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Regularity for Very Weak Solutions of $\mathcal{A}$-Harmonic Equation with Weight

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ABSTRACT. This paper deals with very weak solutions of the $\mathcal{A}$-harmonic equation

$$\mathrm{div}\mathcal{A}(x, \bigtriangledown u) = 0 \qquad (*)$$

with the operator $\mathcal{A} : \Omega \times R^n \to R^n$ satisfies some coercivity and controllable growth conditions with Muckenhoupt weight. By using the Hodge decomposition with weight, a regularity property is proved: There exists an integrable exponent $r_1 = r_1(\lambda, n, p) < p$, such that every very weak solution $u \in W^{1,r}_{loc}(\Omega, w)$ with $r_1 < r < p$ belongs to $W^{1,p}_{loc}(\Omega, w)$. That is, u is a weak solution to $(*)$ in the usual sense.

1. Introduction and statement of result

Let w be a locally integrable, nonnegative function in R^n. Then a Radon measure μ is canonically associated with the weight w,

$$\mu(E) = \int_E w(x)dx. \tag{1.1}$$

Thus $d\mu(x) = w(x)dx$, where dx is the n-dimensional Lebesgue measure. In what follows, the weight w and the measure μ are identified via (1.1). Let Ω be an open subset of R^n, $n \geq 2$. Consider the following second order divergence type elliptic

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equation (also called $\mathcal{A}$-harmonic equation or Leray-Lions equation)

$$\text{div}\mathcal{A}(x, \bigtriangledown u(x)) = 0, \tag{1.2}$$

where $\mathcal{A} : \Omega \times R^n \to R^n$ is a Carathéodory function and satisfies
(i) $|\mathcal{A}(x, \xi)| \leq \lambda w(x)|\xi|^{p-1}$,
(ii) $\langle \mathcal{A}(x, \xi), \xi\rangle \geq \lambda^{-1} w(x)|\xi|^p$,
where $1 < p < \infty$, $\lambda \geq 1$ are two fixed constants, and $w(x) \in A_1$ be a Muckenhoupt weight. The prototype of equation (1.2) is the p-harmonic equation with weight

$$\text{div}(w(x)| \bigtriangledown u|^{p-2} \bigtriangledown u) = 0.$$

Definition 1.1. A function $u \in W^{1,r}_{loc}(\Omega, w)$, $\max\{1, p-1\} \leq r < p$ is called a very weak solution of (1.2) if

$$\int_\Omega \langle \mathcal{A}(x, \bigtriangledown u), \bigtriangledown \varphi\rangle dx = 0 \tag{1.3}$$

for all $\varphi \in W^{1,r/(r-p+1)}(\Omega, w)$ with compact support.

Recall that $u \in W^{1,p}_{loc}(\Omega, w)$ is a weak solution of (1.2) if (1.3) holds for all $\varphi \in W^{1,p}(\Omega, w)$ with compact support. The word *very weak* in Definition 1.1 means that the Sobolev integrable exponent r of u is smaller than the *natural* exponent p. Regularity theory for very weak solutions of (1.2) with the operator $\mathcal{A}$ satisfies the conditions (i) and (ii) with $w(x) \equiv 1$ has been considered by Iwaniec and Sbordone [1] by using the theory of Hodge decomposition. J. L. Lewis [2] obtained a similar result by using an alternative method. In this note, we consider the regularity theory for very weak solutions of (1.1) with some general conditions (i) and (ii) and obtain the following result.

Theorem. *Suppose that $w \in A_1$ be a doubling Muckenhoupt weight. There exists $r_1 = r_1(\lambda, n, p) < p$, such that every very weak solution $u \in W^{1,r}_{loc}(\Omega, w)$ with $r_1 < r < p$ belongs to $W^{1,p}_{loc}(\Omega, w)$. That is, u is a weak solution to (1.2) in the usual sense.*

2. Definitions and some preliminary lemmas

Definition 2.1.[3] Given a nonnegative locally integrable function w, we say that w belongs to the A_p class of Muckenhoupt, $1 < p < \infty$, if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w dx\right)\left(\frac{1}{|Q|}\int_Q w^{1/(1-p)} dx\right)^{p-1} = A_p(w) < \infty, \tag{2.1}$$

where the supremum is taken over all cubes Q of R^n. When $p = 1$, replace the inequality (2.1) with

$$Mw(x) \leq cw(x)$$

for some fixed constant c and a.e. $x \in R^n$, where M is the Hardy-Littlewood maximal operator.

It is well-known that $A_1 \subset A_p$ whenever $p > 1$, see [3]. We say that a weight w is doubling, if there is a constant $C > 0$ such that

$$\mu(2Q) \leq C\mu(Q).$$

whenever $Q \subset 2Q$ are concentric cubes in R^n, where $2Q$ is the cube with the same center as Q and with side-length twice that of Q. Given a measurable subset E of R^n, we will denote by $L^p(E, w)$, $1 < p < \infty$, the Banach space of all measurable functions f defined on E for which

$$\|f\|_{L^p(E,w)} = \left(\int_E |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

The weighted Sobolev class $W^{1,p}(E, w)$ consists of all functions f for which f and its first generalized derivatives belong to $L^p(E, w)$. The symbols $L^p_{loc}(\Omega, w)$ and $W^{1,p}_{loc}(\Omega, w)$ are self-explanatory.

Definition 2.2. A function $K(x) : R^n \to R$ is a Calderon-Zygmund kernel if it satisfies the following properties:

$$\|\hat{K}(x)\|_\infty \leq C,$$
$$|K(x)| \leq C|x|^{-n},$$
$$|K(x) - K(x-y)| \leq C\tfrac{|y|}{|x|^{n+1}} \quad \text{for } |x| > 2|y|.$$

For $f \in L^p(R^n)$, $1 < p < \infty$, let

$$Tf(x) = p \cdot v \cdot \int_{R^n} K(x-y) f(y) dy.$$

Let the above operator T be named the Calderon-Zygmund (CZ) singular integral operator. It is well-known that the CZ singular integral operators are bounded on weighted L^p spaces whose weights belong to the A_1 class, see [4], [5].

Lemma 2.1. *Let w be an A_1 weight and T be a CZ singular integral operator. Then there exists a constant C, such that for every $f \in L^p(R^n, w)$,*

$$\|Tf\|_{L^p(R^n,w)} \leq CA_p(w)^{p'} \|f\|_{L^p(R^n,w)},$$

where $p' = p/(p-1)$.

The following lemma comes from [5] which is a Hodge decomposition in weighted spaces.

Lemma 2.2. *Let Ω be a regular domain of R^n (By a regular domain we understand any domain of finite measure for which the estimates in* (2.3) *and* (2.4) *are satisfied.*

For example, a Lipschitz domain is regular.) and $w(x)$ be an A_1 weight. If $u \in W_0^{1,p-\varepsilon}(\Omega,w)$, $1<p<\infty$, $-1<\varepsilon<p-1$, then there exist $\varphi \in W_0^{1,(p-\varepsilon)/(1-\varepsilon)}(\Omega,w)$ and a divergence-free vector field $H \in L^{(p-\varepsilon)/(1-\varepsilon)}(\Omega,w)$ such that

$$|\nabla u|^{-\varepsilon}\nabla u = \nabla\varphi + H \tag{2}$$

and

$$\|\nabla\varphi\|_{L^{(p-\varepsilon)/(1-\varepsilon)}(\Omega,w)} \le CA_p(w)^{\gamma}\|\nabla u\|^{1-\varepsilon}_{L^{p-\varepsilon}(\Omega,w)}, \tag{3}$$

$$\|H\|_{L^{(p-\varepsilon)/(1-\varepsilon)}(\Omega,w)} \le CA_p(w)^{\gamma}|\varepsilon|\|\nabla u\|^{1-\varepsilon}_{L^{p-\varepsilon}(\Omega,w)}, \tag{4}$$

where γ depending on r.

We will also need the following lemma in the proof of the main theorem.

Lemma 2.3.[6] *Suppose that w is a doubling weight and that nonnegative $f \in L^{\tau}_{loc}(\Omega,w)$, $1<\tau<\infty$, satisfies*

$$\left(⨍_Q f^{\tau}d\mu\right)^{1/\tau} \le C_1 ⨍_{2Q} f d\mu$$

for each cube Q such that $2Q\subset\Omega$, where $⨍_Q g d\mu = \frac{1}{\mu(Q)}\int_Q g d\mu$ is the integral mean for g over Q, and the constant $C_1 \ge 1$ independent of the cube Q. Then there exist $q>\tau$ so that

$$\left(⨍_Q f^{q}d\mu\right)^{1/q} \le C_2 \left(⨍_{2Q} f^{\tau} d\mu\right)^{1/\tau},$$

where the constant $C_2 \ge 1$ is independent of the cube Q. In particular, $f \in L^q_{loc}(\Omega,w)$.

The following lemma comes from [7].

Lemma 2.4. *Let $Q=Q(R)$ be any cube with side-length R, $\tau>1$ and $u\in C^1(\overline{Q})$. Then there exist constants $C,\delta^*>0$, such that for all $1\le K\le K^*=\frac{n}{n-1}+\delta^*$,*

$$\left(\frac{1}{\mu(Q)}\int_Q |u-u_Q|^{K\tau}d\mu\right)^{1/K\tau} \le CR\left(\frac{1}{\mu(Q)}\int_Q |\nabla u|^{\tau}d\mu\right)^{1/\tau}, \tag{5}$$

where $u_Q = ⨍_Q u d\mu$.

Obviously, (2.5) can be extended to functions $u\in W^{1,K\tau}(\overline{Q})$ by an approximation argument.

Lemma 2.5.[8] *Let $f(t)$ be a nonnegative bounded function defined for $0\le T_0 \le t \le T_1$. Suppose that for $T_0 \le t < s \le T_1$ we have*

$$f(t) \le A(s-t)^{-\alpha} + B + \theta f(s),$$

where A, B, α, θ are non-negative constants, and $\theta < 1$. Then there exists a constant c, depending only on α and θ such that for every $\varrho, R, T_0 \le \varrho < R \le R_1$ we have

$$f(\varrho) \le c[A(R-\varrho)^{-\alpha} + B].$$

3. Proof of the main theorem

Let $w \in A_1$ be a doubling Muckenhoupt weight and $u \in W^{1,r}_{loc}(\Omega, w)$, $\max\{1, p-1\} < r < p$, be a very weak solution of (1.2). For an arbitrary cube $2Q \subset\subset \Omega$, take a cut-off function $\eta \in C_0^\infty(2Q)$, such that $0 \le \eta \le 1$, $\eta \equiv 1$ in Q and $|\bigtriangledown \eta| \le \frac{C(n)}{R}$, where R is the side-length of Q. Consider the Hodge decomposition of $|\bigtriangledown (\eta(u-C))|^{r-p} \bigtriangledown (\eta(u-C)) \in L^{r/(r-p+1)}(2Q, w)$, where C is a constant to be determined later. We have by Lemma 2.2 that

$$|\bigtriangledown (\eta(u-C))|^{r-p} \bigtriangledown (\eta(u-C)) = \bigtriangledown\varphi + H, \tag{3.1}$$

where H is divergence-free and the following estimates hold

$$\|\bigtriangledown \varphi\|_{L^{r/(r-p+1)}(2Q,w)} \le CA_p(w)^\gamma \|\bigtriangledown (\eta(u-C))\|^{r-p+1}_{L^r(2Q,w)}, \tag{3.2}$$

$$\|H\|_{L^{r/(r-p+1)}(2Q,w)} \le CA_p(w)^\gamma |\varepsilon| \|\bigtriangledown (\eta(u-C))\|^{r-p+1}_{L^r(2Q,w)}. \tag{3.3}$$

If we set

$$E(\eta, u) = |\bigtriangledown (\eta(u-C))|^{r-p} \bigtriangledown (\eta(u-C)) - |\eta \bigtriangledown u|^{r-p} \eta \bigtriangledown u,$$

then by an elementary inequality from [9],

$$||X|^{-\varepsilon}X - |Y|^{-\varepsilon}Y| \le \frac{2^\varepsilon(1+\varepsilon)}{1-\varepsilon}|X-Y|^{1-\varepsilon}, \quad 0 \le \varepsilon < 1, X, Y \in R^n,$$

one can derive that

$$|E(\eta, u)| \le \frac{2^{p-r}(p-r+1)}{r-p+1}|(u-C) \bigtriangledown \eta|^{r-p+1}.$$

It follows from Definition 1.1 and (3.1) that

$$\int_\Omega \langle \mathcal{A}(x, \bigtriangledown u), |\bigtriangledown (\eta(u-C))|^{r-p} \bigtriangledown (\eta(u-C))\rangle dx = \int_\Omega \langle \mathcal{A}(x, \bigtriangledown u), H\rangle dx.$$

This implies, by the definition of $E(\eta, u)$, that

$$\begin{aligned} &\int_\Omega \langle \mathcal{A}(x, \bigtriangledown u), |\eta \bigtriangledown u|^{r-p} \eta \bigtriangledown u\rangle dx \\ = &\int_\Omega \langle \mathcal{A}(x, \bigtriangledown u), H\rangle dx - \int_\Omega \langle \mathcal{A}(x, \bigtriangledown u), E(\eta, u)\rangle dx \\ = &I_1 + I_2. \end{aligned} \tag{3.4}$$

The left-hand side of the above inequality can be estimated by using the condition (ii),

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), |\eta \nabla u|^{r-p} \eta \nabla u \rangle dx \geq \lambda^{-1} \int_Q |\nabla u|^r d\mu. \tag{3.5}$$

$|I_1|$ can be estimated by the condition (i), Hölder's inequality, (3.3) and Young's inequality,

$$\begin{aligned} |I_1| &= \left| \int_\Omega \langle \mathcal{A}(x, \nabla u), H \rangle \right| \\ &\leq \lambda \int_{2Q} |\nabla u|^{p-1} |H| d\mu \leq \lambda \| \nabla u\|^{p-1}_{L^r(2Q,w)} \|H\|_{L^{r/(r-p+1)}(2Q,w)} \\ &\leq CA_p(w)^\gamma \lambda |p-r| \| \nabla u\|^{p-1}_{L^r(2Q,w)} \| \nabla (\eta(u-C))\|^{r-p+1}_{L^r(2Q,w)} \\ &\leq CA_p(w)^\gamma \lambda (p-r) \left[C(\varepsilon) \| \nabla u\|^r_{L^r(2Q,w)} + \varepsilon \| \nabla (\eta(u-C))\|^r_{L^r(2Q,w)} \right]. \end{aligned}$$

Take $C = u_{2Q}$. By Lemma 2.4 with $K = \frac{n}{n-1}$ and $\tau = \frac{r}{K} = \frac{(n-1)r}{n}$, one obtain by Young's inequality that

$$\begin{aligned} &\| \nabla (\eta(u-C))\|^r_{L^r(2Q,w)} = \int_{2Q} |\nabla (\eta(u-C))|^r d\mu \\ &\leq 2^{r-1} \int_{2Q} \left(|\eta \nabla u|^r + |(u-C) \nabla \eta|^r \right) d\mu \\ &\leq 2^{r-1} \int_{2Q} |\nabla u|^r d\mu + \frac{C 2^{r-1}}{\mu(2Q)^{1/(n-1)}} \left(\int_{2Q} |\nabla u|^{(n-1)r/n} d\mu \right)^{n/(n-1)}. \end{aligned} \tag{3.6}$$

Thus

$$\begin{aligned} |I_1| \leq\ & CA_p(w)^\gamma \lambda (p-r) C(\varepsilon) \int_{2Q} |\nabla u|^r d\mu \\ & + CA_p(w)^\gamma 2^{r-1} \lambda (p-r) \varepsilon \int_{2Q} |\nabla u|^r d\mu \\ & + \frac{CA_p(w)^\gamma 2^{r-1} \lambda (p-r)}{\mu(2Q)^{1/(n-1)}} \left(\int_{2Q} |\nabla u|^{(n-1)r/n} d\mu \right)^{n/(n-1)}. \end{aligned} \tag{3.7}$$

Finally, we estimate $|I_2|$. Take r sufficiently close to p such that $\frac{2^{p-r}(p-r+1)}{r-p+1} < 2$. Thus

$$\begin{aligned} |I_2| &= \left| \int_{2Q} \langle \mathcal{A}(x, \nabla u), E(\eta, u) \rangle \right| \leq 2\lambda \int_{2Q} |\nabla u|^{p-1} |(u-C) \nabla \eta|^{r-p+1} d\mu \\ &\leq 2\lambda \| \nabla u\|^{p-1}_{L^r(2Q,w)} \|(u-C) \nabla \eta\|^{r-p+1}_{L^r(2Q,w)} \\ &\leq \lambda \varepsilon \| \nabla u\|^r_{L^r(2Q,w)} + C(\varepsilon) \lambda \|(u-C) \nabla \eta\|^r_{L^r(2Q,w)}. \end{aligned}$$

By Lemma 2.4 again, one can derive that

$$\|(u-C) \nabla \eta\|^r_{L^r(2Q,w)} \leq \frac{C}{\mu(2Q)^{1/(n-1)}} \left(\int_{2Q} |\nabla u|^{(n-1)r/n} d\mu \right)^{n/(n-1)}.$$

Thus

$$|I_2| \leq \lambda\varepsilon \| \bigtriangledown u\|^r_{L^r(2Q,w)} + \frac{C(\varepsilon)\lambda}{\mu(2Q)^{1/(n-1)}} \left(\int_{2Q} | \bigtriangledown u|^{(n-1)r/n} d\mu \right)^{n/(n-1)}. \tag{3.8}$$

Adding all the estimates (3.4), (3.5), (3.7) and (3.8), we obtain

$$\begin{aligned} \int_Q | \bigtriangledown u|^r d\mu \; &\leq CA_p(w)^\gamma \lambda^2 (p-r) C(\varepsilon) \int_{2Q} | \bigtriangledown u|^r d\mu \\ &+ CA_p(w)^\gamma 2^{r-1} \lambda^2 (p-r)\varepsilon \int_{2Q} | \bigtriangledown u|^r d\mu \\ &+ \frac{CA_p(w)^\gamma 2^{r-1}\lambda^2(p-r)}{\mu(2Q)^{1/(n-1)}} \left(\int_{2Q} | \bigtriangledown u|^{(n-1)r/n} d\mu \right)^{n/(n-1)} \\ &+ \lambda^2\varepsilon \int_{2Q} | \bigtriangledown u|^r d\mu + \frac{C(\varepsilon)\lambda^2}{\mu(2Q)^{1/(n-1)}} \left(\int_{2Q} | \bigtriangledown u|^{(n-1)r/n} d\mu \right)^{n/(n-1)}. \end{aligned}$$

Take r sufficiently close to p and ε small enough, such that $\theta = CA_p(w)^\gamma \lambda^2 (p-r)C(\varepsilon) + CA_p(w)^\gamma 2^{r-1}\lambda^2(p-r)\varepsilon + \lambda^2\varepsilon < 1$ (in fact, this inequality can decide the value of r_1), by Lemma 2.5, there exists a constant c, such that

$$\int_Q | \bigtriangledown u|^r d\mu \leq \frac{Cc}{\mu(2Q)^{1/(n-1)}} \left(\int_{2Q} | \bigtriangledown u|^{(n-1)r/n} d\mu \right)^{n/(n-1)}.$$

Divided by $\mu(Q)$ in both sides of the above inequality yields

$$\fint_Q | \bigtriangledown u|^r d\mu \leq C \left(\fint_{2Q} | \bigtriangledown u|^{(n-1)r/n} d\mu \right)^{n/(n-1)}. \tag{3.9}$$

We are now in a position of using Lemma 2.3 to improve the degree of integrability of $| \bigtriangledown u|$. Accordingly, there exists $q > r$, such that $| \bigtriangledown u| \in L^q_{loc}(\Omega, w)$. For u fixed, we denote by I the set of all exponents $q \in [r, p]$ such that $| \bigtriangledown u| \in L^q_{loc}(\Omega)$, where $r_1 < r < p$. By assumption, the set I contains r. We shall prove that I coincides with the interval $[r, p]$. Obviously the set I is closed. Now inequality (3.9) and Lemma 2.3 imply that $| \bigtriangledown u|$ actually belongs to $L^{\tilde{p}}_{loc}(\Omega, w)$ with some exponent $\tilde{p} > r$. In conclusion, the set I is closed and open and thus coincides with the interval $[r, p]$. By Sobolev Imbedding theorem we have $u \in W^{1,p}_{loc}(\Omega, w)$. This completes the proof of the theorem.

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