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Meromorphic Function Sharing Two Small Functions with Its Derivative

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Abstract. In this paper, we deal with the problem of uniqueness of meromorphic functions that share two small functions with their derivatives, and obtain the following result which improves a result of Yao and Li: Let $f(z)$ be a nonconstant meromorphic function, $k > 5$ be an integer. If $f(z)$ and $g(z) = a_1(z)f(z) + a_2(z)f^{(k)}(z)$ share the value 0 CM, and share $b(z)$ IM, $\overline{N}_E(r, f = 0 = f^{(k)}) = S(r)$, then $f \equiv g$, where $a_1(z),\ a_2(z)$ and $b(z)$ are small functions of $f(z)$.

1. Introduction and main results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We say that two meromorphic functions f and g share a finite value a IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities).

Denote by $\overline{N}(r, f = b = g)$ the reduced counting function of the common zeros of $f - b$ and $g - b$ ignoring the multiplicities, and $\overline{N}_E(r, f = b = g)$ the reduced counting function of the common zeros of $f-b$ and $g-b$ with the same multiplicities. We say that f and g share b IM^* provided that

$$\overline{N}(r, \frac{1}{f-b}) - \overline{N}(r, f = b = g) = S(r, f)$$

and

$$\overline{N}(r, \frac{1}{g-b}) - \overline{N}(r, f = b = g) = S(r, f).$$

Similarly, we say that f and g share b CM^* provided that

$$\overline{N}(r, \frac{1}{f-b}) - \overline{N}_E(r, f = b = g) = S(r, f)$$

and

$$\overline{N}(r, \frac{1}{g-b}) - \overline{N}_E(r, f = b = g) = S(r, f).$$

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It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna Theory, as found in [4], [5]. In 1986, Frank-Weissenborn proved the following result.

Theorem A([1]). *Let f be a nonconstant meromorphic function, a, b be two distinct finite complex number. If f and $f^{(k)}$ share the value a, b CM, then $f \equiv f^{(k)}$.*

Frank asked the following question.

Question 1. Does the Theorem A hold if we replace the condition that f and $f^{(k)}$ share b CM by the condition that f and $f^{(k)}$ share b IM?

The following example given by Ping-Li shows that the answer to Question 1 is, in general, negative. Let a_1 be any finite constant, $a_2 = a_1 + \sqrt{2}i$, ω be a nonconstant solution of the Riccati differential equation

$$\omega' = (\omega - a_1)(\omega - a_2)$$

and let

$$f = (\omega - a_1)(\omega - a_2) - \frac{1}{3}.$$

It is easy to verify that

$$f'' = 6\omega' f,$$

$$f'' + \frac{1}{6} = 6(f + \frac{1}{6})^2.$$

Since 0 is the Picard value of ω', then 0 must be a CM shared value of f and f''. It is easy to see that f and f'' share the value $-\frac{1}{6}$ IM, but $f \not\equiv f''$.

In 1990, Yang proved the following result.

Theorem B([3]). *Let f be a nonconstant entire function, $k \geq 2$ be an integer, $a \neq 0$ be a finite constant. If 0 is the Picard value of f and $f^{(k)}$, and if f and $f^{(k)}$ share a IM, then $f = e^{Az+B}$, A, B be two constants, where $A^k = 1$, and so that $f \equiv f^{(k)}$.*

It is natural to ask what results can be obtained if $f^{(k)}$ is replaced by a differential polynomial of f, and the values 0 and a are replaced by the small functions of f? In 2006, Yao and Li proved the next result.

Theorem C([7]). *Let $f(z)$ be a nonconstant meromorphic function, $a_1(z)$, $a_2(z)$ and $b(z)$ be small functions of $f(z)$, and let $g(z) = a_1(z)f + a_2(z)f'$. If f and g share the value 0 CM^*, and share the function $b(z)$ IM^*, then $f \equiv g$ or f takes one of the following two forms:*

(1) *$f = \frac{b}{h-1}$ and $a_1 b + a_2 b' = -b$, where h satisfies $\frac{h'}{h} = -\frac{1}{a_2}$.*

(2) *$f = \frac{2b}{1-h}$ and $a_1 b + a_2 b' = 0$, where h satisfies $\frac{h'}{h} = -\frac{2}{a_2}$.*

In this paper, we obtained the following results.

Theorem 1. *Let f be a nonconstant meromorphic function, $k(k > 5)$ be a positive integer, $a_1(z)$, $a_2(z)$ and $b(z)$ be small functions of f, and let $g(z) = a_1(z)f + a_2(z)f^{(k)}$. If f and g share the value 0 CM, share the function $b(z)$ IM, and $\overline{N}_E(r, f = 0 = f^{(k)}) = S(r)$, then $f \equiv g$.*

Theorem 2. *Let f be a nonconstant meromorphic function, k be a positive integer, $a_1(z)$, $a_2(z)$ and $b(z)$ be small functions of f, and let $g(z) = a_1(z)f + a_2(z)f^{(k)}$. If f and g share the value 0 CM, share the function $b(z)$ IM, if $\overline{N}_E(r, f = 0 = f^{(k)}) = S(r)$ and $\Theta(\infty, f) > \frac{5}{6}$, then $f \equiv g$.*

Corollary 1. *Let f be a nonconstant entire function, and $g(z) = a_1(z)f+a_2(z)f^{(k)}$. If f and g share the value 0 CM, and share the function $b(z)$ IM, then $f \equiv g$, where $a_1(z)$, $a_2(z)$ and $b(z)$ are defined as in Theorem 2.*

Theorem 3. *Let f be a nonconstant meromorphic function, $a_1(z)$, $\cdots$, $a_k(z)$ $(k > 2)$ and $b(z)$ be small functions of f, and let $L(f) = W(a_1, a_2, \cdots, a_k, f)$, where $W(a_1, a_2, \cdots, a_k, f)$ is the Wronskian of a_1, $\cdots$, a_k, f. If f and $L(f)$ share $b(z)$ IM and*

$$N(r, \frac{1}{f}) + N(r, \frac{1}{L}) = S(r, f),$$

then $f = L(f)$.

2. Lemmas

Lemma 1([8]). *Let f be a nonconstant meromorphic function, k be a positive integer, then*

$$N(r, \frac{1}{f^{(k)}}) < N(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f), \tag{2.1}$$

$$N(r, \frac{f^{(k)}}{f}) < k\overline{N}(r, f) + k\overline{N}(r, \frac{1}{f}) + S(r, f), \tag{2.2}$$

$$N(r, \frac{f^{(k)}}{f}) < k\overline{N}(r, f) + N(r, \frac{1}{f}) + S(r, f). \tag{2.3}$$

Suppose that f and g share the value a IM, and let z_0 be a a-point of f of order p, a a-point of g of order q. We denote by $N_L(r, \frac{1}{f-a})$ the counting function of those a-points of f where $p > q$, and we denote by $\overline{N}_L(r, \frac{1}{f-a})$ the corresponding counting function that ignores the multiplicities.

Lemma 2([6]). *Let f be a nonconstant meromorphic function. If f and g share the value 1 IM, then*

$$\overline{N}_L(r, \frac{1}{f^{(k)} - 1}) < \overline{N}(r, \frac{1}{f^{(k)}}) + \overline{N}(r, f) + S(r, f). \tag{2.4}$$

Lemma 3. *Let f be a nonconstant meromorphic function, $a_1(z)$, $a_2(z)$ and $b(z)$ be small functions of f, and let $g(z) = a_1 f + a_2 f^{(k)}$, where k is a positive integer. If f and g share the value 0 CM, and share the function b IM, $\overline{N}_E(r, f = 0 = f^{(k)}) = S(r)$ and if $f \not\equiv g$, then*

$$\overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) = S(r).$$

Proof. Since f and g share 0, b IM, and

$$\overline{N}(r, f) = \overline{N}(r, g) + S(r, f),$$

from the second fundamental theorem, we have

$$S(r, f) = S(r, g)(= S(r)).$$

Noticing that f and g share the value 0 CM and $\overline{N}_E(r, f = 0 = f^{(k)}) = S(r)$, we have

$$\overline{N}(r, f = 0 = g) \leq N(r, \frac{1}{a_2}) \leq S(r)$$

or

$$\overline{N}(r, f = 0 = g) \leq N(r, a_2) \leq S(r).$$

So we get

$$\overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) = S(r).$$

□

Lemma 4([2]). *Let f be a transcendental meromorphic function, $a_1(z)$, $a_2(z), \cdots, a_k(z)$ $(k > 2)$ be linearly independent small functions of f, $L(f) = W(a_1, a_2, \cdots, a_k, f)$ be the Wronskian of $a_1, \cdots, a_k, f$. Then*

$$k\overline{N}(r, f) \leq N(r, \frac{1}{L}) + (1 + \varepsilon)N(r, f) + S(r, f),$$

where ε is any given positive number.

3. Proof of Theorem 1

From the second fundamental theorem, Lemma 1 and Lemma 3, we have

$$\begin{aligned} T(r, g) &\leq \overline{N}(r, g) + \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{g - b}) + S(r) \\ &\leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{g - b}) + S(r). \end{aligned}$$

Since f and g share the small function $b(z)$ IM, we obtain

$$\begin{aligned}
T(r,g) &\leq \overline{N}(r,f)+\overline{N}(r,\frac{1}{\frac{g}{f}-1})+S(r)\\
&\leq \overline{N}(r,f)+\overline{N}(r,\frac{1}{a_1+\frac{a_2f^{(k)}}{f}-1})+S(r)\\
&\leq \overline{N}(r,f)+T(r,\frac{f^{(k)}}{f})+S(r)\\
&\leq \overline{N}(r,f)+N(r,\frac{f^{(k)}}{f})+S(r)\\
&\leq (k+1)\overline{N}(r,f)+S(r)\\
&\leq N(r,g)+S(r)\\
&\leq T(r,g)+S(r),
\end{aligned}$$

so we have

(3.1) $$T(r,g)=(k+1)\overline{N}(r,f)+S(r),\ N(r,f)=\overline{N}(r,f)+S(r).$$

Let $G=\frac{g}{b}$, $F=\frac{f}{b}$ and

$$H=\frac{G''}{G'}-2\frac{G'}{G-1}-\frac{F''}{F'}+2\frac{F'}{F-1}.$$

By the Lemma of logarithmic derivatives, we have $m(r,H)=S(r)$. Since f and g share the value b IM, and share 0 CM, we know that F and G share the value b IM*, and share 0 CM*, then

(3.2)
$$N(r,H)=\overline{N}(r,F)+\overline{N}_L(r,\frac{1}{F-1})+\overline{N}_L(r,\frac{1}{G-1})+\overline{N}_0(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'})+S(r),$$

where $\overline{N}_0(r,\frac{1}{F'})$ denotes the reduced counting function of F' which are not the zeros of F and $F-1$. $\overline{N}_0(r,\frac{1}{G'})$ are similarly defined. From the second fundamental theorem, we have

(3.3) $$T(r,F)\leq \overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{F-1})-\overline{N}_0(r,\frac{1}{F'})+S(r,F),$$

(3.4) $$T(r,G)\leq \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{G-1})-\overline{N}_0(r,\frac{1}{G'})+S(r,G).$$

If $H\not\equiv 0$, by calculation, we know that the common simple zeros of $F-1$ and $G-1$ are the zeros of H, it follows that

(3.5) $$N_E^{1)}(r,\frac{1}{F-1})\leq N(r,\frac{1}{H})\leq T(r,H)=N(r,H)+S(r)$$

and

$$\begin{aligned}\overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1}) &= 2N_E^{1)}(r,\frac{1}{F-1})+2\overline{N}_L(r,\frac{1}{F-1})\\ &\quad +2\overline{N}_L(r,\frac{1}{G-1})+2\overline{N}_E^{(2}(r,\frac{1}{F-1})\\ &\leq \overline{N}(r,F)+3\overline{N}_L(r,\frac{1}{F-1})+3\overline{N}_L(r,\frac{1}{G-1})\\ &\quad +N_E^{1)}(r,\frac{1}{F-1})+2\overline{N}_E^{(2}(r,\frac{1}{F-1})\\ &\quad +\overline{N}_0(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'}). \qquad (3.6)\end{aligned}$$

Combining (3.2) − (3.6), we have

$$\begin{aligned}T(r,F)+T(r,G) &\leq 3\overline{N}(r,F)+3\overline{N}_L(r,\frac{1}{F-1})+2\overline{N}_L(r,\frac{1}{G-1})\\ &\quad +2\overline{N}_E^{(2}(r,\frac{1}{F-1})+N_E^{1)}(r,\frac{1}{F-1})+S(r)\\ &\leq 3\overline{N}(r,F)+\overline{N}_L(r,\frac{1}{F-1})+2\overline{N}_L(r,\frac{1}{G-1})+N(r,\frac{1}{F-1})+S(r)\\ &\leq 3\overline{N}(r,F)+\overline{N}_L(r,\frac{1}{F-1})+2\overline{N}_L(r,\frac{1}{G-1})+T(r,F)+S(r).\end{aligned}$$

Therefore

$$T(r,G)\leq 3\overline{N}(r,F)+\overline{N}_L(r,\frac{1}{F-1})+2\overline{N}_L(r,\frac{1}{G-1})+S(r).$$

From Lemma 2 and Lemma 3, we get

$$T(r,G)=(k+1)\overline{N}(r,f)\leq 6\overline{N}(r,f)+S(r). \qquad (3.7)$$

Since $k\geq 6$, we get from (3.1) that $T(r,f)=S(r,f)$, which is impossible. Hence, $H\equiv 0$. By integration two times, we have

$$\frac{1}{G-1}=\frac{A}{F-1}+B, \qquad (3.8)$$

where $A\neq 0$ and B are constants. We rewrite (3.8) in the following forms

$$F=\frac{(B-A)G+(A-B-1)}{BG-(B+1)},$$

$$G=\frac{(B+1)F+(A-B-1)}{BF+(A-B)}.$$

We distinguish the following three cases.

Case 1. If $B \neq 0, -1$, then

$$\overline{N}(r, \frac{1}{G - \frac{B+1}{B}}) = \overline{N}(r, F).$$

By the second fundamental theorem, Lemma 1 and the definitions of F and G, we have

$$\begin{aligned} T(r,G) &< \overline{N}(r,G) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{G - \frac{B+1}{B}}) + S(r) \\ &< 2\overline{N}(r,f) + \overline{N}(r, \frac{1}{g}) + S(r) \\ &< 2\overline{N}(r,f) + S(r). \end{aligned}$$

From the assumption and (3.1), this is impossible.

Case 2. If $B = -1$, then

$$G = \frac{A}{-F + A + 1}, \qquad F = \frac{(A+1)G - A}{G}.$$

If $A \neq -1$, then

$$\overline{N}(r, \frac{1}{G - \frac{A}{A+1}}) = \overline{N}(r, \frac{1}{F}).$$

By the same reasoning as in Case 1, we get a contradiction. Thus $A = -1$, and so $FG \equiv 1$, $fg = b^2$. We obtain

$$N(r,f) + N(r, \frac{1}{f}) = S(r).$$

It follows that

$$\begin{aligned} 2T(r, \frac{f}{b}) &= T(r, \frac{f^2}{b^2}) = T(r, \frac{b^2}{f^2}) + O(1) \\ &= T(r, \frac{g}{f}) + O(1) = T(r, a_1 + a_2 \frac{f^{(k)}}{f}) + O(1) \\ &= S(r,f). \end{aligned}$$

This is impossible.

Case 3. If $B = 0$, by the similar discussion as the Case 2, if $A \neq 1$, we get a contradiction. Therefore $A = 1$, and so $f \equiv g$. The proof of Theorem 1 is thus completed.

4. Proof of Theorem 2

From the proof of Theorem 1, if $H \not\equiv 0$, we obtain from (3.7) that

$$T(r,f) \leq 6\overline{N}(r,f) + S(r,f).$$

This contradicts the assumption that $\Theta(\infty, f) > \frac{5}{6}$. Hence $H \equiv 0$. By the same reasoning as in the proof of Theorem 1, we have $f \equiv g$.

Question 2. Is it true that $f \equiv g$ if $1 < k \leq 5$?

5. Proof of Theorem 3

If $f \not\equiv L$, then $\frac{L}{f} \not\equiv 1$. Let z_0 be the common zero of $f - b$ and $L - b$, not a zero or a pole of b, then $\frac{L(z_0)}{f(z_0)} = 1$. Since f and $L(f)$ share b IM, from the lemma of logarithmic derivatives, we get

$$\begin{aligned}\overline{N}(r, \frac{1}{L-b}) &\leq \overline{N}(r, \frac{1}{\frac{L}{f}-1}) + S(r,f) \\ &\leq T(r, \frac{L}{f}) \leq N(r, \frac{L}{f}) + S(r,f) \\ &\leq N(r, \frac{1}{f}) + k\overline{N}(r,f) + S(r,f).\end{aligned}$$

From the second fundamental theorem,

$$\begin{aligned}T(r,L) &\leq \overline{N}(r,L) + \overline{N}(r,\frac{1}{L}) + \overline{N}(r,\frac{1}{L-b}) + S(r,f) \\ &\leq \overline{N}(r,f) + k\overline{N}(r,f) + S(r,f) \\ &\leq N(r,f) + k\overline{N}(r,f) + S(r,f) \\ &\leq T(r,L) + S(r,f),\end{aligned}$$

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{f-b}) + S(r,f) \\ &\leq \overline{N}(r,f) + k\overline{N}(r,f) + S(r,f)\end{aligned}$$

$$= (k+1)\overline{N}(r,f) + S(r,f). \tag{5.1}$$

So we get

$$\overline{N}(r, \frac{1}{L-b}) = k\overline{N}(r,f) + S(r,f), \tag{5.2}$$

and

$$N(r,f) = \overline{N}(r,f) + S(r,f). \tag{5.3}$$

We know that the poles of f "almost all" are simple. Let

$$\alpha = \frac{L'}{L} - (k+1)\frac{f'}{f}.$$

By the lemma of logarithmic derivatives, we get $m(r,\alpha) = S(r,f)$. Since the poles of f "almost all" are simple. By calculation, we know that the simple pole are not the pole of α. Therefore $N(r,\alpha) = S(r,f)$. Hence we have $T(r,\alpha) = S(r,f)$. We distinguish the following two cases.

Case 1. If f is a rational function, since $T(r,\alpha) = S(r,f)$, then α must be a constant, and $L = f^{k+1}Ce^{\alpha z}$, where C is a nonzero constant. If $\alpha \neq 0$, then L is not a rational function, which is a contradiction. Hence $\alpha = 0$, and thus $L = Cf^{k+1}$. Since $T(r,b) = S(r,f)$, $b \neq 0$, f and $L(f)$ share b IM , the equation $C\omega^{k+1} - b = 0$ have $k+1$ different roots. We select a root ω_0 of this equation such that $\omega_0 \neq b$, and f assumes the value ω_0 which is possible. Since $k+1 \geq 3$, and f is a rational function. If z_0 is a zero of $f - \omega_0$, then $Cf^{k+1}(z_0) = b$. Since f and $L(f)$ share b IM, we have $f(z_0) = b$, therefore $\omega_0 = b$, which is a contradiction.

Case 2. If f be a transcendental meromorphic function, then by Lemma 4, we know that $\overline{N}(r,f) = S(r,f)$. Hence from (4.1) and (4.3), we get $T(r,f) = S(r,f)$. This is impossible.
Hence $f = L$, the proof of Theorem 3 is thus proved.

Question 3. If we replace L by a more general differential polynomial of f, is it true that $f = L$?

References

[1] F. Weissenborn, *Meromorphe Funktionen, die mit einer ihrer Ableitungen Werte teilen*, Complex Vaiable Theorey Appl., **7**(1986), 33-43.

[2] F. Weissenborn, *On the zeros of linear differential polynoimal of meromorphic functions*, Complex Vaiable Theorey Appl., **12**(1989), 77-81.

[3] L. Z. Yang, *Entire function that share finite values with their derivates*, Bull. Austral. Math. Soc., **41**(1990), 337-342.

[4] H. X. Yi and C. C. Yang, Uniqueness Theory of Meromorphic Functions, Science Press, Beijing, 1995. (Chinese)

[5] H. X. Yi and C. C. Yang, Uniqueness Theory of Meromorphic Functions, Kluwer Academic Publishers, 2003.

[6] H. X. Yi, *Meromorphic function that share one or two value II*, Kodai Math. J., **22**(1999), 264-272.

[7] W. H. Yao and P. Li, *Meromorphic functions sharing two small functions with its derivative*, J. Math. Anal. Appl., **322**(2006), 133-145.

[8] Q. C. Zhang, *The uniqueness of meromorohic function with their derivatives*, Kodai Math. J., **21**(1998), 179-184.