r|||x||$ for all $r \in \mathbb{K}$ and $x \in X$;

(iii) the strong triangle inequality (ultrametric); namely,

$$||x+y|| \leq \max\{||x||, ||y||\} \quad (x, y \in X).$$

Then $(X, ||\ .\ ||)$ is called a non-Archimedean space.
Due to the fact that

$$||x_n - x_m|| \leq \max\{||x_{j+1} - x_j|| : m \leq j \leq n-1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.
In 1897, Hensel [4] discovered the following important example of non-Archimedean numbers:
Let p be a prime number. For any nonzero rational number $a = p^r \dfrac{m}{n}$ such that

m and n are coprime to the prime number p, define the p-adic absolute value $|a|_p = p^{-r}$. Then $|\ |$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\ |$ is denoted by $\mathbb{Q}_p$ and is called the p-adic number field.

During the last three decades theory of non-Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings and superstrings (cf. [8]). Although many results in the classical normed space theory have a non-Archimedean counterpart, but their proofs are essentially different and require an entirely new kind of intuition, cf. [13], [14].

In [13], the stability of some functional equations in non-Archimedean normed spaces are investigated. We develop a new method which seems to be more applicable to treat Hyers–Ulam–Rassias stability of the quartic functional equation in non-Archimedean normed linear spaces.

2. Quartic stability

Throughout this section we will assume that V and X are linear spaces over a non-Archimedean field $\mathbb{K}$. For a function $f: V \to X$, we define

$$Df(x,y) = f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y).$$

J. M. Rassias in [17] has shown that for every quartic mapping $q: V \to X$,

$$q(2x) = 2^4 q(x) \qquad (x \in V).$$

We need to the following generalization of this result:

Lemma 2.1. *Let $q: V \to X$ be a quartic function, then for each integer $n \in \mathbb{K}$,*

$$q(nx) = n^4 q(x) \qquad (x \in V). \tag{2.1}$$

Proof. If we substitute $x = y = 0$ in (1.1), we see that $q(0) = 0$. This proves (2.1) for $n = 0$. For the case $n = 1$, (2.1) is obvious. Lemma 1.3 of [17] proves the result for $n = 2$. Let $k \geq 2$ and for each $n = 0, 1, \cdots, k$, (2.1) holds. Put $y = (k-1)x$ in (1.1), to obtain

$$q((k+1)x) + q(-(k-2)x) = 4q(kx) + 4q(-(k-1)x) + 24q(x) - 6q((k-1)x)$$

for all $x \in V$. By Lemma 1. 2 of [17], q is even, so that by induction hypothesis

$$q\big((k+1)x\big) + (k-2)^4 q(x) = 4k^4 q(x) + 4(k-1)^4 q(x) + 24q(x) - 6(k-1)^4 q(x) \tag{2.2}$$

for all $x \in V$. By (1.2) and (2.2), it follows that

$$q\big((k+1)x\big) = (k+1)^4 q(x) \qquad (x \in V).$$

This proves the Lemma. $\square$

Now we are ready to state the main result of the paper:

Theorem 2.2. *If X is a complete non-Archimedean normed space and $f : V \to X$ is an even function such that $f(0) = 0$ and for which there exists a function $\varphi : V \times V \to [0, \infty)$ such that for some positive integer $k \geq 2$, $\lim_{n\to\infty} |k|^{4n}\varphi(k^{-n}x, k^{-n}y) = 0$ uniformly on V^2 and*

$$\|Df(x, y)\| \leq \varphi(x, y) \quad \forall x, y \in V, \tag{2.3}$$

then there exists a unique quartic mapping $q : V \to X$ and an integer m such that

$$||q(x) - f(x)|| \leq \max\{|k|^{4(j-1)}\psi_k(k^{-j}x) : 1 \leq j \leq m\} \qquad \forall x \in V,$$

where

$$\psi_1(x) = 0, \quad \psi_2(x) = \frac{1}{|2|}\varphi(x, 0), \quad \psi_3(x) = \max\{\varphi(x, x), |2|\varphi(x, 0)\},$$

$$\psi_4(x) = \max\{\varphi(x, 2x), |4|\psi_3(x), |6|\psi_2(x)\}$$

and for $n \geq 4$,

$$\psi_{n+1}(x) = \max\{\varphi(x, (n-1)x), |4|\psi_n(x), |6|\psi_{n-1}(x), |4|\psi_{n-2}(x), \psi_{n-3}(x)\}$$

Proof. Put $y = 0$ in (2.3) to obtain

$$||2f(2x) - 2^5 f(x)|| \leq \varphi(x, 0) \quad \forall x \in V. \tag{2.4}$$

This means that

$$||f(2x) - 2^4 f(x)|| \leq \frac{1}{|2|}\varphi(x, 0) = \psi_2(x) \quad \forall x \in V. \tag{2.5}$$

By replacing y by x in (2.3), we see that

$$||f(3x) + f(x) - 4f(2x) - 24f(x) + 6f(x)|| \leq \varphi(x, x) \quad \forall x \in V. \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\begin{aligned} ||f(3x) - 3^4 f(x)|| &\leq \max\{||f(3x) - 4f(2x) - 17f(x)||, ||4(f(2x) - 2^4 f(x))||\} \\ &\leq \max\{\varphi(x, x), |2|\varphi(x, 0)\} = \psi_3(x) \quad \forall x \in V. \end{aligned} \tag{2.7}$$

Replacing y by $2x$ in (2.3), we have

$$||f(4x) - 4f(3x) - 4f(-x) - 24f(x) + 6f(2x)|| \leq \varphi(x, 2x) \qquad (x \in V). \tag{2.8}$$

By (2.5) and (2.7) for each $x \in V$, we obtain

$$\begin{aligned} ||f(4x) - 4^4 f(x)|| &\leq \max\Big\{||f(4x) - 4f(3x) - 4f(-x) - 24f(x) + 6f(2x)||, \\ &\qquad |4|||f(3x) - 3^4 f(x)||, |6|||f(2x) - 2^4 f(x)||\Big\} \\ &\leq \max\{\varphi(x, 2x), |4|\psi_3(x), |6|\psi_2(x)\} = \psi_4(x). \end{aligned} \tag{2.9}$$

In general for $n \geq 4$, put $y = (n-1)x$ in (2.3), then for all $x \in V$,

$$||f((n+1)x) - 4f(nx) + 6f((n-1)x) - 4f((n-2)x) + f((n-3)x) - 24f(x)|| \\ \leq \varphi(x, (n-1)x).$$

Therefore for $n \geq 4$ and each $x \in V$,

$$||f((n+1)x) - (n+1)^4 f(x)|| \leq \max \Big\{\varphi(x,(n-1)x), \quad |4|\psi_n(x), |6|\psi_{n-1}(x), \\ |4|\psi_{n-2}(x), \qquad \psi_{n-3}(x)\Big\} = \psi_{n+1}(x).$$

In particular,

$$||f(kx) - k^4 f(x)|| \leq \psi_k(x) \quad \forall x \in V \tag{2.10}$$

and $\lim_{n\to\infty} |k|^{4n}\psi_k(k^{-n}x) = 0$ uniformly on V. By replacing x with $k^{-n}x$ and multiplying both sides of the inequality in $|k^{4(n-1)}|$, we see that

$$||k^{4(n-1)} f(k^{-(n-1)}x) - k^{4n} f(k^{-n}x)|| \leq |k^{4(n-1)}|\psi_k(k^{-n}x) \quad \forall x \in V. \tag{2.11}$$

Since the right side of the above inequality tends to zero as $n \to \infty$, $\{k^{4n} f(k^{-n}x)\}$ is a Cauchy sequence in non-Archimedean complete space X, so it converges to some function

$$q(x) = \lim_{n\to\infty} k^{4n} f(k^{-n}x).$$

For each $x, y \in V$, we have

$$\begin{aligned} ||Dq(x,y)|| \quad &\leq \quad \max\Big\{||q(u) - k^{4n} f(k^{-n}u)||, ||k^{4n} Df(k^{-n}x, k^{-n}y)|| : \\ &\qquad u \in \{2x+y, 2x-y, x+y, x-y, x, y\}\Big\} \\ &\leq \quad \max\Big\{||q(u) - k^{4n} f(k^{-n}u)||, |k^{4n}|\varphi(k^{-n}x, k^{-n}y) : \\ &\qquad u \in \{2x+y, 2x-y, x+y, x-y, x, y\}\Big\}. \end{aligned} \tag{2.12}$$

The inequality (2.12) shows that q is quartic, since the right hand side of (2.12) tends to zero as n tends to ∞. By (2.11), we have

$$\begin{aligned} ||k^{4m} f(k^{-m}x) - f(x)|| \quad &= \quad ||\sum_{j=1}^{m} k^{4j} f(k^{-j}x) - k^{4(j-1)} f(k^{-(j-1)}x)|| \\ &\leq \quad \max\{||k^{4j} f(k^{-j}x) - k^{4(j-1)} f(k^{-(j-1)}x)|| : 1 \leq j \leq m\} \\ &\leq \quad \max\{|k|^{4(j-1)}\psi_k(k^{-j}x) : 1 \leq j \leq m\} \quad \forall m. \end{aligned}$$

Since $\{|k|^{4(j-1)}\psi_k(k^{-j}x)\}$ on V uniformly converges to zero, there is some m such that for each $x \in V$ and $n \geq m$,

$$|k|^{4(n-1)}\psi_k(k^{-n}x) \leq \max\{|k|^{4(j-1)}\psi_k(k^{-j}x); 1 \leq j \leq m\}.$$

It follows that for each $n \geq 1$,

$$||k^{4n}f(k^{-n}x) - f(x)|| \leq \max\{|k|^{4(j-1)}\psi_k(k^{-j}x); 1 \leq j \leq m\}.$$

By taking limit of the left hand side of the above inequality as n tends to ∞, we see that

$$||q(x) - f(x)|| \leq \max\{|k|^{4(j-1)}\psi_k(k^{-j}x); 1 \leq j \leq m\}.$$

Let $q' : V \to X$ be a quartic mapping such that

$$||q'(x) - f(x)|| \leq \max\{|k|^{4(j-1)}\psi_k(k^{-j}x); 1 \leq j \leq m\}.$$

Then for each $x \in V$, by Lemma 2.1, we have

$$\begin{aligned}
||q'(x) - q(x)|| &= ||k^{4n}q'(k^{-n}x) - k^{4n}q(k^{-n}x)|| \\
&\leq \max\{||k^{4n}q'(k^{-n}x) - k^{4n}f(k^{-n}x)||, ||k^{4n}f(k^{-n}x) - k^{4n}q(k^{-n}x)||\} \\
&\leq \max\{|k|^{4(n+j-1)}\psi_k(k^{-(j+n)}x); 1 \leq j \leq m\}.
\end{aligned}$$

As n tends to infinity, the right hand side of the above inequality tends to zero and the uniqueness assertion follows. $\square$

3. Applications

The following result is due to J. M. Rassias [17]:

Theorem 3.1. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume that there exists a constant $c > 0$ such that*

$$||Df(x, y)|| \leq c \quad \forall x, y \in X.$$

Then $q(x) = \lim_{n\to\infty} 2^{-4n}f(2^n x)$ exists and defines a quartic mapping from X to Y such that

$$||f(x) - q(x)|| \leq \frac{17c}{180}$$

for all $x \in X$.

The following example shows that this result is not true in non-Archimedean normed spaces:

Example 3.2. Let $p > 2$ be a prime number and $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $f(x) = 2$ for all $x \in \mathbb{Q}_p$. Since $|2| = 1$,

$$|Df(x, y)| = 1$$

for all $x, y \in \mathbb{Q}_p$. However,

$$|2^{-4n}f(2^n x) - 2^{-4m}f(2^m x)| = 1 \quad \forall x, y \in \mathbb{Q}_p.$$

Hence $\{2^{-4n}f(2^n x)\}$ is not a Cauchy sequence.

However, we have the following version of Rassias's result for non-Archimedean normed spaces.

Corollary 3.3. *Let $p > 2$ be a prime number and $\mathbb{K} = \mathbb{Q}_p$. If $f : V \to X$ satisfies*

$$||Df(x,y)|| < \varepsilon \quad (x, y \in V).$$

for some $\varepsilon > 0$, then there is a unique quartic mapping $q : V \to X$ such that

$$||f(x) + f(-x) - 2f(0) - 2\,q(x)|| < \varepsilon \quad (x \in V). \tag{3.1}$$

In particular if f is even then

$$||f(x) - f(0) - q(x)|| < \varepsilon \quad (x \in V). \tag{3.2}$$

Proof. Let $g(x) = \big(f(x)+f(-x)\big)/2 - f(0)$, then g is an even function with $g(0) = 0$. Since $|2| = 1$, for each $x \in V$ we have

$$\begin{aligned} ||Dg(x,y)|| &\leq \max\{||Df(x,y) - Df(0,0)||, ||Df(-x,-y) - Df(0,0)||\} \\ &\leq \max\{||Df(x,y)||, ||Df(-x,-y)||, ||Df(0,0)||\} < \varepsilon. \end{aligned}$$

Applying Theorem 2.2 for g, $\varphi(x,y) = \varepsilon$ and $k = p$, we can find a unique quartic mapping $q : V \to X$ which satisfies (3.1). If f is even, since $|2| = 1$,

$$||f(x) - f(0) - q(x)|| = |2|.||f(x) - f(0) - q(x)|| = ||f(x) + f(-x) - 2f(0) - 2\,q(x)|| < \varepsilon.$$

$\square$

Remark 3.4. Thanks to the proof of Theorem 2.2, the condition

"$\lim_{n\to\infty} |k|^{4n}\varphi(k^{-n}x, k^{-n}y) = 0$ uniformly on V^2"

of the theorem can be replaced by the following:

(a) $\lim_{n\to\infty} |k|^{4n}\varphi(k^{-n}x, k^{-n}y) = 0$ pointwise on V^2 and

(b) there is some integer m such that

$$\varphi(k^{-n}x, k^{-n}y) \leq \max\{\varphi(k^{-j}x, k^{-j}y) : 1 \leq j \leq m\} \quad (x, y \in V, n \geq m).$$

Corollary 3.5. *Let $f : V \to X$ and $\varphi : V^2 \to [0, \infty)$ satisfy the following conditions:*

(i) *For some $r < 4$, $||Df(x,y)|| \leq ||x||^r + ||y||^r$ for each $x, y \in V$.*

(ii) *There is some integer $k \in \mathbb{K}$ such that $|k| < 1$.*

Then there is a unique quartic mapping $q : V \to X$ *such that*

$$||f(x) - q(x)|| \leq \alpha |k|^{-r} ||x||^r \qquad (x \in V)$$

for some $1 \leq \alpha \leq 2$.

Proof. The argument used in Theorem 2.2 for $\varphi(x, y) = ||x||^r + ||y||^r$ shows that there is some $1 \leq \alpha \leq 2$ such that $\psi_k(x) = \alpha ||x||^r$. Since $r < 4$,

$$\lim_{n\to\infty} |k|^{4(n-1)} \psi_k(k^{-n}x) = \lim_{n\to\infty} \alpha |k|^{n(4-r)-4} ||x||^r = 0 \quad (x \in V)$$

and

$$|k|^{4(n-1)} \psi_k(k^{-n}x) \leq \max\{|k|^{n(j-1)} \psi_k(k^{-j}x) : j \geq 1\} = \alpha |k|^{-r} ||x||^r \quad (x \in V, n \geq 1).$$

Thanks to the proof of Theorem 2.2, we can find a unique quartic mapping $q : V \to X$ such that

$$||f(x) - q(x)|| \leq \alpha |k|^{-r} ||x||^r \qquad (x \in V).$$

□

Acknowledgment. The author thanks the referee for his/her careful reading the paper.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2**(1950), 64–66.

[2] L. Cădariu, *Fixed points in generalized metric space and the stability of a quartic functional equation*, Bul. Ştiinţ. Univ. Politeh. Timiş. Ser. Mat. Fiz., **50(64)(2)**(2005), 25-34.

[3] S. Czerwik, Functional Equations and Inequalities in Several Variables, World Scientific, River Edge, NJ, 2002.

[4] K. Hensel, *Über eine neue Begründung der Theorie der algebraischen Zahlen*, Jahresber. Deutsch. Math. Verein, **6**(1897), 83-88.

[5] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A., **27**(1941), 222-224.

[6] D. H. Hyers, G. Isac and Th. M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Basel, 1998.

[7] S. -M. Jung, Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, 2001.

[8] A. Khrennikov, Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models, Kluwer Academic Publishers, Dordrecht, 1997.

[9] H. -M. Kim, *On the stability for mixed type of quartic and quadratic functional equation*, J. Math. Anal. Appl., **324**(2006), 358-372.

[10] S. H. Lee, S. M. Im and I. S. Hwang, *Quartic functional equations*, J. Math. Anal. Appl., **307(2)**(2005), 387-394.

[11] A. K. Mirmostafaee, M. Mirzavaziri and M. S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Systems, **159**(2008), 730-738.

[12] A. K. Mirmostafaee and M. S. Moslehian, *Fuzzy versions of Hyers–Ulam–Rassias theorem*, Fuzzy Sets and Systems, **159**(2008), 720-729.

[13] M. S. Moslehian and Th. M. Rassias, *Stability of functional equations in non-Archimedean spaces* Appl. Anal. Discrete Math., **1**(2007), 325-334.

[14] L. Narici and E. Beckenstein, *Strange terrain—non-Archimedean spaces*, Amer. Math. Monthly, **88(9)**(1981), 667-676.

[15] A. Najati, *On the stability of a quartic functional equation*, J. Math. Anal. Appl., **340(1)**(2008), 569-574.

[16] C. Park, *On the stability of the orthogonally quartic functional equation* Bull. Iranian Math. Soc., **31(1)**(2005), 63-70.

[17] J. M. Rassias, *Solution of the Ulam stability problem for quartic mappings*, Glas. Mat. Ser. III, **34(54)(2)**(1999), 243-252.

[18] Th. M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math., **62(1)**(2000), 23-130.

[19] Th. M. Rassias, Functional Equations, Inequalities and Applications, Kluwer Academic Publishers, Dordrecht, Boston and London, 2003.

[20] S. M. Ulam, Problems in Modern Mathematics (Chapter VI, Some Questions in Analysis: §1, Stability), Science Editions, John Wiley & Sons, New York, 1964.