A Formula Derivation of Channel Capacity Calculation in a MIMO System

S. M. Humayun Kabir, Eun-Ju Lee and Gi-Wan Yoon, *Member, KIMICS*

Abstract—In this letter, we derive a tight closed-form formula for an ergodic capacity of a multiple-input multiple-output (MIMO) for the application of wireless communications. The derived expression is a simple closed-form formula to determine the ergodic capacity of MIMO systems. Assuming the channels are independent and identically distributed (i.i.d.) Rayleigh flat-fading between antenna pairs, the ergodic capacity can be expressed in a closed form as the finite sum of exponential integrals.

Index Terms—Channel capacity, multiple antennas, MIMO, wireless communications.

I. INTRODUCTION

Multiple-antenna concepts have become the main concentration of wireless communications providing high spectral efficiencies. Through an increased spatial dimension, multiple-input multiple-output is providing dramatic channel capacity gain. Due to an increased number of channel parameter in the receiver side it is very difficult to obtain perfect channel state information (CSI). Considering different constraint such as transmission matrix and constant power constraint E. Telatar [1] derived the analytical formula for ergodic capacity of independent and identically distributed (i.i.d) Rayleigh flat-fading MIMO channels in integral form introducing the Laguerre polynomials.

A closed form solution for the Rayleigh fading channel capacity is obtained under the adaptation polices of optimal power and rate adaptation, constant power with optimal rate adaptation, and channel inversion with fixed rate [2]. In MIMO systems multiple antennas at both transmit and receive ends have recently drawn significant concentration in response to the increasing requirements on data rate and quality in communication systems [2]-[5].

The main objective of this paper is to extend the analysis in [1] to obtain simple closed-form formulas for the ergodic capacity of i.i.d Rayleigh flat-fading MIMO channels. By analyzing the integral channel capacity formula provided in [1], H. Shin [4] extends the integral expression to a closed form solutions. Gans [5] first evaluate the probability density function (pdf) of the received instantaneous SNR under channel estimation error. By using the pdf obtained by Gans [5], we extend the integral formula provided by E. Taleter [1] and obtained a simple closed form ergodic capacity formula. The derived expression for the ergodic channel capacity is simple and provides less complexity by replacing the infinite integration with finite summation.

The remainder of this paper is organized as follows. The next section provides channel model of MIMO systems. In section III, we derived the closed form formulas for the MIMO ergodic capacity. Finally, section IV concludes the paper.

II. CHANNEL MODEL

For simplicity we consider a communication system of n_t transmit and n_r receive antennas. Information transformation is done through mapping the bits in a particular signal constellation. The corresponding mapped symbols are encoded to get transmission matrix X of an OSTBC; $X_{ij}$is the linear combination of the symbols and their complex conjugates which are transmitted through i^{th} transmit antenna in the j^{th} time slot. The rate of an Orthogonal Space Time Block Codes (OSTBC) is defined as the ratio between the number of symbols the encoder takes as its input and the number of space time coded symbols transmitted from each antenna. So, the OSTBC rate is expressed as $R = k/N$, where k symbols are transmitted through N time slots. From [3], Transmission matrix X of an OSTBC is related such that$XX^H = r\sum_{m=1}^{M}|s_m|^2 I_{n_t}$. The constant r depends on the transmission matrix X.

The input-output relation of conventional MIMO systems can be written as

Manuscript received May 2, 2009 ; Revised June 13, 2009.

S. M. Humayun Kabir is with the Department of Information and Communications Engineering, KAIST, Daejeon, Korea (Tel: +82-42-866-6291, Fax: +82-42-866-6227, Email: kabir@kaist.ac.kr)

$$Y = HX + W \tag{1}$$

where the received signal Y is an $n_r \times N$ matrix. H is an $n_r \times n_t$ channel matrix. X is an $n_t \times N$ transmission matrix, and the noise W is an $n_r \times N$ matrix with i.i.d. complex circular Gaussian random variables, with zero mean and σ^2 variance. The MIMO channel model is described by a $n_r \times n_t$ matrix H, where h_{ij} is a complex Gaussian variable describing the channel from the j^{th} transmit antenna to the i^{th} receive antenna. It assumes that the channel matrix H is normalized in a way such that $E[\,\|H\|_F^2\,] = n_r n_t = M$. The average energy of the transmitted symbol from each antenna are assumed to be $\mathbf{E}/n_t$, so that the average power of the received signal at each received antenna is equal to $\mathbf{E}$ and the average SNR per received antenna is $\mathbf{E}/\sigma^2$. Maximum Likelihood (ML) decoder can be used at the receiver end to detect the transmitted symbol with significant accuracy. Using ML decoder at the receiver end, received signal can be expressed as follows

$$y = r\|H\|_F^2 x_n + w_k \tag{2}$$

where w_k represent noise term with zero mean, and $r\|H\|_F^2 \sigma_k{}^2$ variance. In [3] the effective signal-to-noise ratio (SNR) per symbol is expressed as

$$\vartheta = \frac{\bar{\vartheta}}{n_t R}\|H\|_F^2 \tag{3}$$

where $\bar{\vartheta} = E_s/\sigma_k{}^2$ is the average SNR per receive antenna. Also in [3] the correlation coefficient is determined as $\rho = \sqrt{1-\varepsilon^2}$, where $\varepsilon \in [0,1]$ is the measure of the accuracy of the channel estimation. Under channel estimation error, the probability density function of the received instantaneous SNR, evaluated by Gans [5] is expressed as

$$\tau_\vartheta(\vartheta) = \frac{e^{-\frac{n_t R\vartheta}{\bar{\vartheta}(1-\rho^2)}}}{(1-\rho^2)^{1-M}}\frac{n_t R}{\bar{\vartheta}}\,{}_1F_1\left(M; 1; \frac{n_t R}{\bar{\vartheta}}\cdot\frac{\rho^2}{(1-\rho^2)^2}\right)$$

$$= \frac{e^{-n_t R\vartheta/(\bar{\vartheta}(1-\rho^2))}}{(1-\rho^2)^{1-M}}\frac{n_t R}{\bar{\vartheta}}\sum_{k=0}^{M-1}\left(\frac{1}{k!}\right) \times \binom{M-1}{k}\left(\frac{n_t R}{\bar{\vartheta}}\cdot\frac{\rho^2}{(1-\rho^2)^2}\right)^k \tag{4}$$

where $M = n_t n_r$, and ${}_1F_1(a; b; z)$ is the confluent hypergeometric function of the first kind defined as ${}_1F_1(a; b; z) = \sum_{k=0}^{\infty}(a)_k z^k/(b)_k k!$, where $(a)_k$ and $(b)_k$ are Pochhammer symbol [6]. The Pochhammer symbol is defined as $(a)_k = \Gamma(a+k)/\Gamma(a) = a(a+1)\cdots(a+k-1)$ and $\binom{n}{k} = n!/k!\,(n-k)!$ is the binomial coefficient.

III. CLOSED FORM CHANNEL CAPACITY

Considering the channel model discussed in previous sections we drive a closed form channel capacity for MIMO systems. The ergodic capacity in bits/s/Hz of an i.i.d. Rayleigh fading MIMO channel with transmit and receive antennas is given by

$$C = \frac{QR}{\ln(2)}\frac{\rho^2 e^Q}{(1-\rho^2)^{2-N}}\sum_{k=1}^{N}\sum_{j=1}^{k+1} Q^{k-j}\Gamma(j-k-1, Q) \tag{5}$$

where $Q = n_t R/\vartheta$ and equal power is allocated to each transmitted antenna.

Given the pdf of ϑ, the ergodic capacity of the equivalent STBC channel is

$$C = R\int_0^\infty \log_2(1+\vartheta)\,\tau_\vartheta(\vartheta)d\vartheta \tag{6}$$

Inserting equation (4) in the above equation we get

$$C = \frac{R}{\ln(2)}\frac{1}{(1-\rho^2)^{1-N}}\frac{n_t R}{\bar{\vartheta}}\sum_{k=0}^{N-1}\left(\frac{1}{k!}\right) \times \left(\frac{n_t R}{\bar{\vartheta}}\cdot\frac{\rho^2}{(1-\rho^2)^2}\right)^k \int_0^\infty \ln(1+\vartheta)\vartheta^k e^{-\frac{n_t R}{\bar{\vartheta}}}d\vartheta \tag{7}$$

to evaluate the integral of the above equation we use the result of [4]

$$I_l(\alpha) = \int_0^\infty \ln(1+x)\,x^{l-1}e^{-\alpha x}dx \quad \alpha > 0, l = 1,2,3,\cdots$$
$$= (l-1)!\,e^\alpha \sum_{i=0}^{n}\frac{\Gamma(i-l,\alpha)}{\alpha^i} \tag{8}$$

the complementary incomplete gamma function is defined in [6]

$$\Gamma(a, y) = \int_0^\infty e^{-v}v^{a-1}dv \tag{9}$$

Applying equation (8), the integral part in equation (7) can be evaluated as

$$I = k!\,e^{\frac{n_t R}{\bar{\vartheta}}}\sum_{j=1}^{k+1}\frac{\Gamma\left(j-k-1, \frac{n_t R}{\bar{\vartheta}}\right)}{\frac{n_t R}{\bar{\vartheta}}^j} \tag{10}$$

Inserting the integral value of equation (10) in equation (7), we get the closed form ergodic capacity expression as expressed in (5).

Fig. 1 shows simulation result of the derived capacity. Simulation is done for the different number of antennas in both transmitter and receiver side.

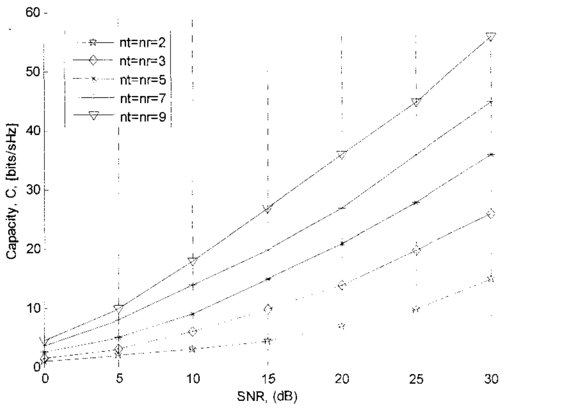


Fig. 1 Closed-form ergodic channel capacity versus average signal to noise ratio. The legend denotes the number of transmit and receive antennas

IV. CONCLUSIONS

In this paper, we derived a formula for the channel capacity, as a closed form solution, of MIMO wireless systems. The derived capacity formula allows us to calculate the channel capacity in a closed form. Consequently, the derived formula can be applied for calculating the channel capacity of the practical MIMO system where large numbers of antennas are used to transmit and receive data through wireless media.

ACKNOWLEDGMENT

This work was supported by the Korea Science and Engineering Foundation (KOSEF) grant funded by the Korea government (MEST) (No. R11-2005-029-06003-0).

REFERENCES

[1] I.E. Teletar, "Capacity of multi antenna Gaussian channel", European Trans. Telecomm. (ETI), vol.10, no.6 pp.586-595, Nov./Dec. 1999.

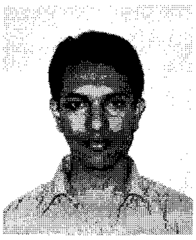
[2] Mohammed-Slim Alouini, and Andrea J. Goldsmith, "Capacity of Rayleigh Fading Channels Under Different Adaptive Transmission and Diversity-Combining Techniques," IEEE Trans. Veh. Technol., vol.39, no.3, pp.187-189,Aug.1990.

[3] Kyung Seung Ahn, R. W. Heath, "Shannon Capacity and Symbol Error Rate of Space-Time Block Codes in MIMO Rayleigh Channels With Channel Estimation Error", IEEE Trans. Wireless Commun., vol. 7, no. 1, pp. 324–333, Jan. 2008.

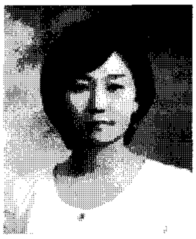
[4] H. Shin and J. H. Lee. "On the capacity of MIMO wireless channels," IEICE Trans. Commun., Vol. E87-B, no.3, pp. 671-677, Mar.2004.

[5] M. J. Gans, "The effect of Gaussian error in maximal ratio combiners," IEEE Trans. Commun. Technol., vol. 19, no. 4, pp. 492–500, Aug. 1971.

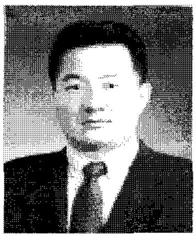
[6] I.S Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products, 5th ed., Academic, San Diego, Ca, 2000.



S. M. Humayun Kabir
Received his B.Sc. in Computer Science and Engineering, Khulna University, Bangladesh, in 2006. Since 2007, he has been an MS student in Communication Electronics Lab, Korea Advanced Institute of Science and Technology, Daejeon, Korea.



Eun-Ju Lee
Received her B.Sc. in Electrical Engineering, Korea University, Korea, in 2002. Since 2008, she has been an MS student in Communication Electronics Lab, Korea Advanced Institute of Science and Technology, Daejeon, Korea



Gi-Wan Yoon
Member KIMICS. Received B.S. degree in Seoul National University (SNU), in 1983 and M.S. degree in KAIST, in 1985, both Seoul, Korea. Received Ph.D. degree in The University of Texas at Austin, USA, in 1994. From 1985 to 1990, he was with LG Group, Seoul, Korea. From 1994 to 1997, he was with digital equipment corporation, MA, USA. Currently, he is a professor at KAIST ICC, Daejeon, Korea. His major research areas of interest include multifunctional intelligent devices & their technologies for RF and wireless applications.