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The Fourier Method — A Course on Differential Equations and Infinite Series

Christensen, Ole
Department of Mathematics, Building 303, Technical University of Denmark,
2800 Lyngby, Denmark; Email: Ole.Christensen@mat.dtu.dk

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We present the thoughts behind a course combining the subjects of differential equations and infinite series. The key point is how to connect the two topics in a course with a natural flow.

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1. INTRODUCTION

The topics differential equations and infinite series are usually taught in different courses. At the Technical University of Denmark, we have recently combined these subjects in one course; the purpose of this article is to demonstrate how this can be done with a natural flow.

The two topics "differential equations" and "infinite series" have a very different status for most students. The theory of infinite series is usually considered to be abstract and difficult; the students in applied science often lack the motivation, i.e., an explanation why infinite series are needed and which kind of problems they are relevant for. On the other hand, differential equations are considered to be "natural", partly because the students are familiar with the topic, and partly because they know that such equations appear in all types of engineering and science. For this reason, the first part of the course deals with differential equations: this gives the students an opportunity to connect the new material to the knowledge they already gained in previous courses on mathematics. As a very important additional benefit, we will show that this order gives a natural approach to infinite series.

The combination of the subjects of differential equations and infinite series makes it possible to formulate a clear "high-light" - a central result that unify most of the topics appearing in the course. Consider a system of differential equations on the form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \\ y = \mathbf{c}^{\top}\mathbf{x}; \end{cases} \tag{1}$$

here $\mathbf{A}$ is a known $n \times n$ matrix, $\mathbf{u}$ is a vector function from $\mathbb{R}$ to $\mathbb{C}$, and $\mathbf{b}$ and $\mathbf{c}$ are vectors in $\mathbb{R}^n$. The main goals in the course are as follows:

(i) Assume that u is a periodic function. Find an expression, e.g., in terms of a Fourier series

$$y(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} \tag{2}$$

for a solution to (2).

(ii) Estimate how many terms one should include in the series (2) in order to obtain a partial sum that approximates y sufficiently well.

The solutions to these questions are obtained at the end of the course, as culmination of almost all results obtained in the course. We will now sketch the key points needed in order to reach this point. Except for the basic theory of differential equations and systems hereof, the material can be found in the book (Christensen & Christensen, 2003).

2. DESCRIPTION OF THE KEY POINTS

The lectures can naturally be split into 4 blocks:

- Differential equations and systems hereof;
- Infinite series;
- Fourier series;
- The Fourier method: how to obtain Fourier series solutions to differential equations of the form (1).

We will now describe the lecture blocks in more detail.

1-4. Lecture: Differential equations.

The first 4 lectures deal with the classical theory for nth order differential equations and systems of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{u}; \tag{3}$$

here $\mathbf{A}$ is a known $n \times n$ matrix, $\mathbf{u}$ is a vector function from $\mathbb{R}^n$ to $\mathbb{C}$, and $\mathbf{x}$ is the solution. In applications, a variant of this system plays an important role: for this reason we will also deal with a system on the form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \\ y = \mathbf{c}^\top \mathbf{x}, \end{cases} \tag{4}$$

where $u : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mathbf{b}$ and $\mathbf{c}$ are vectors in $\mathbb{R}^n$. Note that for the system (4), we consider the scalar-valued function y as the solution - no longer $\mathbf{x}$. For such a system, the transfer function is defined by

$$H(s) = -\mathbf{c}^\top (\mathbf{A} - s\mathbf{I})^{-1}\mathbf{b}. \tag{5}$$

It is well known that if $u(t) = e^{st}$ for some $s \in \mathbb{C}$ for which $\det(\mathbf{A} - s\mathbf{I}) \neq 0$, then (4) has the solution

$$y(t) = H(s)e^{st}.$$

The lectures also introduce the concepts "stability" and "asymptotical stability."

5-8. Lecture: Infinite series.

The fact that the students already are familiar with systems of differential equations gives a very natural way of introducing infinite series. Consider an asymptotically stable system of the form (4): we know how to solve it for functions of the form $u(s) = e^{st}$ for some $s \in \mathbb{C}$. Using the principle of superposition, we are also able to solve any asymptotically stable system for functions of the type

$$u(t) = \sum_{n=-N}^{N} c_n e^{int} \tag{6}$$

for $N \in \mathbb{N}$ and arbitrary coefficients $c_n \in \mathbb{C}$. It is natural to ask if we can solve the system for more general functions u – and the theory for infinite series shows that this indeed is possible. In fact, all periodic functions can be expressed as a limit of functions of the type (6) – and a solution can be obtained by taking the limit of the solutions corresponding to the functions in (6).

This gives a very clear motivation for the study of infinite series. It also makes it clear that we need some tools to estimate how many terms we need in order to obtain a certain approximation of an infinite sum. Among the various well-known tests of convergence, the lectures focus on the integral test (for sums with positive terms) and Leibniz' rule (valid for alternating series). Under the relevant assumptions, appropriate variants of these results tell how many terms we need to include in order to obtain a partial sum which only differs from the exact sum with a given (arbitrary) tolerance. The main philosophy is to use these results (when appropriate) to check which partial sum one needs to consider, and then calculate the partial sum via Maple or another computer program.

9-10. Lecture: Fourier series.

The Fourier series for a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ is

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{7}$$

where the Fourier coefficients are

$$c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z}. \tag{8}$$

The Nth partial sum of the Fourier series is

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}.$$

A key question in Fourier analysis is how well the partial sums of the Fourier series approximates the given function. Fourier's theorem says that if a function f is 2π–periodic and piecewise differentiable, then the Fourier series converges pointwise for all x; and if the function f furthermore is continuous at all $x \in \mathbb{R}$, then

$$|f(x) - S_N(x)| \leq \frac{1}{\sqrt{N}} \frac{1}{\sqrt{\pi}} \sqrt{\int_{-\pi}^{\pi} |f'(t)|^2 dt}, \ \forall x \in \mathbb{R}. \tag{9}$$

In particular, the result shows that the Fourier series converges uniformly under the given assumptions.

For functions f appearing as solutions to differential equations, we usually do not know an explicit expression for f, but only the Fourier coefficients. This makes it difficult to apply (9); in such cases, we can alternatively use that

$$|f(x) - S_N(x)| \leq \sum_{|n|>N} |c_n|, \ \forall x \in \mathbb{R}. \tag{10}$$

The inequality (10) is used to determine how many terms we need to include in the partial sum in order to obtain a certain approximation; technically, this is done via the integral test.

11-12. Lecture: Solution to a system of differential equations.

These lectures contain the main results of the course, which connect the theory for Fourier series with the aim of solving a system of differential equations. We consider again the system (4), which is assumed to be asymptotically stable, with transfer function $H(s)$. Now assume that the function u is 2π-periodic, piecewise differentiable and continuous, with Fourier series

$$u(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}, \quad t \in \mathbb{R}.$$

Then it is proved that (4) has a solution given by the Fourier series

$$y(t) = \sum_{n=-\infty}^{\infty} c_n H(in)e^{int}, \quad t \in \mathbb{R}. \tag{11}$$

The function y in (11) is usually not given explicitly, i.e., the only information available is that the Fourier series of y is given by (11). In practice, we are only able to work with a partial sum of the Fourier series, so it is important to have theoretically results to tell how many terms we need to include in the partial sum in order to obtain a certain precision. In the case discussed here, we can use the estimate (10),

$$\left| y(t) - \sum_{n=-N}^{N} c_n H(in)e^{int} \right| \le \sum_{|n|>N} |c_n H(in)|,$$

combined with the integral test.

Parseval's theorem relates the norm of a 2π-periodic function f with the Fourier coefficients:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

The expression

$$||f|| = \sqrt{\int_0^{2\pi} |f(t)|^2 \, dt}.$$

is called the norm of the function f. Often it is more relevant to search for a partial sum S_N which yield a certain approximation of f in norm (i.e., which makes $||f - S_N||$ sufficiently small) than to try to make $|f(x) - S_N(x)|$ small in the uniform sense. We prove that for any $0 < \delta < 1$,

$$\frac{||S_N||}{||f||} \ge \delta \Leftrightarrow \sum_{|n|>N} |c_n|^2 \le \frac{1-\delta^2}{2\pi} ||f||^2.$$

If we want to find a partial sum containing, say, 99 percent of the norm of the norm of f, we can use this with $\delta = 0.99$. The integral test is a useful tool in order to find an appropriate value for N.

The lectures also provide another important link between infinite series and solution of differential equations. We discuss how to find power series solutions to differential equations with variable coefficients, for example, equations of the type

$$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + ty = 0, \;\; y(0) = 1, y'(0) = 0.$$

The idea is to consider functions y having a power series representation,

$$y(t) = \sum_{n=0}^{\infty} c_n t^n, \;\; t \in]-\rho, \rho[; \tag{12}$$

inserting the power series into the differential equation leads to a set of equations determining how one has to choose the coefficients c_n in order to obtain a solution. The method is often applied in engineering literature.

3. CONCLUDING REMARKS

We have described the ideas behind a course offered at teh Technical University of Denmark, aiming at students on the third semester. The course combines analysis of differential equations and infinite series. The connection of these two topics in one course allows to motivate the study of the abstract infinite series in a very natural fashion.

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