

On the Relationship between Zero-sums and Zero-divisors of Semirings

ANDREW J. HETZEL*

Department of Mathematics, Tennessee Technological University, Box 5054, Cookeville, Tennessee 38505 USA
e-mail: ahetzel@tntech.edu

REBECA V. LEWIS LUFU

Department of Mathematics, Tennessee Technological University, Box 5054, Cookeville, Tennessee 38505 USA
e-mail: rlewis@tntech.edu

ABSTRACT. In this article, we generalize a well-known result of Hebisch and Weinert that states that a finite semidomain is either zerosumfree or a ring. Specifically, we show that the class of commutative semirings S such that S has nonzero characteristic and every zero-divisor of S is nilpotent can be partitioned into zerosumfree semirings and rings. In addition, we demonstrate that if S is a finite commutative semiring such that the set of zero-divisors of S forms a subtractive ideal of S , then either every zero-sum of S is nilpotent or S must be a ring. An example is given to establish the existence of semirings in this latter category with both nontrivial zero-sums and zero-divisors that are not nilpotent.

1. Introduction

This article is devoted to an exploration of how ideal-theoretic considerations in commutative semirings, particularly finite commutative semirings, impact the multiplicative behavior of those elements of the semiring that have additive inverses in the semiring. The general question as to the algebraic nature of these so-called “zero-sums” of a semiring is one of the most central in the theory of semirings. We are especially motivated by a result of Hebisch and Weinert [9, Corollary 3.4, p. 81] that establishes that the class of finite semidomains can be partitioned by the antipodal properties of being zerosumfree (that is, only the zero element is a zero-sum of the semiring) and being a ring (where, by definition, every element is a zero-sum of the semiring). Of course, there exist infinite semidomains with nontrivial zero-sums that are not rings; for example, the polynomial semiring $X\mathbb{Z}[X] + \mathbb{N}$, where

* Corresponding author.

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\mathbb{Z} is the ring of integers and \mathbb{N} is the semiring of nonnegative integers. However, we demonstrate here that the Hebisch-Weinert result can be generalized, even to certain infinite semirings, by means of certain ideal-theoretic properties connected with the set of zero-divisors of a semiring.

In Section 2, we focus on a class of semiring, properly containing the class of semidomains, that is representable by the property that every zero-divisor of the semiring is nilpotent. We begin by presenting several well-known fundamental facts concerning prime ideals of a semiring in Theorem 2.1 and Corollary 2.3 that prove useful throughout this article. Proposition 2.4 then shows that the semirings being considered in this section can, in fact, be conveniently characterized by the property that the zero ideal is a primary ideal of the semiring. Next, Proposition 2.6 along with Corollary 2.7 underscore the influence that the (principal) ideal structure of a semiring can have on the behavior of the zero-sums of the semiring. After presenting some basic information in Proposition 2.8 on the general relationships that exist amongst the sets of zero-divisors, non-(multiplicatively) cancellative elements, (multiplicatively) cancellative elements, and units of a semiring, we develop additional information along these lines in the context of certain semirings of nonzero characteristic in Theorem 2.9 and Corollary 2.10. Moreover, Corollary 2.10 reveals that, in this context, equality of the set of zero-divisors of the semiring and the set of non-cancellative elements of the semiring characterizes when every element of the semiring is a zero-sum of the semiring, that is, when the semiring is a ring. Theorem 2.12 then answers a natural question suggested by Corollary 2.10 by showing that, in a finite semiring that is not a ring, any zero-sum of the semiring must, in fact, be more specialized than simply an arbitrary non-cancellative element of the semiring—it must be a zero-divisor of the semiring. With much heuristic evidence established then, we present the main theorem of this section in Theorem 2.13. Theorem 2.13 reveals that for semirings of nonzero characteristic, those that are members of the class of semiring being considered in this section are either zerosumfree or rings.

In Section 3, we shift our focus to another class of semiring, also properly containing the class of semidomains, that is representable by the property that the set of zero-divisors of the semiring is a subtractive ideal of the semiring. We are motivated in this regard by the recent success of studies of the notion of “primal ideal” for (commutative) rings in such articles as [6] and [7]. Here, we extend the usual definition of “primal ideal” for rings to create the (inequivalent) notions of “primal ideal” and “s-primal ideal” for semirings in Definition 3.1. In parallel then with the characterization (Proposition 2.4) of the relevant property of semirings in Section 2 by means of the zero ideal, we couch the relevant property of semirings in this section in terms of the property that the zero ideal of the semiring is an s-primal ideal of the semiring. We first demonstrate that s-primal ideals need not themselves be subtractive ideals in Example 3.2 by utilizing a novel extension of the notion of “idealization” for rings. We next provide some additional elementary facts in Propositions 3.3 and 3.4 concerning primal ideals for semirings. Proposition 3.5 and Corollary 3.6, in conjunction with Example 3.7, then reveal that, in fact, the class of semiring being considered in this section properly contains the class

of semiring considered in Section 2. However, Example 3.7 also shows that the main theorem (Theorem 2.13) of Section 2 cannot be generalized to semirings of nonzero characteristic for which the zero ideal is an s-primal ideal of the semiring. Nevertheless, the main theorem, Theorem 3.9, of this section reveals that for finite semirings that are not rings and for which the zero ideal is an s-primal ideal of the semiring, the zero-sums of the semiring must be found amongst the nilpotent elements of the semiring. Moreover, we are able to provide in Corollary 3.10 an extension of the Hebisch-Weinert result itself based upon the notion of “s-primal ideal”. We conclude this article then with a semiring-theoretic analogue of the “Prime Avoidance Lemma” in Lemma 3.11 and a theorem (Theorem 3.12) that gives a context for when the theory of finite semirings for which the zero ideal is an s-primal ideal of the semiring reduces to the motivating theory of finite semidomains.

Throughout this article, we adopt the standard [8] definition for “semiring”. Specifically, $(S, +, \cdot)$ (or simply S if the operations of $+$ and \cdot are understood) is called a *semiring* if $(S, +)$ is an (additive) abelian monoid with identity 0, (S, \cdot) is a (multiplicative) monoid with identity 1, multiplication distributes over addition from both the left and the right, and $0a = 0 = a0$ for all $a \in S$. Moreover, we will assume that all semirings S in this article are commutative (that is, (S, \cdot) is an abelian monoid) with $1 \neq 0$. In addition, we note that a subsemiring T of the semiring S , by definition [8], contains both the 0 and the 1 of S .

We now give some definitions that are used frequently throughout this article. Let S be a commutative semiring. A nonempty subset I of S is called an *ideal* of S if $a + b \in I$ and $sa \in I$ whenever $a, b \in I$ and $s \in S$. The ideal I of S is called a *prime ideal* of S if I is a proper ideal of S such that either $a \in I$ or $b \in I$ whenever $a, b \in S$ such that $ab \in I$. The *radical* of the ideal I is given by $\text{rad}(I) = \{a \in S \mid \text{there exists a positive integer } n \text{ such that } a^n \in I\}$. The ideal I of S is called *subtractive* if $a, b \in S$ such that both $a + b \in I$ and $b \in I$ imply that $a \in I$. Given a nonempty subset A of S , the *ideal generated by* A is $(A) = \{s_1a_1 + s_2a_2 + \cdots + s_na_n \mid s_i \in S, a_i \in A\}$. In particular, for $a \in S$, the set $(a) = aS = \{as \mid s \in S\}$ is an ideal of S called the *principal ideal of* S *generated by* a .

Following [9], an element a in a semiring S is called a *zero-sum* of S if there exists an element $b \in S$ such that $a + b = 0$ (note here that, unlike in [9], we include 0 in the set of zero-sums of a semiring). In such a case, the element b is unique and is designated by $-a$. A semiring S is called *zerosumfree* if 0 is the only zero-sum of S . An element a of a commutative semiring S is called a *zero-divisor* of S if there exists $0 \neq b \in S$ such that $ab = 0$ (note here that we include 0 in the set of zero-divisors of a semiring). The collection of all zero-divisors of a commutative semiring S will be denoted by $\text{ZD}(S)$. Furthermore, the subset $\{a \in S \mid \text{there exists a positive integer } n \text{ such that } a^n = 0\}$ of $\text{ZD}(S)$ consisting of the nilpotent elements of S will be denoted by $\text{Nil}(S)$. A semiring with $1 \neq 0$ is called a *semidomain* if it is commutative and does not have any nonzero zero-divisors. That is, a semidomain is precisely a commutative entire semiring with $1 \neq 0$. It is clear that the commutative semiring S is a semidomain if and only if

$\{0\}$ is a prime ideal of S . Finally, throughout this article, by “cancellative element”, we shall always mean multiplicatively cancellative element. That is, an element a of a commutative semiring S will be called a *cancellative element* of S if for all $b, c \in S$, it is the case that $b = c$ whenever $ab = ac$.

Any unexplained terminology is standard, as in [8].

2. Semirings S for which $\text{Nil}(S) = \text{ZD}(S)$

One of the most natural generalizations of “semidomain” can be discovered by considering the class of (commutative) semirings S for which every zero-divisor of S is nilpotent. Rings with this property were valuable in the studies of “going-down rings” and “quasi-going-up rings” conducted in [3] and [10], respectively, and, more recently, in the study of “quasilocal going-down rings” in [4]. Moreover, there are numerous examples of semirings with this property; in particular, if p is a prime and n is a positive integer, each of $\mathbb{Z}/p^n\mathbb{Z}$ and $\mathcal{I}(\mathbb{Z}/p^n\mathbb{Z})$, the semiring of ideals of $\mathbb{Z}/p^n\mathbb{Z}$ under the usual addition and multiplication of ideals, is a semiring such that every zero-divisor of the semiring is a nilpotent element of the semiring. In fact, each member of the latter class of examples is a semiring that is not a ring and is a semidomain precisely when $n = 1$.

Since, for a semiring S , the set of nilpotent elements of S is precisely the radical of the zero ideal of S , we begin with a fundamental fact found in [8, Proposition 7.28] concerning the connection between the radical of an ideal of a commutative semiring and the prime ideals of the semiring.

Theorem 2.1(Krull’s Theorem). *Let S be a semiring. If I is an ideal of S , then $\text{rad}(I) = \bigcap \{P \mid P \text{ is a prime ideal of } S \text{ such that } P \supseteq I\}$. In particular, $\text{Nil}(S) = \bigcap \{P \mid P \text{ is a prime ideal of } S\}$.*

Theorem 2.1 highlights the fact that there exists an intimate relationship between the (prime) ideal structure of a semiring S and the set $\text{Nil}(S)$. (We remark that the set $\text{ZD}(S)$ also enjoys a close relationship with the prime ideal structure of S in that $\text{ZD}(S)$ is a set-theoretic union of prime ideals of S ; cf. [12, Theorem 2, p. 2] and [8, Proposition 7.12, p.87]. However, we shall not need this fact for our purposes in this article.) Moreover, the following generalization of “prime ideal” will prove useful as an alternative way of characterizing semirings S for which $\text{Nil}(S) = \text{ZD}(S)$.

Definition 2.2. As in [8], a proper ideal I of a semiring S is called a *primary ideal* of S if whenever $a, b \in S$ such that $ab \in I$ and $a \notin I$, it must be the case that there exists a positive integer n for which $b^n \in I$.

We thus have the following immediate corollary (cf. [8, Corollary 7.29]) of Theorem 2.1 in case the ideal I is a primary ideal of the semiring S .

Corollary 2.3. *Let I be a primary ideal of a semiring S . Then $\text{rad}(I)$ is the smallest prime ideal of S containing I . In particular, if I is a prime ideal of S , then $I = \text{rad}(I)$.*

We now take this opportunity to make a convenient piece of notation. Let I be an ideal of a semiring S . Put $\text{ZD}_S(I) = \{a \in S \mid \text{there exists } s \in S \setminus I \text{ such that } sa \in I\}$, the set of so-called “non-prime” elements of S to I . It is straightforward to see that $I \subseteq \text{rad}(I) \subseteq \text{ZD}_S(I)$. Proposition 2.4, which is well-known in the ring-theoretic context (cf. [2, p. 50]), provides characterizations in terms of the notions of “primary ideal” and “prime ideal” of when certain equalities amongst these algebraic structures occur.

Proposition 2.4. *Let I be an ideal of the semiring S .*

- (a) *I is a primary ideal of S if and only if $\text{rad}(I) = \text{ZD}_S(I)$. In particular, $\{0\}$ is a primary ideal of S if and only if $\text{Nil}(S) = \text{ZD}(S)$.*
- (b) *I is a prime ideal of S if and only if $I = \text{ZD}_S(I)$.*

Proof. (a) (\Rightarrow) It is sufficient to show that if I is a primary ideal of S , then $\text{ZD}_S(I) \subseteq \text{rad}(I)$. Suppose then that I is a primary ideal of S , and let $a \in \text{ZD}_S(I)$. Then there exists $b \in S \setminus I$ such that $ab \in I$. Since I is a primary ideal of S , there exists a positive integer n such that $a^n \in I$, whence $a \in \text{rad}(I)$, as desired.

(\Leftarrow) Suppose that $\text{rad}(I) = \text{ZD}_S(I)$, and let $a, b \in S$ such that $ab \in I$ but $a \notin I$. Then $b \in \text{ZD}_S(I)$, and so $b \in \text{rad}(I)$. Thus, there exists a positive integer n such that $b^n \in I$. Therefore, I must be a primary ideal of S .

(b) This equivalence follows by combining Corollary 2.3 with part (a) above. \square

We next briefly turn our attention in Propositions 2.5 and 2.6 and Corollary 2.7 below to collecting some elementary facts concerning the impact of certain intersections between the set of zero-sums of a semiring and several other special subsets of a semiring. Moreover, these results inaugurate a focus on the motivating idea of this article, namely, determining relationships that exist between the zero-sums and zero-divisors of a semiring S that provide sufficient conditions for S to be a ring.

Proposition 2.5. *Let S be a semiring. Then the following are equivalent:*

- (1) *S is a ring,*
- (2) *some unit of S is a zero-sum of S ,*
- (3) *the 1 of S is a zero-sum of S .*

Proof. Each of the implications (1) \Rightarrow (2), (1) \Rightarrow (3), and (3) \Rightarrow (2) are patent. Therefore, it suffices to show (2) \Rightarrow (1). Suppose (2). Let $a \in S$, and choose a unit u of S that is a zero-sum of S . Let $v \in S$ such that $uv = 1$. Then $(-u)va + a = ((-u)v + uv)a = (-u + u)va = 0$, whence a is a zero-sum of S . Therefore, S is a ring, to complete the proof. \square

Proposition 2.6. *Let S be a semiring that is not a ring. Let a be a zero-sum of S such that $-a \in aS$. Then a is a zero-divisor of S .*

Proof. Let S be a semiring that is not a ring, and let a be a zero-sum of S such that $-a \in aS$. Choose $s \in S$ such that $-a = as$. Then $a(s + 1) = 0$. However, since S is not a ring, $s + 1 \neq 0$ by Proposition 2.5. Therefore, a must be a zero-divisor of S . \square

Corollary 2.7. *Let S be a semidomain such that there exists a nonzero zero-sum a of S with the property that $-a \in aS$. Then S is a domain.*

For several of the ensuing results, we let $\text{Noncan}(S)$ denote the set of non-(multiplicatively) cancellative elements of the semiring S and $\text{Can}(S)$ denote the set of (multiplicatively) cancellative elements of the semiring S . We also denote the set of units of the semiring S by $\mathcal{U}(S)$. Proposition 2.8 gives some basic containment relationships amongst these sets along with the set $\text{ZD}(S)$ of zero-divisors of the semiring S .

Proposition 2.8. *Let S be a semiring.*

(a) $\text{ZD}(S) \subseteq \text{Noncan}(S)$ and $\mathcal{U}(S) \subseteq \text{Can}(S)$.

(b) If S is finite, then $\text{Can}(S) = \mathcal{U}(S)$.

Proof. (a) Let $a \in \text{ZD}(S)$. Then there exists $0 \neq b \in S$ such that $ab = 0 = a0$. Therefore, $a \in \text{Noncan}(S)$, as desired.

Let $u \in \mathcal{U}(S)$. Then there exists $v \in \mathcal{U}(S)$ such that $vu = 1$. Let $a, b \in S$ such that $ua = ub$. Then $a = vua = vub = b$. Therefore, $u \in \text{Can}(S)$, as desired.

(b) Let S be a finite semiring. Part (a) above establishes that $\mathcal{U}(S) \subseteq \text{Can}(S)$. For the reverse containment, let $a \in \text{Can}(S)$. Consider the function $\lambda: S \rightarrow S$ given by $s \mapsto as$. Since a is a cancellative element of S , λ is injective. Since S is finite, λ must then be surjective. Thus, there exists $b \in S$ such that $ab = 1$. Therefore, $a \in \mathcal{U}(S)$, as desired. \square

Theorem 2.9 along with Corollary 2.10 below provide for a (rather substantial) collection of semirings S in which the containment $\text{ZD}(S) \subseteq \text{Noncan}(S)$ given in part (a) of Proposition 2.8 cannot be replaced with an equality of sets. In particular, Corollary 2.10 reveals that, within this collection, information about the set of zero-divisors of a semiring once again furnishes information about the set of zero-sums of the semiring (see, in particular, Proposition 2.6 and Corollary 2.7 above). For each of these results, we make use of the notion of the “characteristic” of a semiring, introduced in [1]. Specifically, the semiring S has *characteristic* 0 if the characteristic (or basic) subsemiring $B(S) = \{n1 \mid n \in \mathbb{N}\}$ of S is isomorphic to \mathbb{N} and *characteristic* (n, i) if $B(S)$ is isomorphic to $B(n, i)$.

Theorem 2.9. *Let S be a semiring of characteristic (n, i) , where $i > 1$. Then $\text{Noncan}(S) \neq \text{ZD}(S)$.*

Proof. Put $k = n - i + 1$ and $a = 1 + 1 + \cdots + 1$ (k times). Let $B(S)$ be the characteristic (or basic) subsemiring of S . Then $a \in B(S) \subseteq S$. Since $2 \leq k \leq n - 1$, it follows that a is a nonunit of $B(S)$. By Proposition 2.8(b), a is a non-cancellative element of $B(S)$, whence a is a non-cancellative element of S .

Now, suppose that there exists a nonzero element $b \in S$ such that $ab = 0$. Then $kb = 0$, and so, in particular, b is a zero-sum of S . However, since the characteristic of S is (n, i) , we have that $nb = ib$. Thus, $(k - 1)b = (n - i)b = 0$, and so $b = b + (k - 1)b = kb = 0$, a contradiction. Therefore, $a \in \text{Noncan}(S) \setminus \text{ZD}(S)$. The result follows. \square

Corollary 2.10. *Let S be a semiring of characteristic (n, i) , where $i \neq 1$. Then S is a ring if and only if $\text{Noncan}(S) = \text{ZD}(S)$.*

Proof. The “if” assertion follows immediately from Theorem 2.9 and the fact that $i = 0$ precisely when S is a ring. Conversely, since it is well-known that non-cancellative elements of a ring are zero-divisors of the ring, the “only if” assertion follows from Proposition 2.8(a). \square

Remark 2.11. (a) Theorem 2.9 is best possible, in the sense that “characteristic (n, i) , where $i > 1$ ” cannot be replaced with either “characteristic $(n, 1)$ ” or “characteristic 0”. For example, $S = \mathcal{I}(\mathbb{Z}/p^k\mathbb{Z})$, the semiring of ideals of $\mathbb{Z}/p^k\mathbb{Z}$, where p is a prime, under the usual addition and multiplication of ideals, has characteristic $(2, 1)$, but $\text{Noncan}(S) = S \setminus \{\mathbb{Z}/p^k\mathbb{Z}\} = \text{ZD}(S)$.

Now, suppose $n > 2$. Then the direct product $T = \mathbb{Z}/(n-1)\mathbb{Z} \times B(2, 1)$ has characteristic $(n, 1)$, but $\text{Noncan}(T) = \{(a, 0) \mid a \in \mathbb{Z}/(n-1)\mathbb{Z}\} \cup \{(b, 1) \mid b \in \text{ZD}(\mathbb{Z}/(n-1)\mathbb{Z})\} = \text{ZD}(T)$. (In general, it is straightforward to verify that the characteristic of $B(m, j) \times B(n, i)$ is $([m-j, n-i] + \max(i, j), \max(i, j))$, where $[m-j, n-i]$ denotes the least common multiple of $m-j$ and $n-i$.) For the “characteristic 0” case, let V be a half domain (that is, V is a subsemiring of an integral domain) of characteristic 0 that is not itself a domain (for example, take $V = \mathbb{N}$). Then $\text{Noncan}(V) = \{0\} = \text{ZD}(V)$.

(b) We note here that, in fact, any of the semirings $B(n, i)$, where $n > 2$ and $i > 0$, themselves contain non-(multiplicatively) cancellative elements that are not zero-divisors of the semiring. Since $i > 0$, it is clear that each such semiring is a semidomain, whence $\text{ZD}(B(n, i)) = \{0\}$ for each such $B(n, i)$. However, each of these $B(n, i)$ ’s must contain nonzero non-cancellative elements. For otherwise, by Proposition 2.8(b), a counterexample would be a finite semifield of order > 2 that is not a field, contradicting either [13, Theorem 5 (1), p. 333] or [1, Theorem 6, p. 576].

By Corollary 2.10, there is certainly no lack of finite semirings S for which $\text{Noncan}(S) \not\subseteq \text{ZD}(S)$. In light of Propositions 2.5 and 2.8(b), one might be led, a priori, to believe that there exist finite semirings S (necessarily not rings) with the property that there exist zero-sums in the set $\text{Noncan}(S) \setminus \text{ZD}(S)$. Theorem 2.12 below establishes that this is not the case.

Theorem 2.12. *Let S be a semiring of nonzero characteristic. Then either every zero-sum of S is a zero-divisor of S or S is a ring.*

Proof. Suppose that S is a semiring of nonzero characteristic, say (n, i) . Let a be a zero-sum of S . Since $a \in S$, it must be the case that $na = ia$. However, since a is a zero-sum of S , it follows that $(1 + 1 + \cdots + 1)a = (n - i)a = 0$, where there are $n - i$ 1’s in the summation. If this summation of 1’s is zero, then 1 is a zero-sum of S and, therefore, S is a ring by Proposition 2.5. If, instead, this summation of 1’s is nonzero, then a must be a zero-divisor of S . \square

We now provide the main theorem of this section in Theorem 2.13. Theorem

2.13 reveals that, for semirings S for which every zero-divisor of S is nilpotent, those which have nonzero characteristic must be of one of two extremes, namely, either every element of S is a zero-sum of S or only 0 is a zero-sum of S . This represents a substantial generalization of the result of Hebisch and Weinert [9, Corollary 3.4, p. 81] for finite semidomains. Moreover, we recover this result of Hebisch and Weinert in Corollary 2.14.

Theorem 2.13. *Let S be a semiring of nonzero characteristic. If $\{0\}$ is a primary ideal of S , then S is either zerosumfree or a ring.*

Proof. Suppose that S is a semiring of nonzero characteristic, say (n, i) , and suppose that $\{0\}$ is a primary ideal of S . Assume that S is not zerosumfree. Choose then a nonzero zero-sum $a \in S$. Since S has characteristic (n, i) , we have that $na = ia$, whence $(1 + 1 + \cdots + 1)a = (n - i)a = 0$, where there are $n - i$ 1's in the summation. Since a is nonzero and $\{0\}$ is a primary ideal of S , there exists a positive integer k such that $(1 + 1 + \cdots + 1)^k = 0$, and so $1 + 1 + \cdots + 1 = 0$, where there are $(n - i)^k$ 1's in the last summation. Thus, 1 is a zero-sum of S and, therefore, S is a ring by Proposition 2.5. \square

Corollary 2.14. *If S is a finite semiring such that $\{0\}$ is a primary ideal of S , then S is either zerosumfree or a ring. In particular, if S is a finite semidomain, then S is either zerosumfree or a ring.*

3. Semirings S for which $\text{ZD}(S)$ is a subtractive ideal of S

In this section, we consider another generalization of “semidomain” in the class of (commutative) semirings S for which the zero-divisors of S form a subtractive ideal of S . It turns out (Corollary 3.6) that such a class of semiring actually contains the class of semiring for which $\{0\}$ is a primary ideal of the semiring considered in the previous section. However, Example 3.7 reveals that the main theorem (Theorem 2.13) of the previous section cannot be generalized along these lines. Nevertheless, the main theorem, Theorem 3.9, of this section asserts positive information about the zero-sums in finite semirings S with the property that the zero-divisors of S form a subtractive ideal of S .

To parallel with the characterization (Proposition 2.4) of the relevant property of semirings in Section 2 using primary ideals, we make the following definitions (cf. [5]) of “primal ideal” and “s-primal ideal” for a (commutative) semiring.

Definition 3.1. Let S be a semiring, and let I be a proper ideal of S . If $\text{ZD}_S(I)$ is an ideal of S , then we call I a *primal ideal* of S . In such a case, the ideal $\text{ZD}_S(I)$ is referred to as the *adjoint ideal* of I . Moreover, if $\text{ZD}_S(I)$ is a subtractive ideal of S , then we call I an *s-primal ideal* of S .

Of course, since there exist non-subtractive prime ideals, it is clear that there exist primal ideals that are not s-primal ideals. Nevertheless, there is a key similarity between the theory of s-primal ideals for semirings and its counterpart for

rings. In particular, for a(n s-)primal ideal I of a ring R , it is easy to see that the set $\text{ZD}_R(I)$ corresponds to the set of zero-divisors in the factor ring R/I . For an s-primal ideal I of a semiring S , it turns out that the set $\text{ZD}_S(I)$ also corresponds to the set of zero-divisors in the Bourne factor semiring S/I since $S/I = S/(0/I)$, where $0/I$ is a proper subtractive ideal of S contained in $\text{ZD}_S(I)$. However, Example 3.2 below shows that I itself need not be a subtractive ideal of the semiring S even if I is an s-primal ideal of S .

Example 3.2. There exists a semiring S and an ideal I of S such that I is an s-primal ideal of S , but I is not itself a subtractive ideal of S . Moreover, it may be arranged that S is finite. Let T be a semiring and E a T -semimodule. Then one can create a semiring extension of T appropriately dubbed (from its ring-theoretic counterpart) an “idealization” of T and denoted by $T(+)$ E (for background on idealizations in the context of rings and modules, see [11], [14]). The semiring $T(+)$ E has the additive structure of $T \oplus E$ and has multiplication given by $(t_1, e_1)(t_2, e_2) = (t_1t_2, t_1e_2 + t_2e_1)$ (compare with a Dorroh extension of T).

Now, let T be a semidomain, let U be a subsemiring of T , and suppose there exists a nonzero, nonidentity (multiplicatively and additively) idempotent element a of T such that $a + 1 = a$ and a is “prime” to U , that is, for each $0 \neq u \in U$, $ut = a$ if and only if $t = a$, where $t \in T$. We remark that each of these conditions can be met by taking T to be an idempotent semidomain for which there exists a nonzero, nonidentity element a of T such that $a + 1 = a$ and taking U to be the characteristic (or basic) subsemiring of T . Note that T is a U -semimodule under the multiplication of T . Put $S = U(+)$ T , and put $I = \{(0, 0), (0, a)\}$. The fact that I is an ideal of S follows from the facts that a is idempotent and a is prime to U . Moreover, I is not a subtractive ideal of S , as $(0, a) + (0, 1) = (0, a)$ and $(0, 1) \notin I$. However, observe that every element of S of the form $(0, t)$, where $t \in T$, is nilpotent with index of nilpotency 2. Thus, $\{(0, t) \mid t \in T\} \subseteq \text{ZD}_S(I)$. Now, suppose that $(u, t) \in \text{ZD}_S(I)$ such that $u \neq 0$. Then there exists $(v, w) \in S \setminus I$ such that $(vu, vt + uw) = (v, w)(u, t) \in I$. Thus, $vu = 0$ and, since T is a semidomain and $u \neq 0$, we have that $v = 0$. Hence, either $uw = 0$ or $uw = a$. In the former case, it follows that $w = 0$, a contradiction to the fact that $(v, w) \notin I$. In the latter case, it follows that $w = a$, again a contradiction to the fact that $(v, w) \notin I$. We conclude then that $\text{ZD}_S(I) = \{(0, t) \mid t \in T\}$. Moreover, it is straightforward then to verify that $\text{ZD}_S(I)$ is a subtractive ideal of S , whence I is an s-primal ideal of S .

We next provide some basic facts in Propositions 3.3 and 3.4 below concerning primal ideals for semirings, the statements of which are identical, *mutatis mutandis*, to the corresponding statements for primal ideals of rings (see [5]).

Proposition 3.3. *Let S be a semiring. If I is a primary ideal of S , then I is a primal ideal of S .*

Proof. The result follows by combining Corollary 2.3 and Proposition 2.4(a). \square

Proposition 3.4. *Let S be a semiring. If I is a primal ideal of S , then $\text{ZD}_S(I)$ is*

a prime ideal of S .

Proof. Since $\text{ZD}_S(I)$ is an ideal of S , by definition, it is sufficient then to show that $\text{ZD}_S(I)$ is a proper ideal of S and whenever $a, b \in S$ such that $ab \in \text{ZD}_S(I)$ and $a \notin \text{ZD}_S(I)$, it must be the case that $b \in \text{ZD}_S(I)$. Suppose that $1 \in \text{ZD}_S(I)$. Then, by definition, there exists $s \in S \setminus I$ such that $s = s \cdot 1 \in I$, a contradiction. Thus, $\text{ZD}_S(I)$ is a proper ideal of S . Now, let $a, b \in S$ such that $ab \in \text{ZD}_S(I)$ and $a \notin \text{ZD}_S(I)$. Then there exists $s \in S \setminus I$ such that $(ab)s \in I$. However, since $a \notin \text{ZD}_S(I)$, it must be the case that $bs \in I$, whence $b \in \text{ZD}_S(I)$, as desired. \square

Thanks to Proposition 3.3, it is evident that if $\{0\}$ is a primary ideal of a semiring S , then $\text{ZD}(S)$ is an ideal of S . However, Proposition 3.5 asserts that, in this context, more can be said about the ideal $\text{ZD}(S)$. In particular, Corollary 3.6 formalizes the connection between the class of semiring considered in Section 2 and the class of semiring being considered in this section.

Proposition 3.5. *Let S be a semiring. If I is a subtractive primary ideal of S , then $\text{rad}(I) = \text{ZD}_S(I)$ is a subtractive (prime) ideal of S .*

Proof. Suppose that I is a subtractive primary ideal of S . The fact that $\text{rad}(I) = \text{ZD}_S(I)$ follows from Proposition 2.4(a). We show then that $\text{rad}(I)$ is a subtractive ideal of S . Let $a, b \in S$ such that $a + b \in \text{rad}(I)$ and $b \in \text{rad}(I)$. Let n be the smallest positive integer such that $b^n \in I$ and let m be a positive integer such that $(a + b)^m \in I$. By the Binomial Theorem, $a^m b^{n-1} + ma^{m-1} b^n + \dots + b^{m+n-1} = b^{n-1}(a + b)^m \in I$. As $b^n \in I$ and I is a subtractive ideal, we then have that $a^m b^{n-1} \in I$. By assumption, $b^{n-1} \notin I$. Thus, since I is a primary ideal of S , there exists a positive integer k such that $a^{mk} = (a^m)^k \in I$, whence $a \in \text{rad}(I)$. Therefore, $\text{rad}(I)$ is a subtractive ideal. The fact that $\text{rad}(I)$ is also a prime ideal follows from Corollary 2.3. \square

Corollary 3.6. *Let S be a semiring. If I is a subtractive primary ideal of S , then I is an s -primal ideal of S . In particular, if $\{0\}$ is a primary ideal of S , then $\{0\}$ is an s -primal ideal of S , that is, if $\text{Nil}(S) = \text{ZD}(S)$, then $\text{ZD}(S)$ is a subtractive ideal of S .*

In light of Corollary 3.6, Example 3.7 below demonstrates that the conditions “ $\text{Nil}(S) = \text{ZD}(S)$ ” and “ $\text{ZD}(S)$ is a subtractive ideal of S ” for a semiring S are logically inequivalent. As in [8], we call an ideal I of the semiring S a *strong ideal* of S if whenever $a, b \in S$ such that $a + b \in I$, then both $a \in I$ and $b \in I$. In addition, a semiring S is called *simple* if $a + 1 = 1$ for all $a \in S$, and a semiring S is called *idempotent* if it is both additively and multiplicatively idempotent, that is, if $a + a = a = a^2$ for all $a \in S$.

Example 3.7. There exists a finite semiring S such that $\{0\}$ is an s -primal ideal of S , but $\{0\}$ is not a primary ideal of S . Moreover, it may be arranged that either $\text{ZD}(S)$ is a strong ideal of S or there are nontrivial zero-sums of S . Let T be a finite semidomain such that there exists a nonzero principal subtractive prime

ideal aT of T (for example, take T to be either the idempotent, simple semidomain $\{0, 1, a\}$ or the semidomain $B(3, 1)$ with $a = 2$). Let X be an indeterminate over T . Since $T[X] \cong \oplus_{\mathbb{N}} T$ as additive monoids, it is straightforward to verify that the ideal $I = (X^2, aX)$ is a subtractive ideal of $T[X]$. Put $S = T[X]/I$, the Bourne factor semiring of $T[X]$ by I . Observe that $\text{Nil}(S) = \{tX/I \mid t \in T\}$ and $\text{ZD}(S) = \{(t_1X + t_0)/I \mid t_1 \in T, t_0 \in aT\}$. Thus, $\text{Nil}(S) \subsetneq \text{ZD}(S)$, and so, by Proposition 2.4(a), $\{0\}$ is not a primary ideal of S . However, since aT is a subtractive ideal of T , it follows that $\text{ZD}(S)$ is a subtractive ideal of S . Thus, $\{0\}$ is an s-primal ideal of S . Now, since T must be zerosumfree (see [9, Corollary 3.4, p. 81]), it follows that each of $\{0\}$ and $\text{Nil}(S)$ is automatically a strong ideal of S . Moreover, by taking aT to be a strong ideal of T , we can arrange that $\text{ZD}(S)$ is also strong ideal of S . If instead we take a to be a sum of 1's (which necessarily means that aT is not a strong ideal of T), then we can arrange that there are nontrivial zero-sums of S ; in particular, $X/I \in \text{Nil}(S)$ would be a nontrivial zero-sum of S .

Remark 3.8. (a) It is worth noting that the finite semirings S developed in Example 3.7 are necessarily not rings. That is to say, if R is a finite ring, then $\{0\}$ is a(n s-) primal ideal of R precisely when $\{0\}$ is a primary ideal of R . For let R be a finite ring, and let $\{0\}$ be a primal ideal of R . Then the set of zero-divisors is a prime ideal of R (see Proposition 3.4 above). However, it is well-known that every prime ideal of a finite ring is maximal ideal of the ring (cf. [2, Proposition 8.1, p. 89]) and every element of a finite ring is either a zero-divisor of the ring or a unit of the ring (see Proposition 2.8(b) and Corollary 2.10 above). Thus, the set of zero-divisors of R must be the only prime ideal of R , whence, by Theorem 2.1, every zero-divisor of R is nilpotent. By Proposition 2.4(a), this means that $\{0\}$ must be a primary ideal of R . Therefore, unlike in the theory of finite rings, each of the studies of primal ideals and s-primal ideals are significant generalizations of the study of primary ideals in the more general context of finite semirings.

(b) As illustrated in Example 3.7, the fact that there exist finite semirings S for which $\{0\}$ is an s-primal ideal of S but for which there exist nontrivial zero-sums of S shows that both Theorem 2.13 and Corollary 2.14 are best possible, in the sense that “ $\{0\}$ is a primary ideal of S ” cannot be replaced in either result with “ $\{0\}$ is an s-primal ideal of S ”.

We now provide the main theorem, Theorem 3.9, of this section. With respect to Theorem 2.12, Theorem 3.9 shows that if one further assumes that the set of zero-divisors of the finite semiring S forms a subtractive ideal of S , then no zero-sums of S may be found outside the set of nilpotent elements of S without S being a ring. (Hence, the nontrivial zero-sum X/I of the semiring S in Example 3.7 must necessarily be nilpotent.)

Theorem 3.9. *Let S be a finite semiring such that $\{0\}$ is an s-primal ideal of S . Then either every zero-sum of S is nilpotent or S is a ring.*

Proof. Let a be a zero-sum of S , and suppose that S is not a ring. Then, by Theorem 2.12, a must be a zero-divisor of S , whence $-a$ must also be a zero-divisor of S . Now,

note that the set $\{a^n \mid n \geq 1\} \subseteq S$ is finite as S is finite. Thus, there exist positive integers $m > n$ such that $a^m = a^n$. We then have that $a^n(1 + (-a)(a^{m-n-1})) = a^n + (-a)(a^{m-1}) = a^m + (-a)(a^{m-1}) = a^{m-1}(a + (-a)) = 0$. If $a^n \neq 0$, then $1 + (-a)(a^{m-n-1})$ is a zero-divisor of S . However, since $\{0\}$ is an s -primal ideal of S , it follows that 1 is a zero-divisor of S , a contradiction. Therefore, $a^n = 0$, and so a is nilpotent, as desired. \square

Corollary 3.10. *Let S be a reduced finite semiring—that is, S is a finite semiring that has no nonzero nilpotent elements. Suppose that $\{0\}$ is an s -primal ideal of S . Then S is either zerosumfree or a ring. In particular, if S is a finite semidomain, then S is either zerosumfree or a ring.*

We conclude this paper with Theorem 3.12, which provides a context in which the study of reduced finite semirings S where $\{0\}$ is a(n s -)primal ideal of S is equivalent to the study of finite semidomains. We first provide a semiring-theoretic analogue to the Prime Avoidance Lemma in Lemma 3.11.

Lemma 3.11(Prime Avoidance Lemma for Semirings). *Let $P_1, P_2, P_3, \dots, P_n$ ($n \geq 2$) be subtractive ideals of a semiring S , with P_1 and P_2 not necessarily prime, but P_3, P_4, \dots, P_n prime. Let I be an ideal of S . If $I \subseteq \cup_{i=1}^n P_i$, then for some k it must be the case that $I \subseteq P_k$.*

Proof. Deny. Without loss of generality, assume that I is not contained in the union of any collection of $n - 1$ of the P_i 's. For each $i = 1, 2, \dots, n$ choose then an element $a_i \in I \setminus (P_1 \cup P_2 \cup \dots \cup P_{i-1} \cup P_{i+1} \cup \dots \cup P_n)$. By hypothesis, $a_i \in P_i$ for each i . First, assume $n = 2$, with $I \not\subseteq P_1$ and $I \not\subseteq P_2$. Then $a_1 \in P_1$ and $a_2 \notin P_1$, whence $a_1 + a_2 \notin P_1$, as P_1 is a subtractive ideal of S . Similarly, $a_1 + a_2 \notin P_2$, as P_2 is a subtractive ideal of S . Thus, $a_1 + a_2 \notin I$, contradicting the fact that I is an ideal of S containing both a_1 and a_2 .

Now, assume that $n > 2$. Observe that $a_1 a_2 \dots a_{n-1} \in P_1 \cap P_2 \cap \dots \cap P_{n-1}$, but $a_n \notin P_1 \cup P_2 \cup \dots \cup P_{n-1}$. Put $a = a_1 a_2 \dots a_{n-1} + a_n$. Since each of P_1, P_2, \dots, P_{n-1} is a subtractive ideal of S , the element a cannot be element of $P_1 \cup P_2 \cup \dots \cup P_{n-1}$; for otherwise, a_n would be an element of some P_i , $i = 1, 2, \dots, n - 1$, and hence an element of their union. Now, for each $i = 1, 2, \dots, n - 1$ we have that $a_i \notin P_n$, and so $a_1 a_2 \dots a_{n-1} \notin P_n$ since P_n is prime. But, $a_n \in P_n$, and so $a \notin P_n$ since P_n is a subtractive ideal of S . Thus, $a \in I$, but $a \notin P_1 \cup P_2 \cup \dots \cup P_n$, a contradiction. \square

Theorem 3.12. *Let S be a reduced finite semiring—that is, S is a finite semiring that has no nonzero nilpotent elements—such that $\{0\}$ is a (n s -)primal ideal of S . Suppose that every prime ideal of S is a subtractive ideal of S . Then S is a semidomain.*

Proof. By Proposition 3.4, $\text{ZD}(S)$ must be a prime ideal of S . Moreover, since S is finite, there can be only finitely many other prime ideals of S , say P_1, P_2, \dots, P_n . By Theorem 2.1, we have that $\{0\} = \text{Nil}(S) = \text{ZD}(S) \cap P_1 \cap P_2 \cap \dots \cap P_n$. Now, let $0 \neq a \in \text{ZD}(S)$. Then there exists a $0 \neq b \in S$ such that $ab = 0 \in P_i$ for each $i = 1, 2, \dots, n$. Thus there exists a j such that $b \notin P_j$; for otherwise, $b \in \text{ZD}(S) \cap P_1 \cap P_2 \cap \dots \cap P_n = \{0\}$, a contradiction. However, since $ab \in P_j$ and P_j is a prime ideal of S , we have that $a \in P_j$. Therefore, $\text{ZD}(S) \subseteq P_1 \cup P_2 \cup \dots \cup P_n$. By Lemma 3.11, there exists a k such that

$ZD(S) \subseteq P_k$. Thus, $\{0\} = ZD(S) \cap P_1 \cap P_2 \cap \cdots \cap P_{k-1} \cap P_{k+1} \cap \cdots \cap P_n$. Continuing in this fashion, we have that $ZD(S) \subseteq P_i$ for each $i = 1, 2, \dots, n$. Therefore, $ZD(S) = ZD(S) \cap P_1 \cap P_2 \cap \cdots \cap P_n = \{0\}$, and so S is a semidomain. \square

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