IJR International Journal of Railway
Vol. 2, No. 2 / June 2009, pp. 70-79

The Korean Society for Railway

Uniqueness of an Optimal Run-up for a Steep Incline of a Train

Xuan Vu†

Abstract

An optimal driving strategy of a train in a long journey on a nonsteep track has four phases: an initial power phase, a long hold speed phase, a coast phase and a final brake phase. The majority of the journey is speed holding.
On a track with steep gradients, it becomes necessary to vary the strategy around steep sections of track because it is not possible to hold a constant steep on steep track. Instead we must interrupt the speed hold phase with a power phase. The aim of this paper is to show that there is a unique power phase that satisfies the necessary conditions for an optimal journey. The problem is developed and solved for various cases, from a simple single steep gradient to a complicated multiple steep gradient section. For each case, we construct a set of new conditions for optimality of the power phase that minimises the energy used during the power phase subject to a weighted time penalty. We then use the new necessary conditions to develop a calculate scheme for finding an optimal power phase for a steep incline. We also present an example to confirm the uniqueness of an optimal power phase.

Keywords **:** *Energy effcient, Optimal driving strategies*

1. Introduction

The optimal journey for a long journey on a non-steep track has four phases: an initial power phase, a long speed holding phase, a coast phase and a final brake phase. The majority of the journey is speed holding.

When a train comes to a steep gradient section, it becomes necessary to vary the strategy around steep sections of track because it is not possible to hold a constant speed on steep track. Instead, we must interrupt the speed hold phase with a power phase that starts somewhere before the steep section and finishes somewhere beyond the steep section. The aim of this paper is to show that there is a unique power phase that satisfies the necessary conditions for an optimal journey. We consider the optimal control when the speed holding phase is interrupted by a single steep uphill section. For simplicity we assume the track gradient is piecewise constant, and comprises a non-steep gradient, a steep uphill gradient, and another non-steep gradient.

In this paper we first formulate the problem of finding the optimal power phase, and present a new condition for optimality of the power phase that minimises the energy used during the power phase subject to a weighted time penalty. We then derive key necessary conditions for an optimal power phase, and prove that the optimal hold-power-hold phase exists and is unique. Finally we support the proof with some examples.

2. Background

The problem we discuss and solve in this paper is mainly based on the results of a long term research of the Scheduling and Control Group at the University of South Australia. Their research results were used to build an in-cab advice system for long haul trains. They named it Freightmiser. Freightmiser helps to improve timekeeping to scheduled target times and reduce fuel consumption, while satisfying all requirements on speed limits, safe-working systems and train-handling considerations. It has been marketed by TTG Transportation Technology, a consulting company on train technologies. It has currently been used on freight trains operated by Pacific National Freight Company, Australia and has been on trial with various rail companies in Australia, India and UK. The research of the group was described by Howlett *et al* in [4,5,7,11] and in numerous papers by Howlett but most of theoretical works were presented in [3,7].

† *TTG Transportation Technology, Sydney 2000, Australia*
E-mail: xuan.vu@ttgtt.com.au (Xuan Vu)

Before discussing the problem, for convenience we firstly review the previous work of the Scheduling and Control Group.

3. Previous Work

3.1 The equation of motion

The equations of motion for a point mass train are

$$v\frac{dv}{dx}=\frac{p}{v}-q-r(v)+g(x) \quad (1)$$

and

$$\frac{dt}{dx}=\frac{1}{v} \quad (2)$$

where x is the position of the train, $\upsilon(x)$ is the speed of the train and $t(x)$ is the time at which the train is at location x. In this model $0\leq p\leq P$ for some fixed P, is the tractive force per unit mass applied at the wheels, $0\leq q\leq Q$ for some fixed Q, is the braking force per unit mass, $r(v)$ is the resistance force per unit mass and $g(x)$ is the gradient force per unit mass. The resistance force is defined by the formula

$$r(v)=a+bv+cv^2$$

The total time taken for the train to travel from $x=0$ to $x=X$ is

$$T=t(X)-t(0)=\int_0^X\frac{1}{v(x)}dx$$

Howlett and Pudney [5] show that the motion of a train with distributed mass on a given gradient profile is the same as the motion of a point mass train on a modified gradient profile. The motion of a train with length S and mass M can be modelled as

$$v\frac{dv}{dx}=\frac{p}{v}-q-r(v)+\frac{1}{M}\int_0^S\rho(s)g(x-s)ds \quad (3)$$

where x is the position of the front of the train and $\rho(s)$ is the mass per unit length at distance s from the front of the train. This equation can be rewritten in the same form as (1) if the modified gradient acceleration is defined as

$$\bar{g}(x)=\frac{1}{M}\int_0^S\rho(s)g(x-s)ds$$

In the next sections we will consider only point mass trains.

3.2 The Cost Function

The mechanical work done by the locomotive as the train travels from $x=0$ to $x=X$ is

$$J(x)=\int_0^x\frac{P}{v}d\xi$$

in which case

$$\frac{dJ}{dx}=\frac{P}{v}$$

We ignore the (negative) work done by the brakes since this energy is not recovered. We wish to minimise the cost function J subject to the state equations (1) and (2). The boundary conditions for the problem are

$$v(0)=v(X)=0 \text{ and } t(0)=0,\ t(X)=T$$

3.3 Hamiltonian equation

We use Pontryagin's Maximum Principle to find optimal control strategy. The Maximum Principle requires us to maximise the Hamiltonian for the system, which is defined as

$$H=-\frac{p}{v}+\frac{\alpha}{v}\left[\frac{p}{v}-q+g(x)-rv\right]+\frac{\beta}{v} \quad (4)$$

where α and β, are adjoint variables. The adjoint variables evolve according to the equations

$$\frac{d\alpha}{dx}=-\frac{\partial H}{\partial v}=-\frac{p}{v^2}+\frac{\alpha}{v^3}$$

$$(2p-qv+g(x)v-r(v)v+r'(v)v^2)+\frac{\beta}{v^2} \quad (5)$$

and

$$\frac{d\beta}{dx}=-\frac{\partial H}{\partial t}=0 \quad (6)$$

We want to maximise H subject to

$$0\leq p\leq P \text{ and } 0\leq q\leq Q$$

where P is the maximum available driving power, and Q is the maximum available braking force. Thus we define

$$H=\frac{1}{v}\left(\frac{\alpha}{v}-1\right)p-\frac{\alpha}{v}q+\frac{\alpha}{v}(g(x)-r(v))$$

$$+\lambda p+\mu(P-p)+\rho q+\sigma(Q-q) \quad (7)$$

where μ, λ, ρ σ are non-negative Lagrange multipliers. In order to maximize H we apply the Karush-Kuhn-Tucker conditions

$$\frac{\partial H}{\partial p}=\frac{1}{v}\left(\frac{\alpha}{v}-1\right)+\lambda-\mu=0 \quad (8)$$

and

$$\frac{\partial H}{\partial p}=-\frac{\alpha}{v}+\rho-\sigma=0 \quad (9)$$

with the complementary slackness conditions

$$\lambda p = \mu(P-p) = pq = \sigma(Q-q) = 0. \quad (10)$$

There are two critical values of α:$\alpha=v$ and $\alpha=0$. So we must consider the Hamiltonian in five cases: $\alpha>v$, $\alpha=v$, $0<\alpha<v$, $\alpha=0$ and $\alpha<0$. Using a new adjoint variable $\theta=\alpha/v$ we obtain the optimal controls summarised in the table below:

adjoint	mode	control
$\theta>1$	power	$p=P, q=0$
$\theta=1$	hold	$p=r(v)-g(x), q=0$
$0<\theta<1$	coast	$p=0, q=0$
$\theta<0$	brake	$p=0, q=Q$

These necessary conditions for an optimal journey are discussed more carefully in [7,9]. They have been applied in the Freightmiser technology.

4. Problem Formulation

The equation of motion for a train in full power is

$$v\frac{dv}{dx} = \frac{P}{v} - r(v) + g(x) \quad (11)$$

where P is the maximum power per unit mass, $r(v)$is the resistance force per unit mass and $g(x)$is the gradient force per unit mass. The adjoint equation of the system, as defined in previous section, is

$$\frac{d\theta}{dx} - \frac{\psi(v)+P}{v^3}\theta = (-1)\frac{P+\psi(V)}{v^3} \quad (12)$$

where

$$\psi(v) = v^2 r'(v) \quad (13)$$

and $v = v(x)$ is the solution to (11). From Section 1, we know that for an optimal journey a change from hold to power and a change from power to hold each requires $\theta=1$. For convenience, we define $\eta=\theta-1$, so that power starts and finishes at $\eta=0$. From (12), the modified adjoint equation is

$$\frac{d\eta}{dx} - \frac{\psi(v)+P}{v^3}\eta = \frac{\psi(v)-\psi(V)}{v^3} \quad (14)$$

Suppose the optimal holding speed for the entire journey is V. Howlett [7] and Howlett and Leizarowitz [8] show that when the hold phase for an optimal journey is interrupted by a steep uphill section, the optimal control requires a power phase that starts before the start of the steep section and finishes beyond the steep section. During this power phase, the speed of the train increases from the hold speed V before the start of the steep section, decreases to below speed V on the steep section, and returns to speed V after the steep section. Intuitively, we want to keep the "average" speed of the train during the power phase the same as the holding speed for the overall journey. Fig. 1 indicates the problem: we want to find an optimal point p at which to start the power phase so that speed increases before the start of the steep section at b, decreases through speed V on the steep uphill interval $[b, c]$, and increases back to speed V at some point q beyond the steep section.

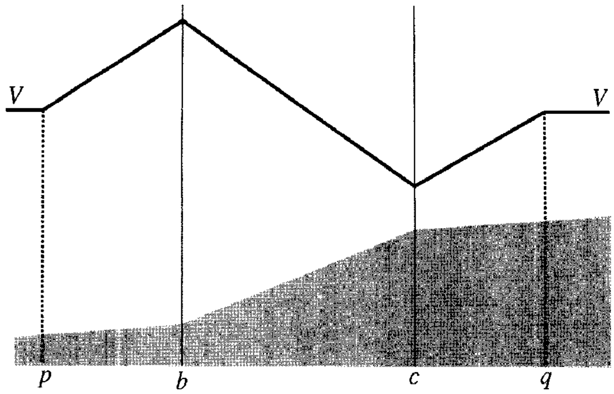

Fig. 1 Optimal Speed profile for a Single Steep Gradient.

5. Necessary Conditions

As mentioned in previous section, for an optimal journey we need

$$\theta = 1 \Leftrightarrow \eta = 0 \text{ and } \frac{d\theta}{dx} = 0 \Leftrightarrow \frac{d\eta}{dx} = 0$$

at $x=p$ and $x=q$. Let $v_0(x)$ be the optimal speed profile. Integrating the modified adjoint equation (14) over $[p, q]$ gives

$$\int_p^q \frac{\psi(v_0)-\psi(V)}{v_0^3} I_p(x)dx = 0 \quad (15)$$

where the integrating factor $I_p(x)$is defined for $x \in [p, q]$ by the formula

$$I_p(x) = Cexp\left(-\int_p^x \frac{\psi(v_0)+P}{v_0^3}d\xi\right) \quad (16)$$

Equation (15) for $p<b<c<q$ was used by Howlett [7] as a standard necessary condition to specify the optimal power phase on a steep uphill section. We will use a variational argument to find an alternative necessary condition.

6. An Alternative Necessary Condition

If the speed changes from the optimal profile $v_0(x)$ by an innitesimal increment to a new profile $(v_0+\delta v)(x)$ then the equation for the new profile is

$$(v_0+\delta v)\frac{d(v_0+\delta v)}{dx} = \frac{P}{v_0+\delta v} - r(v_0+\delta v)+g(x) \tag{17}$$

By applying a Maclaurin series expansion, subtracting the original equation (14) for v_0 and neglecting second and higher order terms we obtain the perturbation equation

$$\delta v\frac{dv_0}{dx}+v_0\frac{d\delta v}{dx} = (-1)\left[\frac{P}{v_0^2}+r'(v_0)\right]\delta v \tag{18}$$

We call δv a first order variation of the speed. The equation for the first order variation is derived rigorously on page 163 of the book by Birkhoff and Rota [1]. If we rewrite (18) in the form

$$\frac{d}{dx}(v_0\delta_v) = (-1)\left[\frac{P+\psi(v_0)}{v_0^3}\right](v_0\delta v) \tag{19}$$

and integrate using the initial condition $v_0(p)=V$ we obtain

$$(v_0\delta v)(x) = V\delta v(p)I_p(x)$$

where $I_p(x)$ is given by (16). By substitution into (15) we have

$$\int_p^q \frac{\psi(v_0)-\psi(V)}{v_0^2}\cdot\delta v\cdot dx = 0$$

or

$$\int_p^q\left[\frac{-\psi(V)}{v_0^2}+r'(v_0)\right]\cdot\delta v\cdot dx = 0 \tag{20}$$

where $\delta v=\delta v(x)$ is the first order variation. This is an alternative necessary condition for the optimal power phase on a steep uphill section. The expression (20) takes the form of a first order variation for an integral cost function. Suppose we define

$$J_0(v) = \int_p^q\left[\frac{\psi(V)}{v}+r(v)\right]dx \tag{21}$$

We have the following result.

Theorem 1 *Let $v(x)$ be a solution to (11) and define*

$$J(v) = J_0(v)-(q-p)\varphi'(V) \tag{22}$$

where $p<b<c<q$ are chosen so that $v(p)=v(q)=V$. A necessary condition for a minimum of J is

$$\int_p^q\left[\frac{-\psi(V)}{v^2}+r'(v)\right]\cdot\delta v\cdot dx = 0$$

where δv is the first order variation of v.

The proofs of the theorems can be found in [13,14].

7. Key Equations

Suppose a train is powering on a steep uphill section $[b, c]$. The gradient function is defined as

$$g(x) = \begin{cases} \Upsilon_0 \text{ if } x\in[p,b) \\ \Upsilon_1 \text{ if } x\in[b,c] \\ \Upsilon_2 \text{ if } x\in(c,q] \end{cases} \tag{23}$$

The gradient accelerations γ_0 and γ_2 could be either positive or negative but we assume they are not steep. We assume γ_1 is steep at hold speed V.

Theorem 2 *The necessary conditions for minimising the cost function* (22) *for a train travelling on the section defined in* (23) *are*

$$[P-\varphi(v_b)+\Upsilon_0 v_b]\mu = [\varphi(v_b)-\varphi'(V)v_b+\psi(V)](\Upsilon_0-\Upsilon_1)$$

and

$$[P-\varphi(v_c)+\Upsilon_2 v_c]\mu = [\varphi(v_c)-\varphi'(V)v_c+\psi(V)](\Upsilon_2-\Upsilon_1)$$

where $\mu(v)>0$ *defined by*

$$\mu = \lambda-(\varphi'(V)-\Upsilon_1)$$

Proof:

The equation of the motion of the train is

$$v\frac{dv}{dx} = \frac{P}{v}-r(v)+g(x) \tag{24}$$

with $v(p)=v(q)=V$. We choose the starting point p with $p<b$ and $v(p)=V$ and then find $q<c$ such that $v(q)=V$. By separating the variables in (24) and integrating we obtain

$$p = b-\int_V^{v_b}\frac{v^2dv}{P-v[r(v)-\Upsilon_0]} \tag{25}$$

where v_b is the speed of the train at the bottom of the steep section, and

$$q = c+\int_{v_c}^{V}\frac{v^2dv}{P-v[r(v)-\Upsilon_2]} \tag{26}$$

where v_c is the speed of the train at the crest of the steep section. If we integrate (24) from b to c then we have

$$c-b = \int_{v_c}^{v_b}\frac{v^2dv}{v[r(v)-\Upsilon_1]-P} \tag{27}$$

Integrating both sides of (24) from p to b gives

$$\frac{1}{2}v_b^2-\frac{1}{2}V^2 = \int_p^b\frac{P}{v}dx-\int_p^b r(v)dx+\Upsilon_0(b-p) \tag{28}$$

Integrating from b to c gives

$$\frac{1}{2}v_c^2-\frac{1}{2}v_b^2 = \int_b^c\frac{P}{v}dx-\int_b^c r(v)dx+\Upsilon_1(c-b) \tag{29}$$

and integrating from c to q gives

$$\frac{1}{2}V^2-\frac{1}{2}v_c^2 = \int_c^q\frac{P}{v}dx-\int_b^c r(v)dx+\Upsilon_1(c-b) \tag{30}$$

By combining (28)~(30) and rearranging, we get

$$\int_p^q r(v)dx = \int_p^q \frac{P}{v}dx + \Upsilon_0(b-p) + \Upsilon_1(c-b) + \Upsilon_2(q-c) \quad (31)$$

From the proof of Theorem (1), the cost function (22) is defined by

$$J_0 = \int_p^q \left[\frac{\psi(V)}{v} + r(v) - \varphi'(V)\right]dx$$

If we consider a small variation $\delta v(x)$ to the optimal speed profile and let $v = v_0 + \delta v$, then from (21) we obtain

$$J_0(v_0+\delta v) = \int_{p_0+\delta p}^{q_0+\delta q}\left[\frac{\psi(V)}{v_0+\delta v} + r(v_0-\delta v)\right]dx \quad (32)$$

By substituting (31) into (32) we obtain

$$J_0 = [\psi(V)+P]\int_p^q \frac{dx}{v}\Upsilon_0(b-p) + \Upsilon_1(c-b)$$

$$+\Upsilon_2(q-c) - \varphi'(V)(q-p). \quad (33)$$

From (24) we can write

$$\frac{1}{v}dx = \begin{cases} \dfrac{vdv}{P-v[r(v)-\Upsilon_0]} & if\, x \in [p,b) \\ \dfrac{-vdv}{v[r(v)-\Upsilon_1]-P} & if\, x \in [b,c] \\ \dfrac{vdv}{P-v[r(v)-\Upsilon_2]} & if\, x \in (c,q] \end{cases} \quad (34)$$

From (34) we define

$$A(v_b,v_c) = \int_p^q \frac{dx}{v}dv$$

$$= \int_V^{v_b} \frac{vdv}{P-v[r(v)+\Upsilon_0]} + \int_{v_c}^{v_b} \frac{vdv}{v[r(v)+\Upsilon_1]-P}$$

$$+\int_{v_c}^{V} \frac{vdv}{P-v[r(v)+\Upsilon_2]} \quad (35)$$

Hence using (25) and (26) we obtain

$$J(v_b,v_c) = [\psi(V)+P]A(v_b,v_c)$$

$$+\Upsilon_0\int_V^{v_b}\frac{v^2dv}{P-v[r(v)-\Upsilon_0]} + \Upsilon_1(c-b)$$

$$+\Upsilon_2\int_{v_c}^{V}\frac{v^2dv}{P-v[r(v)-\Upsilon_2]}$$

$$-\varphi'(V)\left(c-b+\int_V^{v_b}\frac{v^2dv}{P-v[r(v)-\Upsilon_0]}\right)$$

$$+\varphi'(V)\int_{v_c}^{V}\frac{v^2dv}{P-v[r(v)-\Upsilon_2]} \quad (36)$$

We need to minimise $J(v_b, v_c)$subject to

$$c-b = \int_{v_c}^{v_b}\frac{v^2dv}{v[r(v)+\Upsilon_1]-P}$$

We define the Lagrangian function

$$\mathcal{J}(v_b,v_c) = J(v_b,v_c) + \lambda\left[c-b-\int_{v_c}^{v_b}\frac{v^2dv}{v[r(v)+\Upsilon_1]-P}\right] \quad (37)$$

where λ is the Lagrange multiplier. Applying the Karush-Kuhn-Tucker conditions we have

$$\frac{\partial\mathcal{J}}{\partial v_b} = 0 \quad and \quad \frac{\partial\mathcal{J}}{\partial v_c} = 0$$

and the complementary slackness conditions

$$\lambda\left[c-b-\int_{v_c}^{v_b}\frac{v^2dv}{v[r(v)-\Upsilon_1]-P}\right] = 0$$

If we weaken the equality constraint (27) to

$$c-b \le \int_{v_c}^{v_b}\frac{v^2dv}{v[r(v)-\Upsilon_1]-P} \quad (38)$$

then we can also guarantee that λ is non-negative and our solution is unchanged because the control that minimises energy will not travel further than the required distance c–b.

So we have

$$\frac{\partial\mathcal{J}}{\partial v_b} = [\psi(V)+P](\Upsilon_1-\Upsilon_0) + \lambda(P-\varphi(v_b)+\Upsilon_0 v_b)$$

$$-[\varphi'(V)-\Upsilon_0](P-\varphi(v_b)+\Upsilon_1 v_b) \quad (39)$$

and hence $\partial\mathcal{J}/\partial v_b = 0$ gives

$$(P-\varphi(v_b)+\Upsilon_0 v_b)\lambda = (\psi(V)+P)(\Upsilon_0-\Upsilon_1)$$

$$-(\varphi'(V)-\Upsilon_0)(\varphi(v_b)-\Upsilon_1 v_b - P) \quad (40)$$

Similarly with $\partial\mathcal{J}/\partial v_C = 0$ we have

$$(P-\varphi(v_b)+\Upsilon_2 v_c)\lambda = (\psi(V)+P)(\Upsilon_2-\Upsilon_1)$$

$$-(\varphi'(V)-\Upsilon_2)(\varphi(v_c)-\Upsilon_1 v_c - P). \quad (41)$$

Let

$$\mu = \lambda - (\varphi'(V)-\Upsilon_1). \quad (42)$$

Then we can rewrite (40) and (41) as

$$[P-\varphi(v_b)+\Upsilon_0 v_b]\mu = [\varphi(v_b)-Lv(v_b)](\Upsilon_0-\Upsilon_1) \quad (43)$$

and

$$[P-\varphi(v_c)+\Upsilon_2 v_c]\mu = [\varphi(v_c)-Lv(v_c)](\Upsilon_2-\Upsilon_1) \quad (44)$$

where $L_V(v) = \varphi(V)+\varphi'(V)(v-V)$. The line $y=L_V(v)$ is the tangent to the convex curve $y=\varphi(v)$. Equations (27), (43) and (44) are necessary conditions for an optimal solution.

We now wish to show that $\mu > 0$. Since Υ_0 and Υ_2 are non-steep gradient accelerations, we have

$$P-\varphi(v_b)+\Upsilon_0 v_b>0 \tag{45}$$

and

$$P-\varphi(v_c)+\Upsilon_2 v_c>0 \tag{46}$$

Since $\varphi(v)$ is convex and and $L_V(v)$ is the tangent to $\varphi(v)$ at $v=V$ it follows that

$$\varphi(v)-L_V(v)\geq 0$$

Since $\Upsilon_j-\Upsilon_0$ for $j=0, 2$, we can use (43), (45) and (46) to conclude that μ is positive.

8. Existence And Uniqueness

8.1 Geometric approach

We can write (43) and (44) as

$$\varphi(v_b) = L_{\mu, \Upsilon_0}(v_b) \tag{47}$$

and

$$\varphi(v_c) = L_{\mu, \Upsilon_2}(v_c) \tag{48}$$

where

$$L_{\mu, \Upsilon_i}(v) = \left[\Upsilon_i+\frac{(\varphi'(V)-\Upsilon_i)(\Upsilon_i-\Upsilon_1)}{\mu+\Upsilon_i-\Upsilon_1}\right]v_b$$

$$+P-\frac{(\varphi(V)+P)(\Upsilon_i-\Upsilon_1)}{\mu+\Upsilon_i-\Upsilon_1}$$

for $i=0, 2$. The right hand sides of (47) and (48) are linear functions. The straight line $y = L_{\mu, \Upsilon_0}(v)$ passes through the fixed point

$$P_0 = \left(\frac{P+\psi(V)}{\varphi'(V)-\Upsilon_0}, P+\frac{[P+\psi(V)]\Upsilon_0}{\varphi'(V)-\Upsilon_0}\right)$$

and the straight line $y = L_{\mu, \Upsilon_2}(v)$ passes through the fixed point

$$P_2 = \left(\frac{P+\psi(V)}{\varphi'(V)-\Upsilon_2}, P+\frac{[P+\psi(V)]\Upsilon_2}{\varphi'(V)-\Upsilon_2}\right)$$

Since $\varphi(v)$is convex, equations (47) and (48) each have at most two solutions for v.

If we let $\mu=0$, the lines become

$$y = \varphi'(V)v-\psi(V) = \varphi'(V)(v-V)+\varphi(V) = L_V(v)$$

The line $y=L_V(v)$ is the tangent to the curve $y=(v)$ at the point $v=V$. Thus (47) and (48) imply that $v_b=v_c=V$ and the fixed points are on the common tangent.

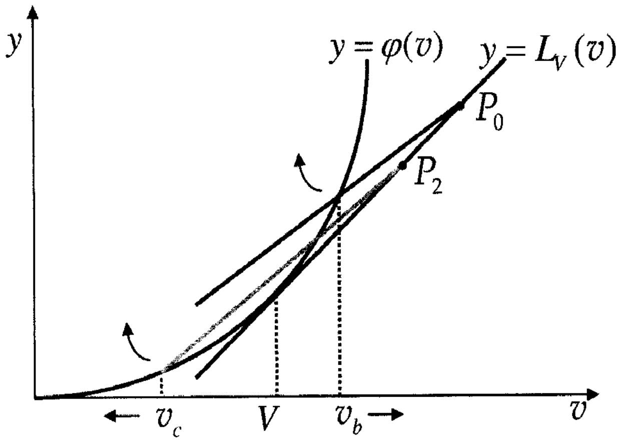

Fig. 2 Illustration of the Geometric Proof.

Fig. 2 shows the tangent $y=L_V(v)$ in purple, the line L_{μ, Υ_0} in red and the line L_{μ, Υ_2} in green.

When $\mu > 0$ the lines $L_{\mu, \Upsilon_j}(v)$ for each $j=0, 2$ cut the curve $y=\varphi(v)$ at two points v_{1, Υ_i} and v_{2, Υ_i} with $v_{1, \Upsilon_i} < V < v_{2, \Upsilon_i}$ for each $i=0, 2$. Since $v_c < V < v_b$ there is only one possible solution to each equation.

Consider the slope of the lines (47) and (48)

$$s_i(\mu) = \frac{(\Upsilon_i-\Upsilon_1)\varphi'(V)+\Upsilon_i\mu}{\Upsilon_i-\Upsilon_1+\mu}, \text{ for } i=0, 2.$$

Since $\Upsilon_i-\Upsilon_1>0$ and $\Upsilon_i<0$ for then we can easily see that s_i is a monotone decreasing function. So if μ increases the slopes of the two lines $y=L_{\mu, \gamma 0}(v)$ and $y=L_{\mu, \gamma 2}(v)$ decrease. It follows that the solution v_b to the equation $\varphi(v_b)=L_{\mu, \gamma 0}(v_b)$ increases and the solution v_c to the equation $\varphi(v_c)=L_{\mu, \gamma 2}(v_c)$ decreases as shown in Fig. 2. However from the constraint (27) we can see if v_b increases then v_c also increases. Therefore there is precisely one value of μ for which the necessary conditions (43), (44) and (27) are satised. Thus the solution to equations (43), (44) and (27) is unique.

8.2 Algebraic approach

8.2.1 Existence of the solution

For an optimal strategy we must satisfy the conditions (27), (43) and (44). It is not easy to solve this system explicitly so we use a numerical iteration. Given a value of v_b we can use (27) to calculate v_c. Now we can calculate $\mu=M(v_c)$ from (44)

$$M(v_c) = \frac{[\varphi(v_c)-L_V(v_c)](\gamma_2-\gamma_1)}{P-\varphi(v_c)+\gamma_2 v_c} \tag{49}$$

If v_b, v_c and μ are optimal they must satisfy (43). That is, we require $f(v_b)=0$ where

$$f(v_b) = [P-\varphi(v_b)+\gamma_0 v_b]M(v_c)$$

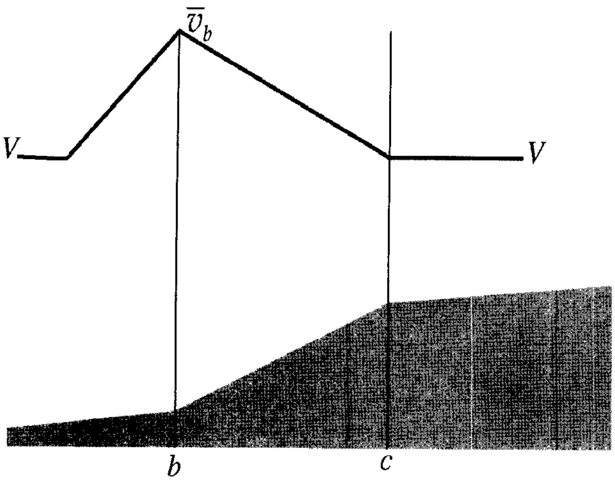

Fig. 3 Speed profile when $v(b)=\bar{v}_b$

$$-[\varphi(v_b)-L_V(v_b)](\gamma_0-\gamma_1) \quad (50)$$

Theorem 3 *A solution to the equation*

$$f(v_b)=0$$

where f is defined by (50) *and* v_b *is the speed at* $x=b$, *exists in its domain* $(V, \bar{v}b)$.

Proof:

Since $v_c<V<v_b$, the possible upper bound $\bar{v}_b$ for v_b can be determined by setting $v_c=V$ and using (27) to calculate the corresponding v_b. Thus $v_b=\bar{v}_b$ is the solution to the equation

$$c-b=\int_V^{vb}\frac{v^2dv}{v(r(v)-\gamma_1)-P}$$

The minimum possible value for v_b is V and the domain of $f(v_b)$ is $(V, \bar{v}_b)$. The speed profile when $v_b=V$ is illustrated in Fig. 3.

We can prove the existence by observing that f is a continuous function of v_b, and showing that either $f(V)<0$ and $f(\bar{v}_b)>0$ or $f(V)>0$ and $f(\bar{v}_b)$. First we need to check the sign of the two ends of the range of v_b. At $v_b=V$ we have

$$\varphi(v_b)-L_V(v_b)=\varphi(v_b)-[\varphi(V)+\varphi'(V)(v_b-V)]=0 \quad (51)$$

when $v_b=V$. That is,

$$f(V)=[P-\varphi(V)+\gamma_0V]M(\underline{vc}).$$

Since γ_0 is non-steep at speed V then

$$P-\varphi(V)+\gamma_0V>0$$

We now check the sign of $M(v_c)$. We have, as in (49),

$$M(v_c)=\frac{[\varphi(v_c)-L_V(v_c)](\gamma_2-\gamma_1)}{P-\varphi(v_c)+\gamma_2v_c}$$

Recall that γ_2 is also non-steep gradient acceleration at speed $v\leq V$. Since $v_c\leq V$,

$$P-\varphi(v_c)+\gamma_2v_c>0$$

That means $M(v_c)>0$ for all $v<V$ and so $f(V)>0$.

When $v_c=V$, $v_b=\bar{v}_b$ and (44) gives $M(v_c)=0$. Then

$$f(\bar{v}_b)=-[\varphi(\bar{v}_b)-L_V(\bar{v}_b)](\gamma_1-\gamma_0)<0 \quad (52)$$

Note that the maximum speed at b, $\bar{v}_b$ might be greater than the limiting speed at b, $v_L(P,\gamma_0)$. If that is the case then we need to set $\bar{v}_b=v_L(P,\gamma_0)$. Since

$$\varphi(v_L(P,\gamma_0))+\gamma_1vL(P,\gamma_0)-P=0$$

then

$$f(v_L(P,\gamma_0))=-[\varphi(v_L(P,\gamma_0))-L_V(v_L(P,\gamma_0))]$$

$$(\gamma_1-\gamma_0)<0.$$

Therefore there exists at least one solution to $f(v_b)=0$ in the interval $v_b\in(V,\bar{v}_b)$

8.2.2 Uniqueness of the solution

If we can prove f is monotonic decreasing then we can prove the solution of $f(v_b)=0$ is unique. Consider

$$\frac{df(v_b)}{dv_b}=[P-\varphi(v_b)+\gamma_0v_b]\frac{dM(v_c)}{dv_c}\cdot\frac{dv_c}{dv_b}$$

$$-(\varphi'(v_b)-\gamma_0)M(v_c)-(\gamma_0-\gamma_1)(\varphi'(v_b)-\varphi'(V)) \quad (53)$$

By differentiating (49) we have

$$\frac{dM(v_c)}{dv_c}=(\gamma_2-\gamma_1)$$

$$\frac{\varphi'(v_c)[P+\psi(V)-\varphi'(V)v_c]-\varphi'(V)[P-\varphi(v_c)]}{[P-\varphi(v_c)+\gamma_2v_c]^2} \quad (54)$$

Let

$$\pi(v)=(\varphi'(v)-\gamma_2)(\psi(V)+P)(\varphi'(V)+\gamma_2)\psi(v)+P) \quad (55)$$

So now we can write (54) as

$$\frac{dM}{dv_c}=\frac{\pi(v_c)(\gamma_2-\gamma_1)}{[P-\varphi(v_c)+\gamma_2v_c]^2} \quad (56)$$

Since

$$\psi'(v)=v\varphi''(v)$$

and

$$P>v[r(v)-\gamma_2]=\varphi(v)-v\gamma_2$$

and

$$v\varphi'(v)=\psi(v)+\varphi(v)$$

then we have

$$\pi'(v)=\varphi''(v)(\psi(V)+P)-(\varphi'(V)-\gamma_2)\psi'(v)$$

$$\varphi''(v)(\psi(V)+P)-(\varphi'(V)-\gamma_2)v\psi''(v)$$
$$>\varphi''(v)[\psi'(V)+\varphi(v)-\gamma_2 v-(\varphi'(V)-\gamma_2)v]$$
$$=\varphi''(v)[\psi'(V)+\varphi(v)-v\varphi'(V)]$$
$$=\varphi''(v)[\psi'(V)+\varphi(V)-\varphi'(V)(v-V)-v\varphi'(V)]$$
$$=\varphi''(v)[\psi'(V)+\varphi(V)-V(\varphi')(V)]$$
$$=0. \quad (57)$$

Since $\pi'(v)>0$ and, from (55), $\pi(V)=0$ we can conclude that $\pi(v)>0$ for $v>V$ and $\pi(v)<0$ for $v<V$. Since $v_c<V$ then $\pi(v_c)<0$. Hence using (56) we obtain

$$\frac{dM(v_c)}{dv_c}<0$$

By differentiating (27) with respect to v_c we have

$$\frac{v_c^2}{\varphi(v_c)-\gamma_1 v_c-P}=\frac{v_b^2}{\varphi(v_b)-\gamma_1 v_b-P}\cdot\frac{dv_b}{dv_c}$$

and hence

$$\frac{dv_c}{dv_b}=\left(\frac{v_b}{v_c}\right)^2\cdot\frac{\varphi(v_c)-\gamma_1 v_c-P}{\varphi(v_b)-\gamma_1 v_b-P}>0$$

So the first term of (53) is

$$[P-\varphi(v_b)+\gamma_0 v_b]\frac{dM(v_c)}{dv_c}\cdot\frac{dv_c}{dv_b}<0$$

Now consider

$$M(v_c)=\frac{[\varphi(v_c)-L_V(v_c)](\gamma_2-\gamma_1)}{P-\varphi(v_c)+\gamma_2 v_c}$$

Since $\varphi(v)$ (v) is convex then $\varphi(v_c)-L_V(v_c)>0$. Hence $M(v_c)>0$, and therefore the second term of (53) is negative. Since $v_b>V$ and $\varphi(v)$ is convex then

$$\varphi'(v_b)>\varphi'(V)$$

and so the last term of (53) is also negative. So from (53) we have

$$\frac{df(v_b)}{dv_b}<0$$

Therefore, there is only one solution to the equation $f(v_b)=0$.

9. Numerical Solution

We want to solve equations (27), (43) and (44) for v_b, v_c and μ. For convenience we write them here again. They are

$$[P-\varphi(v_b)+\gamma_0 v_b]\mu=[\varphi(v_b)-L_V(v_b)](\gamma_0-\gamma_1),$$

$$[P-\varphi(v_c)+\gamma_2 v_c]\mu=[\varphi(v_c)-L_V(v_b)](\gamma_2-\gamma_1)$$

and

$$c-b=\int_{v_c}^{v_b}\frac{v^2 dv}{v[r(v)-\gamma_1]-P}$$

respectively. From (43) we define the function $f(v_b)$ by the formula

$$f(v_b)=[P-\varphi(v_b)-\gamma_0 v_b]\mu$$
$$-[\varphi(v_b)-L_V(v_b)](\gamma_1-\gamma_0) \quad (58)$$

From the previous section, we know that $f(V)>0$ and $f(\bar{v}_b)<0$, and that f is monotonic decreasing. We can use a numerical method such as the Bisection method [2] to find the solution to $f(v_b)=0$. For each candidate value of v_b, we must calculate v_c and before we can evaluate f. The value for v_c can be found using a numerical DE solver to solve the equation of motion (27) forwards from $(x=b, v=v_b)$ to $x=c$. The value for μ is calculated using formula (49) which is derived from (44).

Evaluating $f(v)$ requires many calculations. We could speed up the method by using the *regula falsi* method or Brent's method [2] to reduce the number of evaluations of f required.

10. Example

The gradient acceleration γ_V is the gradient on which the train will approach a limiting speed V under power. That is,

$$\frac{P}{V}-r(V)+\gamma_V=0$$

Therefore, we have

$$\gamma_V=\frac{\varphi(V)-P}{V}$$

In our examples we use holding speed $V=20$ and $P=3$. The gradient acceleration that gives speed V as a limiting speed is $\gamma_V=-0.1233$. If $\gamma<\gamma_V$ then the track is steep uphill.

Example 1 *A single constant gradient steep section.*

In this example we simulate a train powering over a constant gradient uphill section. This section starts at $x=5000$ and ends at $x=6000$. The gradient of the track is

$$g(x)=\begin{cases}-0.075 & if\ x<5000\\ -0.2 & if\ x\in[5000,6000]\\ -0.09 & if\ x>6000\end{cases} \quad (59)$$

First, we plotted $f(v_b)$ for various $v_b\in[20,24]$. The left side of Fig. 4 shows the result. We then used the

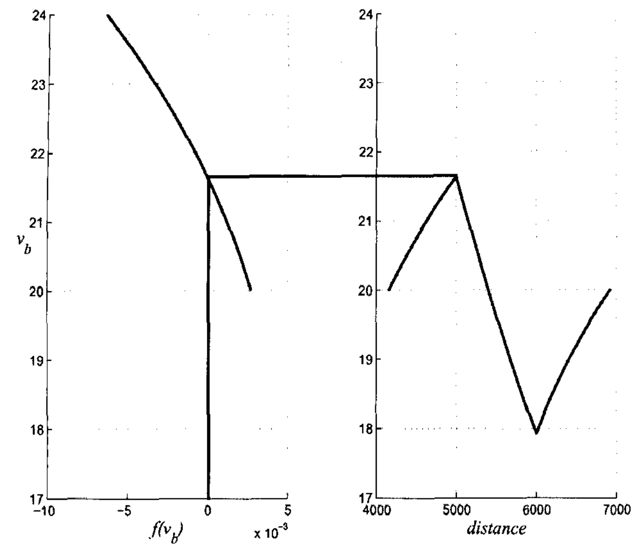

Fig. 4 Result of Example 1

Table 1 Experimental results of hold-power-hold on a single steep uphill.

P	υ_b	J	$f(\upsilon_d)$
3904.15	22.000000	1.5435219085	-0.00080804
4527.70	21.000000	1.6125591341	0.00120800
4238.99	21.500000	1.4985675595	0.00026211
4078.01	21.750000	1.4983069060	-0.00025739
4160.02	21.625000	1.4931503194	0.00000624
4119.40	21.687500	1.4943599371	-0.00012460
4139.81	21.656250	1.4934189882	-0.00005893
4149.94	21.640625	1.4932013580	-0.00002628
4154.98	21.632812	1.4931551063	-0.00001000
4157.50	21.628906	1.4931475411	-0.00000187
4158.76	21.626953	1.4931476388	0.00000218
4158.13	21.627929	1.4931472669	-0.00000015
4157.82	21.628417	1.4931473232	-0.00000086
4157.97	21.628173	1.4931472749	-0.00000035
4158.05	21.628051	1.4931472658	0.00000010

Bisection Method to find the solution to $f(\upsilon_b) = 0$. The optimal speed profile is shown on the right of Fig. 4.

The sequence of estimates for the optimal *vb* are shown in the table 1. Recall that *J* is the objective function for our problem, from (22).

11. Conclusion

For a track with a single steep uphill section, we have used an algebraic argument and a geometry argument to show that there is a unique optimal location before the start of the steep gradient at which the power phase should begin. We have developed a new set of necessary conditions for an optimal power phase for a steep uphill section by minimising a cost function which is a compromise of energy used and time taken. We have also developed a new method for calculating the optimal power phase for the steep uphill section. This method converges quickly to the unique solution.

For a track with two or more gradient uphill sections we are able to prove the existence using the similar approaches but not uniqueness of a solution. However, we are able to develop a numerical scheme for calculating power phases that satisfy the necessary conditions for an optimal strategy and demonstrate uniqueness in numerical examples. The details of the proof and examples can be found in [13,14].

Reference

[1] Birkhoff, G. and Rota G-C.(1978), "Ordinary Differential Equation," *John Wiley and Sons, Inc., 3rd edition.*

[2] Burden, R. G and Faires J. D.(1997), "Numerical Analysis," *Brooks/Cole Publishing Company, 6th edition.*

[3] Howlett P. G.(1990), "An optimal strategy for the control of a train," *Journal Australian Mathematics Society Series B*, Volume 31, pp. 454-471.

[4] Howlett P. G., Milroy I. and Pudney P. J.(1994), "Energy-Efcient Train Control," *Control Engineering Practice*, Volume 2, no. 2, pp. 193-200.

[5] Howlett P. G. and Pudney P. J.(1995), "Energy-Efcient Train Control," *Control Engineering Practice*, Springer-Verlag London Ltd..

[6] Howlett P. G. and Pudney P.J.(2000), "Energy Efcient Driving Strategies for Long Haul Trains," *Proceedings of CORE 2000 Conference on Railway Engineering*, Adelaide, Australia, 21-23.

[7] Howlett P. G.(2000), "The optimal control of the train," *Annals of Operations Research*, Vol. 98, p. 65-87.

[8] Howlett P. G. and A. Leizarowitz(2001), "Optimal strategies for vehicle control problems with finite control sets," *Dynamic of Continuous, Discrete and Impulsive Systems B, Application and Algorithms.*, 8:41-69.

[9] Khmelnitsky E.(2000), "On an Optimal Control Problem of Train Operation," *IEEE Transaction on Automatic Control*, Volume 45, no. 7.

[10] Yee R. and Pudney P. J.(2004), "Saving Fuel on Long-Haul Trains: Freightmiser Initial Trial Results," *Proceedings of CORE 2004 Conference of Railway Engineering*, Darwin, Australia, 20-23.

[11] Cheng J. and Howlett P. G.(1992), "Application of critical velocities to the minimisation of fuel consumption in the control of trains," *Automatica*, vol. 28, No. 1, pp. 165-169.

[12] Benjamin B.R., Long A.M., Milroy I.P., Payne R.L. and Pudney

P.J.(1987), "Control of rail-way vehicles for energy conservation and improved timekeeping," *Proceedings of the Conference on Railway Engineering,* pp. 41-47.

[13] X. Vu(2006), "Analysis of necesssary conditions for the optimal control of a train," *PhD. thesis, University of South Australia*, Adelaide.

[14] X. Vu(2009), "Analysis of necesssary conditions for the optimal control of a train-New necesssary conditions for energy-efcient train control," *VDM Verlag Dr. Muller Aktiengesellschaft & Co. KG*, Germany.