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Some Properties of $(\mathcal{Y})$ Class Operators

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Abstract. In this paper we study some spectral properties of the class $(\mathcal{Y})$ operators and we will investigate on the relation between this class and other usual classes of operators.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded acting $\mathcal{H}$. We say that a bounded linear operator is in the class $\mathcal{Y}_\alpha$ for certain $\alpha \geq 1$ if there exists a positive number k_α such that

$$|TT^* - T^*T|^\alpha \leq k_\alpha^2 (T - \lambda I)^*(T - \lambda I), \ \forall \lambda \in \mathbb{C}.$$

It is shown [7] $\mathcal{Y}_\alpha \subseteq \mathcal{Y}_\beta$ for all α, β such that $1 \leq \alpha \leq \beta$, where $(\mathcal{Y}) := \cup_{\alpha \geq 1} \mathcal{Y}_\alpha$.

It is clear that the class $(\mathcal{Y})$ contains the class of normal operators. For any operators $A \in \mathcal{B}(\mathcal{H})$ set, as usual, $|A| = (A^*A)^{\frac{1}{2}}$ and $[A^*, A] = A^*A - AA^* = |A|^2 - |A^*|^2$ (the self commutant of A), and consider the following standard definitions: A is hyponormal if $|A^*|^2 \leq |A|^2$ (i.e., if $[A^*, A]$ is nonnegative or, equivalently, if $\| A^*x \| \leq \| Ax \|$ for every $x \in \mathcal{H}$), normal if $A^*A = AA^*$, subnormal if it admits a normal extension, quasinormal if A^*A commutes with AA^*, and m-hyponormal if there exists a constant $m \geq 1$, such that

$$m(A - \lambda I)^*(A - \lambda I) - (A - \lambda I)(A - \lambda I)^* \geq 0, \ \forall \lambda \in \mathbb{C}.$$

Let $(\mathrm{N}), (\mathrm{SN}), (\mathrm{QN}), (\mathrm{H})$, and $m - \mathrm{H}$ denote the classes constituting of normal, subnormal, quasinormal, hyponormal and, $m-$hyponormal operators respectively.

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Then

$$(\text{N}) \subset (\text{SN}) \subset (\text{QN}) \subset (\text{H}) \subset m-\text{H}.$$

In the following we will denote the spectrum, the point spectrum, the approximate reduced spectrum and, the approximate spectrum by $\sigma(A), \sigma_p(A), \sigma_{ar}(A)$, and $\sigma_a(A)$ respectively. In this paper we will give some spectral properties of the $(\mathcal{Y})$ class operators and we will investigate on the relation between this class and other usual classes of operators.

Let $\mathcal{A}$ denote a complex Banach Algebra with identity e. A state on $\mathcal{A}$ is a functional $f \in \mathcal{A}^*$ such that $f(e) = 1 = \| f \|$. For $x \in \mathcal{A}$ let

$$W_0(x) = \{f(x) : \text{f is a state on } \mathcal{A}\}$$

be the numerical range of x [9]. $W_0(x)$ is a compact set containing $co\sigma(x)$ (the convex hull of the spectrum of x) [1].

For the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$, if $A \in \mathcal{B}(\mathcal{H})$ then $W_0(A) = \overline{W(A)}$, where

$$W(A) = \{(Ah, h) : h \in \mathcal{H}, \| h \| = 1\}$$

is the special numerical range of $\mathcal{A}$. An element a is finite if $0 \in W_0(ax - xa)$ for each $x \in \mathcal{A}$; $\mathcal{F}(\mathcal{A})$ (or $\mathcal{F}$) denotes the set of all finite elements of $\mathcal{A}$. It is known that $\mathcal{F}$ contains every normal, hyponormal and dominant operators (see [2], [8]). In [4] the author initiated a more general class of finite operators called generalized pair of finite operators defined by

$$\mathcal{GF} = \{(A, B) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) : \| AX - XB \| \geq 1,\ \forall X \in \mathcal{B}(\mathcal{H})\}.$$

It is shown in [7] that

$$(\text{H}) \subset (\mathcal{Y}_1) \subset (m-\text{H}) \subset \cup_{\alpha \geq 2} \mathcal{Y}_\alpha.$$

In this paper we show that the class $(\mathcal{Y}) \subset \mathcal{F}$. For bounded linear operators $T : X \to Y$ and $S : Y \to Z$ on Banach spaces the condition

$$S^{-1}(0) \cap T(X) = 0 \tag{1.1}$$

is equivalent to the equality $(ST)^{-1} = T^{-1}(0)$; when $X = Y = Z$ and $T = S^n$ this is the familiar condition that the operator S "has ascent $\leq n$". Stronger conditions would replace the range $T(X)$ by its closure, either in the norm or in some weaker topology; weaker condition would ask that the intersection of $S^{-1}(0) \cap T(X)$ with some subspace of Y was in some sense nearly zero. Thus Kleinecke [3] showed that if $X = Y = Z = \mathcal{A}$ for a Banach algebra $\mathcal{A}$ and $S = T = \delta_a : x \mapsto ax - xa$ is an inner derivation on $\mathcal{A}$ then

$$S^{-1}(0) \cap T(X) \subseteq Q \tag{1.2}$$

where $Q = QN(\mathcal{A})$ is the quasinilpotents in $\mathcal{A}$. Weber [8] showed for the same S and T that when $\mathcal{A} = \mathcal{B}(\mathcal{H})$ for separable Hilbert space $\mathcal{H}$ then

$$S^{-1}(0) \cap cl_r T(X) \cap J \subseteq Q, \tag{1.3}$$

where cl_r represents the closure in $\mathcal{B}(\mathcal{H})$ with respect to the weak operator topology $\tau = \omega$ and $J = \mathcal{K}(\mathcal{H})$ is the compact operators. In [5] we consider more generally $S = \delta_{A,B} : U \mapsto AU - UB$ with either $T = S$ or $T = \delta_{A^*B^*}$, and find that for example (1.3) holds for $Q = \{0\}$ and $S = \delta_{A,B}$ and $T = \delta_{A^*B^*}$ when J is the finite rank operators and $\tau = \omega$ the weak operator topology, and also when J is the trace class and $\tau = \omega^*$ the ultra weak operator topology. This note continues this study.

2. Main results

Let us begin by the following Berberian techniques: Let $\mathcal{H}$ be a complex Hilbert space. Then there exists a Hilbert space $\mathcal{H}^0 \supset \mathcal{H}$, and an isometric *-isomorphism

$$\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\ (A \mapsto A^0)$$

preserving order, i.e., for all $A, B \in \mathcal{B}(\mathcal{H})$ and for all $\alpha, \beta \in \mathbb{C}$ we have:

1. $\varphi(A^*) = \varphi(A)^*$,
2. $\varphi(\alpha A + \beta B) = \alpha\varphi(A) + \beta\varphi(B)$,
3. $\varphi(I_{\mathcal{H}}) = I_{\mathcal{H}^0}$,
4. $\varphi(AB) = \varphi(A)\varphi(B)$,
5. $\|\varphi(A)\| = \|A\|$,
6. $\varphi(A) \leq \varphi(B)$ if $A \leq B$,
7. $\sigma(\varphi(A)) = \sigma(A),\ \sigma_a(A) = \sigma_a(\varphi(A)) = \sigma_p(\varphi(A))$,
8. if A is a positive operator, then $\varphi(A^\alpha) = |\varphi(A)|^\alpha$ for all $\alpha > 0$.

Lemma 2.1. *If $T \in (\mathcal{Y})$, then $\varphi(T) \in (\mathcal{Y})$.*

Proof. If $T \in (\mathcal{Y})$, then there exists $\alpha \geq 1$ and $k_\alpha > 0$ such that

$$|TT^* - T^*T|^\alpha \leq k_\alpha^2 (T - \lambda I)^*(T - \lambda I),\ \text{for all } \lambda \in \mathbb{C}.$$

It follows from the properties of the map φ that

$$\varphi(|TT^* - T^*T|^\alpha) \leq \varphi(k_\alpha^2 (T - \lambda I)^*(T - \lambda I)),\ \text{for all } \lambda \in \mathbb{C}.$$

By the condition (8) above we have

$$\varphi(|TT^* - T^*T|^\alpha) = |\varphi(|TT^* - T^*T|)|^\alpha$$

For all $\alpha > 0$. Therefore

$$|\varphi(T)\varphi(T^*) - \varphi(T^*)\varphi(T)|^{\alpha} \leq \varphi(k_{\alpha}^2(T-\lambda I)^*(T-\lambda I)), \text{ for all } \lambda \in \mathbb{C}.$$

Hence $\varphi(T) \in (\mathcal{Y})$. □

Now we are ready to give some spectral properties of the class $(\mathcal{Y})$.

Theorem 2.2. *Let $S \in (\mathcal{Y})$.*

(i) *If $\lambda \in \sigma_p(S)$, then $\overline{\lambda} \in \sigma_p(S^*)$, furthermore if $\lambda \neq \mu$, then M_λ (the proper subspace associated to λ) is orthogonal to M_μ.*

(ii) *If $\lambda \in \sigma_a(S)$, then $\overline{\lambda} \in \sigma_a(S^*)$.*

(iii) *If M is an invariant subspace for S and $S|_M$ is normal, then M reduces S.*

(iv) *If there exists a reducing subspace M, then $S|_M \in (\mathcal{Y})$.*

Proof. For (i) and (iii) see [7].

(ii) Let $\lambda \in \sigma_a(S)$ from the condition (7) above, we have

$$\sigma_a(S) = \sigma_a(\varphi(S)) = \sigma_p(S^*).$$

Therefore $\lambda \in \sigma_p(\varphi(S))$. By applying Lemma 2.1 and the above condition (i), we get

$$\overline{\lambda} \in \sigma_p(\varphi(S)^*) = \sigma_p(\varphi(S^*)).$$

Hence $\overline{\lambda} \in \sigma_p(\varphi(S^*)) = \sigma_a(\varphi(S))$.

(iv) Let $S \in (\mathcal{Y})$. Then there exists an integer $n \geq 1$ and $k_n > 0$ such that

$$\| \, |SS^* - S^*S|^{2^{n-1}} x \, \| \leq k_n^2 \, \| \, (S-\lambda I)x \, \| \text{ for all } x \in \mathcal{H}, \text{ for all } \lambda \in \mathbb{C}.$$

Since M reduces S, S can be written respect to the composition $\mathcal{H} = M \oplus M^{\perp}$ as follows:

$$S = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

By a simple calculation we get

$$(SS^* - S^*S)^2 = \begin{bmatrix} (AA^* - A^*A)^2 & 0 \\ 0 & (BB^* - B^*B)^2 \end{bmatrix}.$$

By the uniqueness of the square root, we obtain

$$|SS^* - S^*S| = \begin{bmatrix} |AA^* - A^*A| & 0 \\ 0 & |BB^* - B^*B| \end{bmatrix}.$$

Now by iteration to the order 2^n, it results that

$$|SS^* - S^*S|^{2^{n-1}} = \begin{bmatrix} |AA^* - A^*A|^{2^{n-1}} & 0 \\ 0 & |BB^* - B^*B|^{2^{n-1}} \end{bmatrix}.$$

Therefore for all $x \in M$, we have

$$\begin{aligned} \| \, |SS^* - S^*S|^{2^{n-1}} x \, \| &= \| \, |AA^* - A^*A|^{2^{n-1}} x \, \| \\ &\leq k_n^2 \, \| (S - \lambda I)x \| \\ &= \| (A - \lambda I)x \| \, . \end{aligned}$$

Hence $A \in (\mathcal{Y})_{2^n} \subset (\mathcal{Y})$. □

Now we will prove that the class $(\mathcal{Y})$ is included in the class of finite operator. For this we need the following lemma.

Lemma 2.3. *If $S \in (\mathcal{Y})$, then $\sigma_{ar}(S) \neq \emptyset$.*

Proof. If $S \in (\mathcal{Y})$, then there exists $\alpha \geq 1$ and $k_\alpha > 0$ such that

$$\| \, |SS^* - S^*S|^{\frac{\alpha}{2}} x \, \| \leq k_\alpha^2 \, \| (S - \lambda I)x \| \ \text{ for all } x \in \mathcal{H}, \text{ for all } \lambda \in \mathbb{C}. \tag{2.1}$$

Since

$$(S - \mu I)(S - \mu I)^* = SS^* - S^*S + (S - \mu I)^*(S - \mu I), \text{ for all } \mu \in \mathbb{C},$$

then

$$|\langle (SS^* - S^*S)x, x \rangle| \leq \| \, |SS^* - S^*S|^{\frac{1}{2}} x \, \|^2, \text{ for all } x \in \mathcal{H}.$$

Indeed, consider the polar decomposition of the operator $SS^* - S^*S = VD$, where $D = |SS^* - S^*S|$. Then V is a Hermitian partial isometry which commutes with D because $SS^* - S^*S$ is Hermitian. Hence, for any $x \in \mathcal{H}$ such that $\| x \| = 1$

$$\begin{aligned} \langle (SS^* - S^*S)x, x \rangle &\leq |\langle |SS^* - S^*S|^{\frac{1}{2}} x, |SS^* - S^*S|^{\frac{1}{2}} V^* x \rangle| \\ &\leq \| \, |SS^* - S^*S|^{\frac{1}{2}} x \, \| \| \, |SS^* - S^*S|^{\frac{1}{2}} V^* x \, \| \\ &= \| \, |SS^* - S^*S|^{\frac{1}{2}} x \, \| \| \, V^* |SS^* - S^*S|^{\frac{1}{2}} x \, \| \\ &\leq \| \, |SS^* - S^*S|^{\frac{1}{2}} x \, \|^2 \, . \end{aligned}$$

Consequently

$$\| (S - \mu I)^* x \|^2 \leq \| (S - \mu I)x \|^2 + \| \, |SS^* - S^*S|^{\frac{1}{2}} x \, \|^2, \forall x \in \mathcal{H}, \ \forall \mu \in \mathbb{C}. \tag{2.2}$$

Let $\lambda \in \sigma_a(S)$, then there exists a normed sequence $(x_n)_n \subset \mathcal{H}$, such that $\| (S - \lambda I)x_n \| \to 0$. Therefore for $\lambda = \mu, x_n = x$, for all n we get

$$\| \, |SS^* - S^*S|^{\frac{\alpha}{2}} x_n \, \| \leq k_\alpha^2 \, \| (S - \mu I)x_n \| \, . \tag{2.3}$$

By applying (2.1), (2.2) and (2.3) we deduce that

$$\| (S-\mu I)^* x_n \|^2 \leq (1+k_\alpha^2) \| (S-\mu I)x_n \|^2, \ \forall n.$$

Therefore $\| (S-\mu I)^* x_n \| \to 0$ and $\sigma_{ar}(S) \neq \emptyset$. □

Now we are ready to show that $(\mathcal{Y}) \subset \mathcal{F}$.

Theorem 2.4. *The class $(\mathcal{Y})$ of operators is included in the class of finite operators.*

Proof. It is shown in [2] that if $\sigma_{ar}(A) \neq \emptyset$, then A is finite. it suffices to apply the above lemma. □

Corollary 2.5. *If $S \in (\mathcal{Y})$, then $SS^* - S^*S$ is not invertible.*

Proof. It is well known that $\sigma_a(S)$ is non-empty. Now if $\mu \in \sigma_a(\varphi(S))$, then there exists a normed sequence $\{x_n\}_n \subset \mathcal{H}$ such that $\| (S-\mu I)x_n \| \to 0$. It follows from lemma 2.3 that $\| (S-\mu I)^* x_n \| \to 0$. Since

$$SS^* - S^*S = (S-\mu I)(S-\mu I)^* - (S-\mu I)^*(S-\mu I),$$

$(SS^* - S^*S)x_n \to 0$ and so, $SS^* - S^*S$ is not invertible. □

Let $\mathcal{P}$ denotes a class of operators satisfying the following properties:

1. If $A \in \mathcal{P}$ and M is an invariant subspace for A, then $A|_M \in \mathcal{P}$,
2. If $A \in \mathcal{P}$ and the restriction of A to an invariant subspace M is normal, then M reduces A,
3. If $A|_M \in \mathcal{P}$ and M is of finite dimensional, then $A|_M$ is normal. As a trivial example of the class $\mathcal{P}$ on consider $\mathcal{P} = \{0\}$; an interesting class is $\mathcal{P}$, the class of hyponormal operators.

An operators $A \in \mathcal{B}(\mathcal{H})$ is called dominant by J.G. Stampfli and B.L. Wadhwa [6] if, for all complex λ, $range(A-\lambda I) \subseteq range(A-\lambda I)^*$; or equivalently, if there is a real number M_λ such that

$$\|(A-\lambda I)^* f\| \leq \|(A-\lambda I)f\|, \text{ for all } f \in \mathcal{H}.$$

If there is a real number M such that $M_\lambda \leq M$ for all λ, the dominant operator A is said to be $M-$hyponormal. A $1-$hyponormal is hyponormal.

Theorem 2.6. *Let $A \in \mathcal{B}(\mathcal{H})$. If $T \in \overline{R(\delta_A)}^{\omega} \cap \{A^*\}'$, then*

$$A \in \mathcal{P} \Longrightarrow \{\lambda \in \sigma_p(T^*) : \dim\ker(T^* - \overline{\lambda}I) < \infty\} \subset \{0\}.$$

$$A^* \in \mathcal{P} \Longrightarrow \{\lambda \in \sigma_p(T) : \dim\ker(T - \lambda I) < \infty\} \subset \{0\}.$$

Theorem 2.7. *If A or $A^* \in (\mathcal{Y})$. Then every compact operator in $\overline{R(\delta_A)}^{\omega} \cap \{A^*\}'$ is quasinilpotent.*

Proof. We start with the second assertion. Suppose that $A^* \in (\mathcal{Y})$ and $T \in \overline{R(\delta_A)}^{\omega} \cap \{A^*\}'$. Let $\lambda \in \sigma_p(T)$ such that $E = \ker(T - \lambda I)$ be of finite dimensional, then the subspace E is invariant under T and A^*. Since $A^* \in (\mathcal{Y})$, E reduces A^* by [7]. Let $\mathcal{H} = E \oplus E^{\perp}$, hence we can write

$$A^* = \begin{bmatrix} A_1^* & 0 \\ 0 & A_2^* \end{bmatrix}, \; T = \begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix}.$$

Since $T \in \overline{R(\delta_A)}^{\omega}$, $\lambda I_E \in R(\delta_{A_1})$ and this implies that $\lambda = 0$. Since T is a compact operator in $\overline{R(\delta_A)}^{\omega} \cap \{A^*\}'$, it results from the above theorem that $\sigma(T) = \{0\}$ which implies that T is quasinilpotent. This completes the proof of the second assertion.

Remark that if $T \in \overline{R(\delta_A)}^{\omega} \cap \{A^*\}'$, then $T^* \in \overline{R(\delta_{A^*})}^{\omega} \cap \{A\}'$. Then the first assertion of the theorem follows in exactly the same way as the second. $\square$

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