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A NUMERICAL SCHEME WITH A MESH ON CHARACTERISTICS FOR THE CAUCHY PROBLEM FOR ONE-DIMENSIONAL HYPERBOLIC CONSERVATION LAWS

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Abstract. In this paper, a numerical scheme is introduced to solve the Cauchy problem for one-dimensional hyperbolic equations. The mesh points of the proposed scheme are distributed along characteristics so that the solution on the stencil can be easily and accurately computed. This is very important in reducing errors of the scheme because many numerical errors are generated when the solution is estimated over grid points. In addition, when characteristics intersect, the proposed scheme combines corresponding grid points into one and assigns new characteristic to the point in order to improve computational efficiency. Numerical experiments on the inviscid Burgers' equation have been presented.

1. Introduction

These days, accurate and efficient computational methods are highly demanded for solving problems in computational fluid dynamics (CFD) [1]. For flow problems with complicated structures and a long range of characteristic scale dependencies, high resolution is necessary in order to extract the structural information correctly. In addition, if the flow involves shocks, the numerical scheme should be non-oscillatory near discontinuities and should avoid large dissipations. Weighted essentially non-oscillatory (WENO) schemes are high order numerical schemes, which are not oscillatory near shocks [4, 9]. However, WENO schemes are usually not optimal for computing turbulent flows or aero-acoustic fields, because they can lead to a significant damping of the turbulent or acoustic fluctuations [1]. There have been many researches to improve the dissipation of the WENO scheme including hybrid WENO schemes, which uses WENO scheme locally near discontinuities and applies other methods in smooth regions [5]. In this paper, a numerical scheme is introduced to

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solve the Cauchy problem for one-dimensional hyperbolic equations accurately without any numerical oscillations near discontinuities.

Method of characteristics (e.g. method of Massau) is a well-known analytical method, which determines the solutions along characteristics [6, 7, 8]. But it can be applied only when characteristics do not intersect and when solutions are sufficiently smooth. The moving mesh method [2, 3] is a finite difference scheme, whose grids are allowed to move in time. At each time step, grid points are spread over the spatial domain and solutions are estimated over the points. Recently this moving mesh method has been applied to solve partial differential equations [3], including Navier-Stokes equations [2]. However, for the moving mesh method, when locations of mesh points are changed in time, old and new locations of points may not be on the same characteristics. We propose a numerical scheme, which distributes mesh points along characteristics and, contrary to the moving mesh method, eliminates tangling of characteristics, so that the conservation laws can be solved accurately and efficiently without numerical oscillations or dissipations.

Let us consider the Cauchy problem for a quasi-linear hyperbolic partial differential equation (PDE)

$$\frac{\partial u}{\partial t} + b(x,t,u)\frac{\partial u}{\partial x} = c(x,t,u), \ t > 0 \tag{1}$$

with the initial condition $u(x,0) = u_0(x)$. Mattheij et al [7] recently reviewed analytical and numerical aspects of the equation (1). This PDE can be written as a system of ordinary differential equations (ODEs) [7],

$$\frac{dx}{dt} = b(x,t,u), \tag{2}$$

$$\frac{du}{dt} = c(x,t,u). \tag{3}$$

Equation (2) determines the locations of characteristics in time, while (3) defines the values of the solution along characteristics.

When $b(x,t,u) = b(u)$ and $c(x,t,u) = 0$, we can define a flux function $f(u)$ as

$$f(u) = \int_{u_0}^{u} b(v)dv,$$

where u_0 is some reference value for u, and (1) can be written as the hyperbolic conservation laws [6, 7],

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \ t > 0. \tag{4}$$

We will consider the Cauchy problem for (4) with the initial condition $u(x,0) = u_0(x)$ in this study.

2. Stencil along characteristics

Let us fix $\Delta t > 0$ and set $t = t_0 = 0$. Let $\{x_j,\ j = 0, 1, \ldots, N\}$ be the spatial mesh points at $t = t_0$. We can estimate values of the solution at $(x, t) = (x_j, t_0)$ from the initial condition $u_0(x)$ and let u_j denote $u(x_j, t_0)$. For each j, we can find the characteristic through x_j by solving (2), and let y_j be the spatial location on the curve at $t = t_0 + \Delta t$. In fact, characteristics are straight lines possibly having different slopes at different x_j's since conservation laws are considered. For example, x_j's ($j = 0, 1, \ldots, 4$) at $t = t_0$ in Figure 1 move to y_j's at $t = t_0 + \Delta t$ following characteristics. If no characteristics intersect for $t_0 < t < t_0 + \Delta t$ as in Figure 1, then we set $t_1 = t_0 + \Delta t$ and proceed to $t = t_1 + \Delta t$. Since x_j and y_j are on the same characteristic line and $c(x, t, u) = 0$, the solution u at (y_j, t_1) is $u(y_j, t_1) = u_j$ from (3).

Now let us consider the propagation of grid points from $t = t_1$ to $t = t_1 + \Delta t$. Suppose that some characteristics intersect before $t = t_1 + \Delta t$ so that a shock occurs. For example, characteristics through y_0 and y_3 in Figure 1 intersect at $t = t_0^*$, and those through y_1 and y_3 at $t = t_1^*$. We then find the earliest time when characteristics intersect and a shock first occurs. In Figure 1, t_2^* is the first such moment.

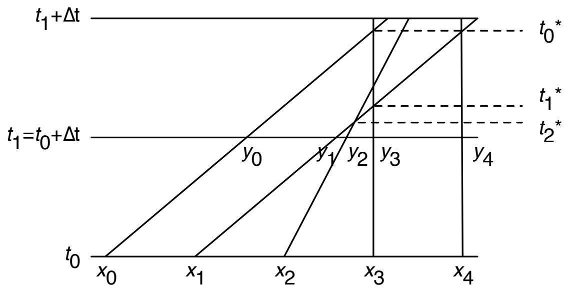

FIGURE 1. Characteristics for the conservation law

Then, we set t_2^* as t_2. We go back to y_j's at $t = t_1$ and let these points propagate along the characteristics up to t_2 only as in Figure 2 to obtain z_j's. In this way no shock occurs for $t_1 < t < t_2$. Note that we have $z_1 = z_2$ since characteristics from y_1 and y_2 intersect. From $t = t_2$ and after, we assign a new characteristic through $z_1 = z_2$ and let it propagate satisfying the Rankine-Hugoniot jump condition. That is, when we consider the propagation from $t = t_2$ to $t_2 + \Delta t$, z_0, z_3 and z_4 in Figure 2 propagate along their characteristics while z_1 and z_2 satisfy the Rankine-Hugoniot jump condition. In fact, when numerically implemented, z_2 can be eliminated from consideration to improve computational efficiency.

The proposed algorithm for (4) for $t \leq T$ can be summarized as follows

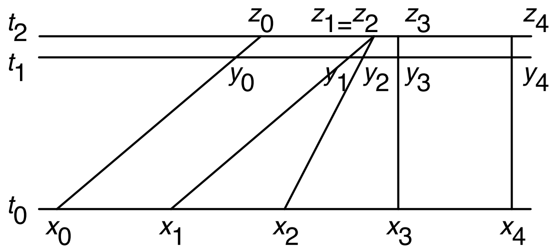

FIGURE 2. Stencil after the shock formulation

(1) Generate spatial mesh points x_j's at $t = 0$.

(2) Estimate values of the solution u_j's on those points using the initial condition.

(3) Repeat the following while $t < T$.
   (a) Estimate new locations of mesh points y_j's at $t + \Delta t$ by moving the mesh points along characteristics, that is, by solving (2).
   (b) Connect points x_j's at t and y_j's at $t + \Delta t$ along characteristics.
   (c) If no curves intersect at step (3b), set t to $t + \Delta t$, rename y_j's as x_j's and go to step (3a).
   (d) If some curves intersect at step (3b),
      (i) Find the earliest time t^* when curves intersect.
      (ii) For the grid points intersecting at t^*, merge the points into one, and from now on let the point propagate satisfying the Rankine-Hugoniot jump condition.
      (iii) For the other mesh points, estimate the locations y_j^*'s at t^*.
      (iv) Set t to t^*, rename y_j^*'s as x_j's and go to step (3a).

At each time step, the moving mesh method spreads grid points over the spatial domain and estimates solutions over those points. However, when locations of mesh points are changed in time, old and new locations for the fixed index may not be on the same characteristics for the moving mesh method. That is, j's spatial grids at t and $t + \Delta t$ may not be on the same characteristics, which imply that values of the solution need to be estimated at each time step even when $c(x, t, u) = 0$ in (3). On the other hand, the proposed scheme solves the conservation laws accurately and efficiently without numerical oscillations or dissipations because characteristics can be easily untangled under the current scheme. The proposed method seems to be appropriate for the Cauchy problem or a Riemann problem whose solution may contain a discontinuity.

3. Computational experiments

Let us consider the inviscid Burgers' equation for $u(x,t)$, where $b(x,t,u) = u$ in (4):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{5}$$

with an initial condition $u(x,0) = u_0(x)$. The spatial domain has been divided into 50 bins and $\Delta t/\Delta x = 0.7$.

Figure 3 shows the solution for the Riemann problem for $T = 2$

$$u(x,0) = \begin{cases} 1 & \text{if } x \in [-5,0] \\ 0 & \text{if } x \in (0,5]. \end{cases}$$

A shock initially occurs at $x = 0$ and propagates following the Rankine-Hugoniot jump condition, $\frac{dx}{dt} = \frac{1}{2}$. For $x < 0$ and $x > 0$, characteristic curves are straight line parallel to $\frac{dx}{dt} = 1$ and $\frac{dx}{dt} = 0$, respectively. Since the stencil is spread along characteristics, there is no numerical diffusion or oscillation.

Note that when we find the earliest moment characteristics intersect, it suffices to consider the earliest moment two neighboring characteristics intersect. For example, in case of Burgers' equation, if x_j and x_{j+1} at $t = t_n$ move to y_j and y_{j+1} at $t = t_n + \Delta t$, and if $y_{j+1} < y_j$, then characteristics intersect at some t^* $(t_n < t^* < t_n + \Delta t)$, where t^* satisfies

$$x_{j+1} - x_j : y_j - y_{j+1} = t^* - t_n : t_n + \Delta t - t^*.$$

That is, we can find t^* by

$$t^* = t_n + \frac{x_{j+1} - x_j}{(x_{j+1} - x_j) + (y_j - y_{j+1})}\Delta t. \tag{6}$$

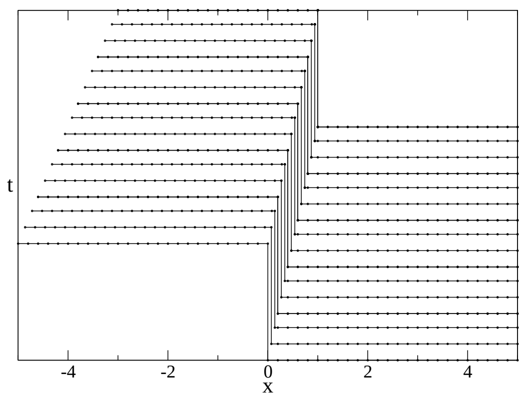

FIGURE 3. The solution of the Burgers' equation with a Riemann initial condition

When the equation (5) is solved for $t \in [0, 1]$ with the initial condition

$$u(x, 0) = \sin(\pi x/2),\ x \in [-1, 1]$$

characteristics move to the left for $x < 0$ and to the right for $x > 0$. That is, a rarefaction is formulated as in Figure 4 (Left). Solutions from the Godunov, WENO and proposed schemes at $t = 1$ have been compared in Figure 4 (Right). The first order accurate Godunov scheme [7] and the fifth order WENO scheme [9, 4] are used in this study. The result from the proposed scheme is as accurate as that from the WENO method, and gives better accuracy than that from the Godunov method. Note that the proposed scheme is a lot simpler than the WENO scheme when implemented, which shows superior efficiency to the WENO method.

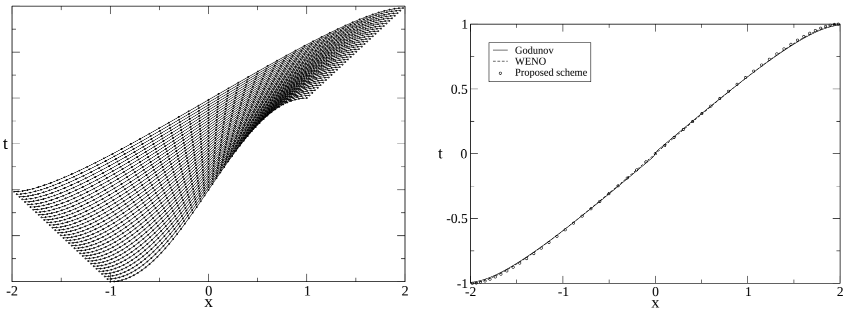

FIGURE 4. The solution of the Burgers' equation. The initial condition is a sine function, which generates a rarefaction. (Left) Solutions from the proposed scheme in time, (Right) Solutions from the Godunov, WENO and proposed schemes at $t = 1$.

When the initial condition is given by a cosine function

$$u(x, 0) = \cos(\pi x),\ x \in [0, 1],$$

the solution propagates as in Figure 5 (Left). The equation is solved for $t \leq 0.32$. Characteristic curves for $x < 0.5$ move to the right and those for $x > 0.5$ to the left. Then shocks occur at $x = 0.5$ and propagate satisfying the Rankine-Hugoniot jump condition, $\frac{dx}{dt} = 0$.

Solutions from the Godunov, WENO and proposed schemes at $t = 0.3$ have been compared in Figure 5 (Right). As in the case of the rarefaction, the proposed scheme is superior to the Godunov method in accuracy and to the WENO scheme in efficiency.

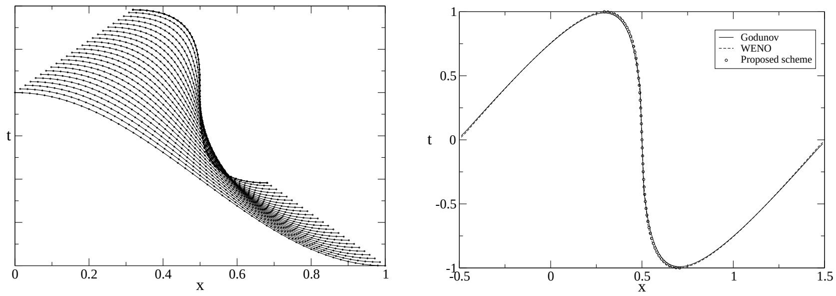

FIGURE 5. The solution of the Burgers' equation. The initial condition is a cosine function, which generates a shock. (Left) Solutions from the proposed scheme in time, (Right) Solutions from the Godunov, WENO and proposed schemes at $t = 0.3$.

4. Conclusions

A numerical scheme is proposed, whose spatial grids move in time along characteristic curves. A numerical oscillation of the solution to the conservation laws is usually caused when several characteristic curves are tangled during the computation. Since the propagation of grids occur only along characteristic curves, they cannot be tangled in the proposed scheme and thus numerical oscillation cannot happen. In addition, numerical dissipation may occur in the proposed scheme not due to the inaccuracy of the scheme but due to the inaccuracy of the ODE solver used. We are working on the hyperbolic problems whose characteristics are not straight lines but curved.

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