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On Comaximal Graphs of Near-rings

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Abstract. Let N be a zero-symmetric near-ring with identity and let $\Gamma(N)$ be a graph with vertices as elements of N, where two different vertices a and b are adjacent if and only if $\langle a\rangle + \langle b\rangle = N$, where $\langle x\rangle$ is the ideal of N generated by x. Let $\Gamma_1(N)$ be the subgraph of $\Gamma(N)$ generated by the set $\{n \in N \ : \ \langle n\rangle = N\}$ and $\Gamma_2(N)$ be the subgraph of $\Gamma(N)$ generated by the set $N\backslash v(\Gamma_1(N))$, where $v(G)$ is the set of all vertices of a graph G. In this paper, we completely characterize the diameter of the subgraph $\Gamma_2(N)$ of $\Gamma(N)$. In addition, it is shown that for any near-ring, $\Gamma_2(N)\backslash M(N)$ is a complete bipartite graph if and only if the number of maximal ideals of N is 2, where $M(N)$ is the intersection of all maximal ideals of N and $\Gamma_2(N)\backslash M(N)$ is the graph obtained by removing the elements of the set $M(N)$ from the vertices set of the graph $\Gamma_2(N)$.

1. Preliminaries

Throughout this paper N is a zero symmetric near-ring with identity. $M(N)$ denotes the intersection of all maximal ideals of N, Max(N) denotes the set of all maximal ideals of N, $\langle x\rangle$ denotes the ideal of N generated by x and $v(G)$ denotes the set of all vertices of a graph G.

For any vertices x, y in a graph G, if x and y are adjacent, we denote it as $x \approx y$. A graph is said to be connected if for each pair of distinct vertices v and w, there is a finite sequence of distinct vertices $v_0 = v, v_2, \cdots, v_n = w$ such that each pair $\{v_i, v_{i+1}\}$ is an edge. Such a sequence is said to be a path and the distance, $d(v, w)$, between connected vertices v and w is the length of the shortest path connecting them. The diameter of a connected graph is the supremum of the distances between vertices. The degree of a vertex v in G is the number of edges of G incident with v. Let G_1 be a subgraph of a graph G and $v \in G_1$. Then $deg_{G_1}(v)$ is the number of edges of G_1 incident with v. An r-partite graph is one whose vertex set can be

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partitioned into r subsets so that no edge has both ends in any one subset. Let V be the set of vertices of a graph G and $V_1 \subseteq V$. Then $G\backslash V_1$ is the graph obtained by removing the vertices of the set V_1 from the vertices set of the graph G. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) graph with part sizes m and n is denoted by $K_{m,n}$. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertices set V_i and edges set E_i. The join of G_1 and G_2 is denoted by $G = G_1 \vee G_2$ with vertices set $V_1 \cup V_2$ and the set of edges is $E_1 \cup E_2 \cup \{x \approx y \ : \ x \in V_1 \ and \ y \in V_2\}$. Following Mason [4], an ideal I of N is called completely reflexive if $ab \in I$ implies $ba \in I$ for $a, b \in N$. In [2], Beck considered $\Gamma(R)$ as a graph with vertices the elements of a commutative ring R, where two different vertices a and b are adjacent if and only if $ab = 0$. He studied finitely colorable rings with this graph structure and in [1], Anderson and Naseer have made further studies of finitely colorable rings. In [6], Sharma and Bhatwadekar defined another graph srtucture on a commutative ring R with vertices the elements of R and where two distinct vertices a and b are adjacent if and only if $\langle a\rangle + \langle b\rangle = R$.

In this paper, we extend the graph structure of rings as defined by Sharma and Bhatwadekar and the results obtained by H. R. Maimani et al. [3] for commutative rings to near-rings (not necessarily commutative). Let N be a near-ring and let $\Gamma(N)$ be a graph with vertices the elements of N and where two different vertices a and b are adjacent if and only if $\langle a\rangle + \langle b\rangle = N$.

Let $\Gamma_1(N)$ be the subgraph of $\Gamma(N)$ generated by the set $\{n \in N \ : \ \langle n\rangle = N\}$ and $\Gamma_2(N)$ be the subgraphs of $\Gamma(N)$ generated by the set $N\backslash v(\Gamma_1(N))$. Then clearly $\Gamma(N) = \Gamma_1(N) \vee \Gamma_2(N)$. If N is a commutative ring, then the set of vertices of $\Gamma_1(N)$ consists of unit elements of N. Other definitions and basic concepts in near-ring theory can be found in G.Pilz [5].

2. Main results

Theorem 2.1. *If $\{P_1, P_2, \cdots, P_n\}$ is a finite family of prime ideals of N with $I \subseteq \cup_{i=1}^n P_i$ for any sub near-ring I of N, then $I \subseteq P_i$ for some i.*

Proof. We may assume that I is not contained in the union of any collection on $n-1$ of the P_i's. If so, we can simply replace n by $n-1$. Thus for each i, we can find an element $a_i \in I$ with $a_i \notin P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cdots \cup P_n$. Take $n = 2$, with $I \nsubseteq P_1$ and $I \nsubseteq P_2$. Then $a_1 \in P_1, \ a_2 \notin P_1$, and so $a_1 + a_2 \notin P_1$. Similarly, $a_1 \notin P_2, \ a_2 \in P_2$, and so $a_1 + a_2 \notin P_2$. Thus $a_1 + a_2 \notin I \subseteq P_1 \cup P_2$, contradicting $a_1, a_2 \in I$. Now assume that $n > 2$ and suppose that $I \nsubseteq P_i$ for all i. Observe that $\langle a_1\rangle\langle a_2\rangle \cdots \langle a_{n-1}\rangle \subseteq P_1 \cap P_2 \cdots \cap P_{n-1}$, but $a_n \notin P_1 \cup P_2 \cdots \cup P_{n-1}$. Now for all $i = 1, 2, \cdots, n-1$, we have $a_i \notin P_n$, and so $\langle a_1\rangle\langle a_2\rangle \cdots \langle a_{n-1}\rangle \nsubseteq P_n$. Then there exists $t \in \langle a_1\rangle\langle a_2\rangle \cdots \langle a_{n-1}\rangle$ such that $x = t + a_n \notin P_n$. Thus $x \in I$ and $x \notin P_1 \cup P_2 \cup \cdots \cup P_n$, a contradiction. □

Lemma 2.2. *Let N be a near-ring. Then the following conditions hold:*
(i) *$\Gamma_1(N)$ is a complete graph.*
(ii) *$a \in M(N)$ if and only if $deg_{\Gamma_2(N)} a = 0$.*

Proof. (i) It is clear from definition.
(ii) Let $a \in M(N)$ and suppose $deg_{\Gamma_2(N)} a \neq 0$, then there exists $b \in \Gamma_2(N)$ such that $\langle a \rangle + \langle b \rangle = N$. On the other hand there exists $M \in$ Max(N) with $b \in M$, and so $M = N$, a contradiction. Conversely, assume that $deg_{\Gamma_2(N)} a = 0$ and suppose that $a \notin M(N)$. Then there exists $M \in$ Max(N) such that $a \notin M$, and so $\langle a \rangle + M = N$. Therefore there exists $b \in M$ such that $\langle a \rangle + \langle b \rangle = N$, a contradiction. □

Corollary 2.3([3], Lemma 2.1)**.** *Let R be a commutative ring with identity. Then the following hold:*
(i) $\Gamma_1(R)$ *is a complete graph.*
(ii) $a \in J(R)$ *if and only if* $deg_{\Gamma_2(R)} a = 0$, *where* $J(R)$ *denotes the Jacobson radical of* R.

Proof. If R is commutative ring with identity, then $J(R)$ is $M(R)$. □

Theorem 2.4. *Let N be a near-ring. Then* $\Gamma_2(N) \backslash M(N)$ *is connected graph and* diam $(\Gamma_2(N) \backslash M(N)) \leq 3$.

Proof. Let $a, b \in \Gamma_2(N) \backslash M(N)$.
Case (i): If $\langle a \rangle \langle b \rangle \nsubseteq M(N)$, then $\langle \langle a \rangle \langle b \rangle \rangle \nsubseteq M(N)$, so there exists $x \in \Gamma_2(N) \backslash M(N)$ such that $\langle \langle a \rangle \langle b \rangle \rangle + \langle x \rangle = N$. Thus $\langle a \rangle + \langle x \rangle = N$ and $\langle b \rangle + \langle x \rangle = N$. So we have the path $a \approx x \approx b$, and so $d(a, b) \leq 2$.
Case (ii): If $\langle a \rangle \langle b \rangle \subseteq M(N)$, then Max(N) $= \mathrm{S_a} \cup \mathrm{S_b}$, where $S_a = \{M \in$ Max(N) : a ∈ M$\}$ and $S_b = \{M \in$ Max(N) : b ∈ M$\}$. Since $a \notin M(N)$, there exists $x \in \Gamma_2(N)$ such that $\langle a \rangle + \langle x \rangle = N$. Then $x \notin M(N)$. Let $M \in$ Max(N) such that $b \notin M$. Then $x \notin M$, and so $\langle b \rangle \langle x \rangle \nsubseteq M(N)$. Therefore by Case (i), $d(b, x) \leq 2$, and so $d(a, b) \leq 3$. □

Corollary 2.5([3], Theorem 3.1)**.** *Let R be a commutative ring with identity. Then* $\Gamma_2(R) \backslash J(R)$ *is connected graph and diam* $(\Gamma_2(R) \backslash J(R)) \leq 3$.

Theorem 2.6. *Let N be a near-ring. Then the following conditions are equivalent:*
(i) $\Gamma_2(N) \backslash M(N)$ *is a complete bipartite graph.*
(ii) *The cardinal number of the set* Max(N) *is 2.*

Proof. i) ⇒ ii) Suppose that $\Gamma_2(N) \backslash M(N)$ is a complete bipartite graph with two parts V_1 and V_2. Set $M_1 = V_1 \cup M(N)$ and $M_2 = V_2 \cup M(N)$. We claim that M_1 and M_2 are maximal ideals of N. Let $x, y \in M_1$.
Consider the following three cases:
Case (i): If $x, y \in M(N)$, then $x - y \in M_1$.
Case (ii): If $x \in M(N)$ and $y \in V_1$, then $x - y \notin M(N)$. If $\langle x - y \rangle = N$, then $\langle x \rangle + \langle y \rangle = N$, a contradiction. If $x - y \in M_2$, then $x - y \in V_2$, and so $\langle x - y \rangle + \langle y \rangle = N$. Thus $\langle x \rangle + \langle y \rangle = N$, a contradiction. Therefore $x - y \in V_1 \subseteq M_1$.
Case (iii): Assume that $x, y \in V_1$. If $x - y \in M(N)$, then there is nothing to prove. Otherwise $x - y \notin M(N)$. Then by same argument of Case (ii), we have $x - y \in M_1$.
Let $x \in M_1$ and $n \in N$. If either $x \in M(N)$ or $n + x - n \in M(N)$, then M_1 is a normal subgroup of N. So, we assume that $x \notin M(N)$ and $n + x - n \notin M(N)$. Since $\langle n + x - n \rangle \subseteq \langle x \rangle$, we have $\langle n + x - n \rangle \neq N$. If $n + x - n \in M_2$, then $n + x - n \in V_2$, and so $\langle n + x - n \rangle + \langle x \rangle = N$ which implies $N = \langle x \rangle$, a contradiction. Therefore $n + x - n \in V_1 \subseteq M_1$. Let $n \in N$ and $x \in M_1$. If either $x \in M(N)$ or $xn \in M(N)$, then M_1 is right ideal of N. Otherwise $x \notin M(N)$ and $xn \notin M(N)$.

Also $\langle xn\rangle \neq N$. Suppose that $xn \in M_2$. Then $xn \in V_2$, and so $\langle xn\rangle + \langle x\rangle = N$. Thus $\langle x\rangle = N$, a contradiction. So $xn \in M_1$. Let $n, n_1 \in N$ and let $x \in M_1$. If either $x \in M(N)$ or $n(n_1 + x) - nn_1 \in M(N)$, then M_1 is a left ideal of N. Otherwise $x \notin M(N)$ and $n(n_1+x)-nn_1 \notin M(N)$. Also $\langle n(n_1+x)-nn_1\rangle \neq N$. Suppose that $n(n_1+x)-nn_1 \in M_2$. Then $n(n_1+x)-nn_1 \in V_2$, and so $\langle x\rangle+\langle n(n_1+x)-nn_1\rangle = N$ which implies $N = \langle x\rangle$, a contradiction. So $n(n_1+x)-nn_1 \in M_1$. So M_1 is an ideal of N. Let $x \in N\backslash M_1$. Then $\langle x\rangle+\langle y\rangle = N$ for all $y \in V_1$ which implies $\langle x\rangle+M_1 = N$, and so M_1 is a maximal ideal of N.
With the same argument, M_2 is a maximal ideal of N. Now, if $M \in$ Max(N), then $M \subseteq M_1 \cup M_2$, and so $M = M_1$ or $M = M_2$ by Theorem 2.1.
ii) $\Rightarrow$ *i)* Let Max(N) $= \{M_1, M_2\}$. Thus the vertices set of $\Gamma_2(N)\backslash M(N)$ is equal to the set $(M_1\backslash M_2) \cup (M_2\backslash M_1)$. Let $a \in M_1\backslash M_2$ and $b \in M_2\backslash M_1$. Then $\langle a\rangle + \langle b\rangle \nsubseteq M_1 \cup M_2$ and so $\langle a\rangle + \langle b\rangle = N$. □

Corollary 2.7([3], Theorem 2.2)**.** *Let R be a commutative ring with identity. Then the following are equivalent:*
(i) *$\Gamma_2(R)\backslash J(R)$ is a complete bipartite graph.*
(ii) *The cardinal number of the set* Max(R) *is equal 2.*

Theorem 2.8. *Let N be a near-ring and let $n > 1$. Then the following hold:*
(i) *If* $|\text{Max(N)}| = n < \infty$, *then the graph $\Gamma_2(N)\backslash M(N)$ is n-partite.*
(ii) *If the graph $\Gamma_2(N)\backslash M(N)$ is n-partite, then* $|\text{Max(N)}| \leq n$. *In this case if the graph $\Gamma_2(N)\backslash M(N)$ is not $(n-1)$-partite, then* $|\text{Max(N)}| = n$.

Proof. The proof is similar to that of Proposition 2.3 of [3]. □

Theorem 2.9. *Let N be a near-ring with* $|\text{Max(N)}| \geq 2$. *Then the following hold:*
(i) *If $\Gamma_2(N)\backslash M(N)$ is a complete n-partite graph, then $n = 2$.*
(ii) *If there exists a vertex of $\Gamma_2(N)\backslash M(N)$ which is adjacent to every other vertex, then $N \cong \mathbb{Z}_2 \times F$, where $\mathbb{Z}_2 = \{0, 1\}$ is the ring under addition modulo 2 and multiplication modulo 2; F is a simple near-ring.*

Proof. (i) Let M_1, M_2 be two maximal ideals of N. Since the elements of $M_i\backslash M(N)$ are not adjacent, and at least one element of $M_1\backslash M(N)$ is adjacent to $M_2\backslash M(N)$, so $M_1\backslash M(N)$ and $M_2\backslash M(N)$ are subsets of two distinct parts of $\Gamma_2(N)$. Suppose $M(N) \subset M_1 \cap M_2$. Then there exists $x \in M_1 \cap M_2$ with $x \notin M(N)$, and so x belongs to $M_1\backslash M(N)$ and $M_2\backslash M(N)$, a contradiction to $M_1\backslash M(N)$ and $M_2\backslash M(N)$ are subsets of two distinct parts of $\Gamma_2(N)$. Thus $M(N) = M_1 \cap M_2$ and hence $|\text{Max(N)}| = 2$. By Theorem 2.6, we have $n = 2$.
(ii) Let $x \in \Gamma_2(N)\backslash M(N)$ such that x is adjacent to every other vertex. Clearly $\langle x\rangle \subseteq M$ for some maximal ideal M of N. Suppose $y(\neq 0) \in M(N)$. Then $x+y \notin M(N)$ and $\langle x+y\rangle \neq N$ which implies $\langle x\rangle+\langle x+y\rangle = N$, and so $M = N$, a contradiction. So $M(N) = 0$. Now, let $y \in M$ with $y \notin \{0, x\}$. Then $N = \langle x\rangle+\langle y\rangle \subseteq M$, a contradiction. Therefore $M = \{0, x\} = \langle x\rangle$ is a maximal ideal of N. Thus for each $s(\neq 0) \in \Gamma_2(N)$, having $\langle x\rangle + \langle s\rangle = N$ implies $N/\langle x\rangle \cong \langle s\rangle$. Thus $\langle s\rangle = F$ is simple and hence $N \cong \mathbb{Z}_2 \times F$. □

Corollary 2.10([3], Proposition 2.4)**.** *Let R be a commutative ring with* $|\text{Max(R)}| \geq 2$. *Then the following hold:*
(i) *If $\Gamma_2(R)\backslash J(R)$ is a complete n-partite graph, then $n = 2$.*

(ii) *If there exists a vertex of* $\Gamma_2(R)\backslash J(R)$ *which is adjacent to every other vertex, then* $R \cong \mathbb{Z}_2 \times F$, *where* F *is a field.*

Lemma 2.11. *Let* N *be a near-ring. Then* $\mathrm{diam}(\Gamma_2(\mathrm{N})\backslash\mathrm{M}(\mathrm{N})) = 1$ *if and only if* $N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. The proof is similar to that of Lemma 3.2 of [3]. □

Theorem 2.12. *Let* N *be a near-ring with atleast two maximal ideals and let* $M(N)$ *be a completely reflexive ideal of* N. *Then* $\mathrm{diam}(\Gamma_2(\mathrm{N})\backslash\mathrm{M}(\mathrm{N})) = 2$ *if and only if one of the following holds:*
(i) $M(N)$ *is a prime ideal.*
(ii) $|\mathrm{Max(N)}| = 2$ *and* $N \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Let $M(N)$ be prime and let $a, b \in \Gamma_2(N)\backslash M(N)$. Then $\langle a\rangle\langle b\rangle \nsubseteq M(N)$, and so by the same argument as in Theorem 2.4, there exists $x \in \Gamma_2(N)\backslash M(N)$ such that $a \approx x \approx y$ is a path. If $\mathrm{diam}(\Gamma_2(\mathrm{N})\backslash\mathrm{M}(\mathrm{N})) = 1$, then by Lemma 2.11, $N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. But $M(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is not a prime ideal, a contradiction.
Next, let $|\mathrm{Max(N)}| = 2$ and $N \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, then by Theorem 2.6, $\Gamma_2(N)\backslash M(N)$ is a complete bipartite graph where at least one of the parts has at least two elements. So $\mathrm{diam}(\Gamma_2(\mathrm{N})\backslash\mathrm{M}(\mathrm{N})) = 2$.

Conversely, let $\mathrm{diam}(\Gamma_2(\mathrm{N})\backslash\mathrm{M}(\mathrm{N})) = 2$ and $M(N)$ is not prime. Let $a, b \notin M(N)$, but $\langle a\rangle\langle b\rangle \subseteq M(N)$. We show that a and b are adjacent. Otherwise there exists $t \in \Gamma_2(N)$ such that $\langle a\rangle + \langle t\rangle = \langle b\rangle + \langle t\rangle = N$. Then there are $x_1 \in \langle a\rangle; x_1^{'} \in \langle b\rangle$ and $y_1, y_1^{'} \in \langle t\rangle$ such that $x_1 + y_1 = x_1^{'} + y_1^{'} = 1$ which implies $x_1x_1^{'} + y_1x_1^{'} + y_1^{'} = 1$. Since $x_1x_1^{'} \in \langle a\rangle\langle b\rangle$ and $y_1x_1^{'} + y_1^{'} \in \langle t\rangle$, we have $\langle\langle a\rangle\langle b\rangle\rangle + \langle t\rangle = N$, which implies $\langle a\rangle\langle b\rangle \nsubseteq M(N)$, a contradiction. Therefore $\langle a\rangle + \langle b\rangle = N$, and so $x + y = 1$ for some $x \in \langle a\rangle$ and $y \in \langle b\rangle$.
Set $S = N/M(N)$ and $a_1 = x + M(N)$ and $b_1 = y + M(N)$. Then $a_1b_1 = 0$ and $a_1 + b_1 = 1_S$. Since $M(N)$ is completely reflexive, we have $\langle a_1\rangle\langle b_1\rangle = 0$. If $z \in \langle a_1\rangle \cap \langle b_1\rangle$, then $\langle z\rangle^2 \subseteq \langle a_1\rangle\langle b_1\rangle = 0$. Since $M(N)$ in semiprime ideal of N, we have $z = 0$. Thus $\langle a_1\rangle \cap \langle b_1\rangle = 0$ and hence $S = \langle a_1\rangle \oplus \langle b_1\rangle$. Let M be a nonzero ideal of $\langle a_1\rangle$ and let $m(\neq 0) \in M$ and $x_1(\neq 0) \in \langle b_1\rangle$. Then by the same argument of a and b, we have $\langle m\rangle + \langle x_1\rangle = S$ which implies $m_1 + x_1^{'} = 1_S$ for some $m_1 \in \langle m\rangle$ and $x_1^{'} \in \langle x_1\rangle$. Now let $t \in \langle a_1\rangle$. Then $m_1t + x_1^{'}t = t$. Since $x_1^{'}t = 0$, we have $t = m_1t \in M$. Thus $\langle a_1\rangle$ is simple. With the same argument, $\langle b_1\rangle$ is simple. Therefore $|\mathrm{Max(S)}| = 2$, and so $|\mathrm{Max(N)}| = 2$. □

Corollary 2.13([3], Proposition 3.3). *Assume that* R *is not local. Then* $\mathrm{diam}(\Gamma_2(\mathrm{R})\backslash\mathrm{J(R)}) = 2$ *if and only if one of the following holds:*
(i) $J(R)$ *is a prime ideal.*
(ii) $|\mathrm{Max(R)}| = 2$ *and* $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$. □

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