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The Hahn-Banach Theorem on Arbitrary Groups

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Abstract. In this paper, one kind of subgroup in arbitrary group which similar to the linear subspace was constructed, and the generalization of the Hahn-Banach theorem on this kind of subgroup in arbitrary groups was obtained.

1. Introduction

The Hahn-Banach theorem is a powerful existence theorem whose form is particularly appropriate to applications in linear problems. In its elegance and power, the Hahn-Banach theorem is a favorite of almost every analyst. The generalization of the Hahn-Banach theorem on groups has been discussed in many articles, much of these discussions were under the assumption of some condition of groups, such as the weakly commutativity in the paper of Z. Gajda and Z. Kominek [3], or groups in class $\mathcal{G}$ during R. Badora [1]. The purpose of this paper is to find the sufficient and necessary condition of Hahn-Banach theorem on arbitrary groups.

Let G be a group, p, f be functionals on $G \to \mathbf{R}$, then p is subadditive and f is additive if and only if

$$p(xy) \le p(x) + p(y), \quad x, y \in G \ \text{ and } \ f(xy) = f(x) + f(y), \quad x, y \in G.$$

Moreover, p is completely commutative if and only if for any $n \in \mathbf{Z}^+$ and any permutation $x_{k_1}, \cdots, x_{k_n}$ of $x_1, \cdots, x_n \in G$, one has

$$p(\prod_{i=1}^{n} x_i) = p(\prod_{i=1}^{n} x_{k_i}).$$

Let $\mathbf{A}$ be all additive functionals f on group G, denote that

$$V_0 = \bigcap_{f \in \mathbf{A}} ker(f)$$

and for every $H < G$, we denote

$$r(H) = \{x | \exists\, l \in \mathbf{Z}^+, x^l \in H\}, \quad V = r(G_0),$$

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where G_0 is the *commutator subgroup* that generated by all the elements $g_1g_2g_1^{-1}g_2^{-1}$ of G.

According to the discussion, we first show that $V = V_0$ in Lemma 3.1, which reveals the relationship between additive functional and subgroup. Second based on the research of $S(\{x_i\})$ (Definition 2.3), for arbitrary group G and it's subgroup H, we give the sufficient and necessary condition that Hahn-Banach theorem can be generalized in Theorem 4.5 and Theorem 4.6. Thus the result of R. Badora ([1], Theorem 3) is able to extend when $V \subseteq H$. Moreover, by our Corollary 4.2, Theorem 4.3 implies when the Hyers theorem on the stability of Cauchy functional equation (see Tabor [7]) still holds true on arbitrary G and when not, that means the conclusion of R. Badora ([1], Theorem 2) is much weaker than what we obtain. Finally as an application of Theorem 4.6, the result on subadditive functional p which is completely commutative was obtained.

2. Preliminaries

Firstly, we give some preparations for the paper.

Proposition 2.1. *$G_0 \lhd G$, for any $x, y \in G$, there exists $v, u \in G_0$ such that*

$$xy = vyx = yxu.$$

Proposition 2.2. *$V \lhd G, V_0 \lhd G$.*

Definition 2.3. For $x_0 \in G, x_0 \notin V$, we denote that

$$S(\{x_0\}) = \{x | \exists\ l, l_0 \in \mathbf{Z}, v \in V, l > 0;\ x^l = v x_0^{l_0}\}$$

Let $\{x_i\} = \{x_1, \cdots, x_n\} \subseteq G$, satisfying

(1) for any $\{x_{i_k}\} \subset \{x_i\}$, $S(\{x_{i_k}\})$ can be defined;

(2) for any $x \in \{x_i\}, \{x_{i_k}\} \subseteq \{x_i\} \setminus \{x\}$, $x \notin S(\{x_{i_k}\})$.

Then we denote

$$S(\{x_i\}) = \{x | \exists\ l, l_i \in \mathbf{Z}, v \in V, l > 0;\ x^l = v \prod x_i^{l_i}\}.$$

Suppose a infinite subset $\{x_\Lambda\} \subseteq G$, satisfying

(1) for any $finite\ subset$ $\{x_k\} \subset \{x_\Lambda\}$, $S(\{x_k\})$ can be defined;

(2) for any $x \in \{x_\Lambda\}$, $finite\ subset$ $\{x_k\} \subset \{x_\Lambda\} \setminus \{x\}$, $x \notin S(\{x_k\})$.

Then we denote

$$S(\{x_\Lambda\}) = \bigcup_{\{x_k\} \subseteq \{x_\Lambda\}} S(\{x_k\}).$$

Proposition 2.4. *Let $\{x_i\} = \{x_1, \cdots, x_n\} \subseteq G$, $S(\{x_i\})$ can be defined, $x_{k_1}, \cdots, x_{k_n}$ is an any permutation of $x_1, \cdots, x_n$, then $S(\{x_{k_i}\}) = S(\{x_i\})$.*

Proof. By Definition 2.3, for any $x \in S(\{x_{k_i}\})$, $\exists\ l, l_{k_i} \in \mathbf{Z}, v \in V, l > 0;\ x^l =$

$v\prod x_{k_i}^{l_{k_i}}$, suppose x_{k_i} is the element $x_j \in \{x_i\}$, denote that $l_j = l_{k_i}$, using Proposition 2.1, we commute x_{k_i} again and again such that

$$x^l = v\prod x_{k_i}^{l_{k_i}} = vu\prod x_i^{l_i} = v'\prod x_i^{l_i}, \quad u \in G_0, \quad v' = vu \in V.$$

So $x \in S(\{x_i\})$, $S(\{x_{k_i}\}) \subseteq S(\{x_i\})$, by the same method we obtain $S(\{x_i\}) \subseteq S(\{x_{k_i}\})$, as the result $S(\{x_i\}) = S(\{x_{k_i}\})$. □

From now on, we ignore the permutation of $x_1, \cdots, x_n$ when discussing $S(\{x_i\})$.

Proposition 2.5. *suppose $\{x_i\} = \{x_1, \cdots, x_n\} \subseteq G$, $S(\{x_i\})$ can be defined, $x \in S(\{x_i\})$, $x^l = v\prod x_i^{l_i}$, $l, l_i \in \mathbf{Z}, v \in V, l > 0$, then l_i/l is uniquely determined by x.*

Proof. For the reduction to absurdity, when $n > 1$, suppose there exist $l', l_i' \in \mathbf{Z}, v' \in V, l' > 0$, such that

$$x^{l'} = v'\prod x_i^{l_i'},$$

where not all the value of $l_i/l - l_i'/l'$ are 0, in other words, $l_il' - l_i'l$ do not all equal to 0. Then, by Proposition 2.1 and the proof of Proposition 2.4, we have

$$e = x^{ll'-l'l} = (v\prod x_i^{l_i})^{l'}(v'\prod x_i^{l_i'})^{-l} = v''\prod x_i^{l_il'-l_i'l}, \ v'' \in V.$$

Notice that there exist $l_kl' - l_k'l \neq 0$, without loss of generality, we can suppose $k = n$ and $l_nl' - l_n'l < 0$, so

$$x_n^{l_n'l-l_nl'} = v''\prod_{i=1}^{n-1} x_i^{l_il'-l_i'l}.$$

Now we get $x_n \in S(\{x_1, \cdots, x_{n-1}\})$, which is contrary to the definition of $S(\{x_i\})$. When $n = 1$, as the same discussion, we get $x_1 \in V$, which is also contrary to the definition. Hence l_i/l is uniquely determined by x. □

Proposition 2.6. *Suppose $\{x_i\} = \{x_1, \cdots, x_n\} \subseteq G$, $S(\{x_i\})$ can be defined, then $S(\{x_i\}) \lhd G$.*

Proof. For $x \in S(\{x_i\})$, $x^l = v\prod x_i^{l_i}$, $v \in V, l, l_i \in \mathbf{Z}, l > 0$, since $V \lhd G$, we have

$$x^{-l} = (\prod_{i=1}^{n} x_{n+1-i}^{-l_{n+1-i}})v^{-1} = v'\prod_{i=1}^{n} x_{n+1-i}^{-l_{n+1-i}}, \ v' \in V.$$

Hence $x^{-1} \in S(\{x_n, \cdots, x_1\}) = S(\{x_i\})$. Moreover, for gxg^{-1}, $g \in G$, we obtain

$$gxg^{-1} = (gxg^{-1}x^{-1})x \in S(\{x_i\}).$$

Suppose $y \in S(\{x_i\})$, then $\exists\, k, k_i \in \mathbf{Z}, u \in V, k > 0; y^k = u\prod x_i^{k_i}$, using Proposition 2.1 over and over, we get

$$(xy)^{kl} = wx^{kl}y^{kl} = w(v\prod x_i^{l_i})^k(u\prod x_i^{k_i})^l = w'\prod x_i^{k_il+l_ik}, \quad w \in G_0, \ w' \in V.$$

Since $kl > 0$, we have $xy \in S(\{x_i\})$, thus $S(\{x_i\}) \lhd G$. □

Proposition 2.7. *Suppose a infinite subset $\{x_\Lambda\} \subseteq G$, where $S(\{x_\Lambda\})$ can be defined, then $S(\{x_\Lambda\}) \lhd G$.*

Proof. This can be proved immediately by the Definition 2.3 and Proposition 2.6. □

Proposition 2.8. *Let $\{x_i\} = \{x_1, \cdots, x_n\} \subseteq G$, where $S(\{x_i\})$ can be defined, suppose $x \in G \setminus S(\{x_i\})$, then $S(\{x_i\}) \cap S(\{x\}) = V$.*

Proof. Firstly, we have $V \subseteq S(\{x_i\}) \cap S(\{x\})$ by Definition 2.3. Moreover, if there exist $y \in S(\{x_i\}) \cap S(\{x\}) \setminus V$, using the similar method in the proof of Proposition 2.5, it can be proved that $x \in S(\{x_i\})$, which is impossible to the condition. □

Proposition 2.9. *Suppose $\{x_i\} = \{x_1, \cdots, x_n\} \subseteq G$ where $S(\{x_i\})$ can be defined, if $x \in G \setminus S(\{x_i\})$, then $S(\{x_i\}) \subseteq S(\{x, x_i\})$.*

3. Some results about additive functionals

The purpose of this section is to study the additive functional on $S(\{x_i\})$.

Lemma 3.1. *Let $\{x_i\} = \{x_1, \cdots, x_n\} \subseteq G$ where $S(\{x_i\})$ can be defined, then for any real number sequence $c_1, \cdots, c_n$, there exist additive functional a on $S(\{x_i\})$, which satisfy $a(x_i) = c_i$ and $a|V = 0$.*

Proof. By Definition 2.3, for any $x \in S(\{x_i\})$,

$$\exists\, l, l_i \in \mathbf{Z}, v \in V, l > 0; x^l = v \prod x_i^{l_i}.$$

Define that

$$a(x) = \sum \frac{l_i}{l} c_i.$$

Following Proposition 2.5, l_i/l is uniquely determined by x, so $a(x)$ is a well-defined functional on $S(\{x_i\})$. Moreover, if $x \in V$, then $l_i/l = 0$, as the result,

$$a|V = 0.$$

Now we prove the additivity of $a(x)$, using the proof of Proposition 2.6, suppose $y \in S(\{x_i\})$, $y^k = u \prod x_i^{k_i}$, $k, k_i \in \mathbf{Z}, u \in V, k > 0$, then the following fact is

$$a(xy) = \frac{\sum(l_i k + k_i l)c_i}{kl} = \sum \frac{l_i c_i}{l} + \sum \frac{k_i c_i}{k} = a(x) + a(z)$$

which prove the additivity of a. It's easy to see $a(x_i) = c_i$, which means $a(x)$ is the additive functional as need. □

Corollary 3.2. *Suppose a infinite subset $\{x_\Lambda\} \subseteq G$ where $S(\{x_\Lambda\})$ can be defined, then for any real number set $\{c_\Lambda\}$, there exist additive functional a on $S(\{x_\Lambda\})$,* *which satisfy* $a(x_\Lambda) = c_\Lambda$ *and* $a|V = 0$.

Lemma 3.3. $V_0 = V$.

Proof. We first prove $V \subseteq V_0$, let $x \in V$, following the Definition of V,

$$\exists\, l \in \mathbf{Z}^+,\ x_i = g_{i1}g_{i2}g_{i1}^{-1}g_{i2}^{-1} \in G_0,\quad x^l = \prod x_i.$$

So for any $f \in \mathbf{A}$,

$$f(x^l) = f(\prod x_i) = \sum f(x_i) = 0,\ \ l(f(x)) = 0,\quad f(x) = 0.$$

Hence $V \subseteq V_0$.
Next, we will show that $V_0 \subseteq V$. If $V = G$, then the proof is finished. If $V \subset G$ and $V \neq G$, then for any $x_0 \in G \setminus V$, by Lemma 3.1, there exist an additive functional $a(x)$ on $S(\{x_0\})$, satisfying $a(x_0) = 1$ and $a|V = 0$. Denote family $\mathcal{F} = \{(S(\{x_\Lambda\}), f_{\{x_\Lambda\}})\}$, where $S(\{x_\Lambda\})$ can be defined and $f_{\{x_\Lambda\}}$ is additive functional on $S(\{x_\Lambda\})$. Moreover, we let all $(S(\{x_\Lambda\}), f_{\{x_\Lambda\}})$ of $\mathcal{F}$ satisfy
(1) $x_0 \in \{x_\Lambda\}$;
(2) $f_{\{x_\Lambda\}}|S(\{x_0\}) = a$.
Then $(S(\{x_0\}), a) \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. Introducing a partial order $\preceq$ by putting $(S(\{x_\Lambda\}), f_{\{x_\Lambda\}}) \preceq (S(\{x_\Gamma\}), f_{\{x_\Gamma\}})$ iff

$$\{x_\Lambda\} \subseteq \{x_\Gamma\},\ f_{\{x_\Gamma\}}|S(\{x_\Lambda\}) = f_{\{x_\Lambda\}}.$$

Let $\mathcal{L} = \{(S(\{x_\Lambda\}), f_{\{x_\Lambda\}}) : \{x_\Lambda\} \in \mathcal{D}\}$ be a linearly ordered subfamily of $\mathcal{F}$, then it's easy to verify that the pair (S, f), where

$$S = \bigcup_{\{x_\Lambda\}\in\mathcal{D}} S(\{x_\Lambda\}),\quad f(x) = f_{\{x_\Lambda\}}(x),\ x \in S(\{x_\Lambda\}), \{x_\Lambda\} \in \mathcal{D}$$

is a upper bound of $\mathcal{L}$, moreover, $(S, f) \in \mathcal{F}$. Hence the Zorn Lemma implies that there exist a maximal element $\{(S(\{x_\Delta\}), f_{\{x_\Delta\}})\}$ in $\mathcal{F}$. We obtain

$$G = S(\{x_\Delta\}).$$

In fact, for the reduction to absurdity, suppose $y \in G \setminus S(\{x_\Delta\})$, we construct a subgroup $S(\{y, x_\Delta\})$. Following the Corollary 3.2, there exist an additive functional a_0 on $S(\{y, x_\Delta\})$, satisfying

$$a_0(y) = 0,\ \ a_0(x) = f_{\{x_\Delta\}}(x), x \in \{x_\Delta\}\ \ and\ \ a_0|V = 0.$$

Now for any $x \in S(\{x_\Delta\})$, there exist $\{x_i\} = \{x_1, \cdots, x_n\} \subseteq \{x_\Delta\}$, $x \in S(\{x_i\})$ and $x^l = v\prod x_i^{l_i}$, $v \in V, l, l_i \in \mathbf{Z}, l > 0$, then

$$a_0(x^l) = a_0(v\prod x_i^{l_i}) = a_0(\prod x_i^{l_i}) = f_{\{x_\Delta\}}(\prod x_i^{l_i}) = f_{\{x_\Delta\}}(v\prod x_i^{l_i}) = f_{\{x_\Delta\}}(x^l),$$

$$a_0(x) = f_{\{x_\Delta\}}(x).$$

So,

$$a_0|S(\{x_\Delta\}) = f_{\{x_\Delta\}}, \quad (S(\{x_\Delta\}), f_{\{x_\Delta\}}) \preceq (S(\{y, x_\Delta\}), a_0)$$

which implies that $S(\{x_\Delta\}) = S(y, \{x_\Delta\})$, and this is contrary to the assumption of $y \in G \setminus S(\{x_\Delta\})$. Thus, $G = S(\{x_\Delta\})$.
As a result, $f_{\{x_\Delta\}}$ is an additive functional on G, where $f_{\{x_\Delta\}}(x_0) = 1$, and by the arbitrariness of x_0, we obtain $V_0 \subseteq V$. Hence $V_0 = V$. □

4. The Hahn-Banach theorem on arbitrary groups

From now on, the Hahn-Banach theorem on arbitrary groups will be discussed.

Theorem 4.1. *Suppose $\{x_\Lambda\} \subseteq G$ where $S(\{x_\Lambda\})$ can be defined, a is an additive functional on $S(\{x_\Lambda\})$, p is subaddtive functional on G where $a \leq p|S(\{x_\Lambda\})$, then $a|V = 0$ if and only if there exist additive functional a_0 on G satisfying $a_0|S(\{x_\Lambda\}) = a$ and $a_0 \leq p|G$.*

Proof. The conclusion is trivial when $G = S(\{x_\Lambda\})$, so we discuss the case $S(\{x_\Lambda\}) \subset G$. Assume that $a|V = 0$. Let $y \in G \setminus S(\{x_\Lambda\})$, then for any $z_1, z_2 \in S(\{x_\Lambda\}), l_1, l_2 \in \mathbf{Z}^+$,

$$l_2 p(z_1 y^{-l_1}) + l_1 p(z_2 y^{l_2}) \geq p((z_1 y^{-l_1})^{l_2} (z_2 y^{l_2})^{l_1}).$$

Using Proposition 2.1, one has

$$(z_1 y^{-l_1})^{l_2} (z_2 y^{l_2})^{l_1} = v y^{l_2 l_1 - l_1 l_2} z_1^{l_2} z_2^{l_1} = v z_1^{l_2} z_2^{l_1},$$

$$v \in G_0 \subseteq V \subseteq S(\{x_\Lambda\}).$$

Thus

$$l_2 p(z_1 y^{-l_1}) + l_1 p(z_2 y^{l_2}) \geq p(v z_1^{l_2} z_2^{l_1}) \geq a(v z_1^{l_2} z_2^{l_1}) = l_2 a(z_1) + l_1 a(z_2),$$

$$\frac{1}{l_1}[a(z_1) - p(z_1 y^{-l_1})] \leq \frac{1}{l_2}[-a(z_2) + p(z_2 y^{l_2})].$$

By the arbitrariness of z_1, z_2, l_1, l_2, we have

$$\sup(\frac{1}{l}[a(z) - p(z y^{-l})]) \leq \inf(\frac{1}{l}[-a(z) + p(z y^{l})]), \quad z \in S(\{x_\Lambda\}), l \in \mathbf{Z}^+.$$

So there exist a real number c such that

$$\sup(\frac{1}{l}[a(z) - p(z y^{-l})]) \leq c \leq \inf(\frac{1}{l}[-a(z) + p(z y^{l})]).$$

Now we consider the subgroup $S(\{x_\Lambda\})$, by Lemma 3.1, Corollary 3.2 and the proof of Lemma 3.3, there exist an additive functional g on $S(\{y, x_\Lambda\})$, satisfying

$$g|V = 0, \quad g|S(\{x_\Lambda\}) = a \ \text{ and } \ g(y) = c.$$

For any $z \in S(\{y, x_\Lambda\})$, when $z \in S(\{x_\Lambda\})$, using $g|S(\{x_\Lambda\}) = a$ we have

$$g(z) \leq p(z),$$

when $z \in S(\{y, x_\Lambda\}) \setminus S(\{x_\Lambda\})$, there exist $\{x_k, y\} \subseteq \{y, x_\Lambda\}, k = 1, \cdots, n$, where satisfy $z^{l_z} = (v \prod x_k^{l_k}) y^{l_y}$, $v \in V, l_z, l_y, l_k \in \mathbf{Z}, l_z > 0$. Thus using the inequality of c just now

$$g(z^{l_z}) = g((v \prod x_k^{l_k}) y^{l_y}) = g(v \prod x_k^{l_k}) + l_y c \leq p((v \prod x_k^{l_k}) y^{l_y}) = p(z^{l_z}) \leq l_z p(z),$$

$$g(z) \leq p(z).$$

On the basis of above, we have an additive functional g on $S(\{y, x_\Lambda\})$, satisfying

$$g|S(\{x_\Lambda\}) = a, \quad g \leq p|S(\{y, x_\Lambda\}).$$

As the same method in Lemma 3.3, by Zorn Lemma we can establish an additive functional a_0 on G, satisfying $a_0|S(\{x_\Lambda\}) = a$ and $a_0 \leq p|G$, which prove the necessity of this theorem. To show the sufficiency, let a_0 be the additive functional satisfy the conditions of this theorem. By $V_0 = V$, we obtain $a|V = a_0|V = a_0|V_0 = 0$. □

Corollary 4.2. *Let p be a subadditive functional on G, then $p|V \geq 0$ if and only if there exist additive functional a on G satisfying $a \leq p|G$.*

One application of above corollary is to discuss the Hyers Theorem (see [1] and [7]) on arbitrary group G.

Theorem 4.3. *Let $c \geq 0$, f is functional on $G \to \mathbf{R}$ satisfying*

$$|f(xy) - f(x) - f(y)| \leq c, \quad x, y \in G,$$

then $-c \leq f|V \leq c$ if and only if there exist an additive functional a on G such that

$$|a(x) - f(x)| \leq c, \quad x \in G.$$

By Theorem 4.3, it is easy to see the necessary and sufficient condition of Heyers theorem. This is much stronger than the result of Badora ([1], Theorem 2). So far, the subgroup of G in our main discussion is always $S(\{x_\Lambda\})$ (here $\{x_\Lambda\}$ can be finite or infinite subset), but by the following proposition, we will reveal the relationship between $S(\{x_\Lambda\})$ and arbitrary subgroup H.

Proposition 4.4. *Let $H < G$, then there exist a subset $\{x_\Lambda\} \subseteq H$, where $S(\{x_\Lambda\})$ can be defined and $H \subseteq S(\{x_\Lambda\}) = r(VH)$, moreover, $r(H) = S(\{x_\Lambda\})$ if $V \subseteq H$.*

Proof. It's the method during the proof of Lemma 3, by Zorn Lemma the existence of $\{x_\Lambda\} \subseteq H$ is confirmed where $S(\{x_\Lambda\})$ can be defined and $H \subseteq S(\{x_\Lambda\})$. At the same time Definition 2.3 implies that $S(\{x_\Lambda\}) = r(VH)$, hence when $V \subseteq H$, $r(H) = S(\{x_\Lambda\})$. □

According to this proposition, Theorem 4.1 can be improved as

Theorem 4.5. *Suppose $V \subseteq H < G$, a is an additive functional on H, p is subaddtive functional on G where $a \leq p|H$, then $a|V = 0$ if and only if there exist additive functional a_0 on G satisfying $a_0|H = a$ and $a_0 \leq p|G$.*

Proof. Assume that $a|V = 0$, then $\exists\ \{x_\Lambda\} \subseteq H$ where $S(\{x_\Lambda\})$ can be defined and $S(\{x_\Lambda\}) = r(H)$, so we have a additive functional f on $S(\{x_\Lambda\}) = r(H)$ by Corollary 3.2 satisfying

$$f|V = 0, \quad f(x_\Lambda) = a(x_\Lambda)$$

from Definition 2.3, it means that

$$f|H = a.$$

On the other hand, for any $x \in S(\{x_\Lambda\})$, $\exists\ l \in \mathbf{Z}^+, x^l \in H$, hence

$$f(x^l) \leq p(x^l) \leq lp(x), \quad f(x) \leq p(x).$$

So by Theorem 4.1, the necessity can be proved. □

To show the sufficiency, let a_0 be the additive functional fulfilling the conditions in this theorem, then Lemma 3.3 implies that

$$a|V = a_0|V = 0.$$

Now we will discuss the condition in arbitrary subgroup H.

Theorem 4.6. *Suppose $H < G$, a is an additive functional on H, p is subaddtive functional on G where $a \leq p|H$, then there exist additive functional a_0 on G satisfying $a_0|H = a$ and $a_0 \leq p|G$ if and only if $a|H \cap V = 0$ and for any $v \in V, x \in H, a(x) \leq p(vx)$.*

Proof. Similar to the proof of Theorem 4.5, assume that $a|H \cap V = 0$ and for any $v \in V, x \in H, a(x) \leq p(vx)$, there exists an additive functional f on $r(VH)$ satisfying

$$f|V = 0, \quad f|H = a.$$

For any $y \in r(VH)$, there exists $y^l = vx$ where $v \in V, x \in H, l \in \mathbf{Z}^+$, so

$$f(y^l) = f(x) \leq p(vx) = p(y^l), \quad f(y) \leq p(y).$$

According to Theorem 4.5, the sufficiency can be proved.
Let a_0 be the additive functional fulfilling the conditions in this theorem, Lemma 3.3 points that

$$a|H \cap V = a_0|H \cap V = 0.$$

At the same time, for any $v \in V, x \in H$,

$$a(x) = a_0(x) = a_0(vx) \leq p(vx)$$

which proves the necessity. $\square$

By the above discussion, Proposition 4.4 also reveals the following conclusion.

Corollary 4.7. *Let a be the additive functional on H where $H < G$, then a can extend to additive functional f on G if and only if $a|H \cap V = 0$.*

Up till now, we have finished our discussion about Hahn-Banach Theorem on arbitrary group. For application, the Hahn-Banach Theorem on commutative subadditive functional p will be discussed in the following.

Lemma 4.8. *Let p be a subadditive functional on G, if p is also completely commutative, then for every $v \in V$, $\exists\ l \in \mathbf{Z}^+$,*

$$\lim_{n\to+\infty} \frac{p(v^{nl})}{nl} = \lim_{n\to+\infty} \frac{p(v^{-nl})}{nl} = 0.$$

Proof. According to the denotation, for any $v \in V$, $\exists\ y_i \in G, y_i = g_{i1}g_{i2}g_{i1}^{-1}g_{i2}^{-1}, l \in \mathbf{Z}^+$,

$$v^l = \prod y_i.$$

Since p is completely commutative, we have

$$p(v^l) = p(\prod y_i) = p(e), \quad p(v^{nl}) = p((\prod y_i)^n) = p(e)$$

and

$$p(v^{-l}) = p((\prod y_i)^{-1}) = p(e), \quad p(v^{-nl}) = p((\prod y_i)^{-n}) = p(e)$$

which implies

$$\lim_{n\to+\infty} \frac{p(v^{nl})}{nl} = \lim_{n\to+\infty} \frac{p(v^{-nl})}{nl} = \lim_{n\to+\infty} \frac{p(e)}{nl} = 0.$$

$\square$

Theorem 4.9. *Suppose $H < G$, a is additive functional on H, p is completely commutative subadditive functional on G, then there exist additive functional a_0 on G, where $a_0|H = a$ and $a_0 \le p|G$.*

Proof. By the denotation, for any $v \in V$, there exist $y_i \in G, y_i = g_{i1}g_{i2}g_{i1}^{-1}g_{i2}^{-1}, l \in \mathbf{Z}^+$,

$$v^l = \prod y_i.$$

Attend that by the complete commutativity of p, for any $x \in H$, there have

$$a(x^l) \le p(x^l) = p((\prod y_i)x^l) = p((vx)^l), \quad a(x) \le p(vx).$$

On the other hand, if $v \in H$ also and $n \in \mathbf{Z}^+$, then

$$(nl)a(v) = a(v^{nl}) \le p(v^{nl}), \quad a(v) \le \frac{p(x^{nl})}{nl}$$

and

$$(-nl)a(v) = a(v^{-nl}) \leq p(v^{-nl}), \quad -a(v) \leq \frac{p(x^{-nl})}{nl}.$$

According to Lemma 4.8, $\lim_{n\to+\infty} \frac{p(v^{nl})}{nl} = \lim_{n\to+\infty} \frac{p(v^{-nl})}{nl} = 0$, thus

$$a(v) \leq 0, -a(v) \leq 0, \quad a(v) = 0.$$

So $a|H \cap V = 0$, which means our proof is finished by Theorem 4.6. $\square$

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