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MIXED VECTOR *FQ*-IMPLICIT VARIATIONAL INEQUALITY WITH LOCAL NON-POSITIVITY

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Abstract. This paper introduces a local non-positivity of two set-valued mappings (F, Q) and considers the existences and properties of solutions for set-valued mixed vector FQ-implicit variational inequality problems and set-valued mixed vector FQ-complementarity problems in the neighborhood of a point belonging to an underlined domain K of the set-valued mappings, where the neighborhood is contained in K.

This paper generalizes and extends many results in [1, 3-7].

1. Introduction

F-complementarity problem (F-CP); finding $x \in K$ such that

$$\langle Tx, x\rangle + F(x) \;=\; 0 \text{ and } \langle Tx, y\rangle + F(y) \;\geq\; 0 \text{ for all } y \in K,$$

and corresponding variational inequality problem;

finding $x \in K$ such that

$$\langle Tx, y - x\rangle + F(y) - F(x) \;\geq\; 0 \text{ for all } y \in K,$$

where K is a nonempty closed and convex cone of a real Banach space X with its dual X^*, $T : K \to X^\star$ is a mapping and $F : K \to (-\infty, +\infty)$ is a positively homogeneous and convex function, were firstly considered in [7].

In 2003, Fang and Huang [1] considered a vector F-complementarity problem with demi-pseudomonotone mappings in Banach spaces by considering the solvability of the problems. Huang and Li [3] studied a scalar F-implicit variational inequality problem and another F-implicit complementarity problem in Banach spaces in 2004. Recently, the result of the scalar case in [3] was extended and generalized to the vector case by Li and Huang [6] . The equivalence between the F-implicit variational inequality problem and F-implicit complementarity problem was presented and some new existence theorems of solutions for F-implicit variational inequality problems were also proved.

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In 2007, Lee, Khan, and Salahuddin [5] generalized some results of [3, 6] to more generalized vector case. They introduced a generalized vector F-implicit complementarity problem and corresponding generalized vector F-implicit variational inequality problem in Banach spaces and proved the equivalence between them under certain assumptions. Furthermore, they derived some new existence theorems of solutions for the generalized vector F-implicit complementarity problems and the generalized vector F-implicit variational inequality problems under some suitable assumptions without any monotonicity.

Recently, the following mixed vector FQ-implicit variational inequality problem (FQ-VI) and corresponding mixed vector FQ-implicit complementarity problems (FQ-CP) for set-valued mappings were considered in [4];

(FQ-VI); find $x \in K$ such that $p - s + w - z \in P(x)$ for any $p \in Q(x, g(y))$, $s \in Q(x, h(x))$, $w \in F(g(y))$, and $z \in F(h(x))$, where $y \in K$.

(FQ-CP); find $x \in K$ such that

(a) $p + w \in P(x)$ for any $p \in Q(x, g(y))$ and $w \in F(g(y))$, where $y \in K$, and

(b) $s + z = 0$ for any $s \in Q(x, h(x))$ and $z \in F(h(x))$, where K is a nonempty closed convex cone of a real Banach space X and $\{P(x) : x \in K\}$ is a family of nonempty pointed closed convex cones with the apex at the origin in a real Banach space Y. Mappings g, $h : K \to K$ are single-valued, $F : K \to 2^Y$ and $Q : K \times K \to 2^Y$ are set-valued.

The following Theorem A and Theorem B in [4] show the equivalence between (FQ-VI) and (FQ-CP) and some existence theorems of solutions for them under some suitable assumptions without monotonicity, respectively.

Theorem A. *Assume that a set-valued mapping $F : K \to 2^Y$ is positively homogeneous, a set-valued mapping $Q : K \times K \to 2^Y$ is also positively homogeneous in the second argument and $g : K \to K$ is surjective. Then (FQ-VI) is equivalent to (FQ-CP).*

Theorem B. *Let K be a nonempty closed convex subset of X and $P : K \to 2^Y$ be upper semicontinuous on K. Assume that*

(a) *g, $h : K \to K$ are continuous, $F : K \to 2^Y$ is lower semicontinuous and $Q : K \times K \to 2^Y$ is lower semicontinuous in two arguments,*

(b) *there exists a single-valued mapping $T : K \times K \to Y$ satisfying*

(b1) *for $x \in K$, $T(x, x) \in P(x)$,*

(b2) *for x, $y \in K$,*

$$a - b + c - d - T(x, y) \in P(x)$$

for any $a \in Q(x, g(y))$, $b \in Q(x, h(x))$, $c \in F(g(y))$ and $d \in F(h(x))$,

(b3) *for $x \in K$ the set $\{y \in K : T(x, y) \notin P(x)\}$ is convex,*

(c) *there exists a nonempty compact convex subset D of K such that for all $x \in K \setminus D$ there exists a $y \in D$ satisfying $a - b + c - d \notin P(x)$ for some $a \in Q(x, g(y))$, $b \in Q(x, h(x))$, $c \in F(g(y))$ and $d \in F(h(x))$.*

Then (FQ-VI) has a solution. Furthermore, the solution set of (FQ-VI) is closed.

This paper introduces a local non-positivity of set-valued mappings (F, Q) and considers the existences and properties of solutions for (FQ-VI) and (FQ-CP) in the neighborhood of a point belonging to an underlined domain K of the set-valued mappings, where the neighborhood is contained in K.

This paper generalizes and extends many results in [1, 3-7].

2. Preliminaries

Remark that $P(x)$, $x \in K$ is a closed set such that
(i) $\lambda P(x) \subset P(x)$, $\lambda > 0$, $x \in K$,
(ii) $P(x) + P(x) \subset P(x)$, $x \in K$,
(iii) $P(x) \cap (-P(x)) = \{0\}$, $x \in K$.

An ordered Banach space $(Y, P(x))$ is a real Banach space with an ordering defined by a closed cone $P(x) \subset Y$ as for any y, $z \in Y$,

$$y \geq z \quad \text{if and only if} \quad y - z \in P(x),$$
$$y \ngeq z \quad \text{if and only if} \quad y - z \notin P(x).$$

Remark that

$$z \leq 0 \quad \text{if and only if} \quad z \in -P(x),$$
$$z \nleq 0 \quad \text{if and only if} \quad z \notin -P(x),$$
$$z \geq 0 \quad \text{if and only if} \quad z \in P(x),$$
$$z \ngeq 0 \quad \text{if and only if} \quad z \notin P(x).$$

Lemma 2.1 ([1]). *Let (Y, P) be an ordered Banach space induced by a pointed closed cone P. Then $x + y \in P$ for x, $y \in P$.*

Definition 2.1 ([4]). Let X, Y be two vector spaces and K be a cone of X. A set-valued mapping $F : K \to 2^Y$ is said to be positively homogeneous if $F(\alpha x) = \alpha F(x)$ for all $x \in K$ and $\alpha \geq 0$. F is said to be linear if $F(\alpha x + \beta y) = \alpha F(x) + \beta F(y)$ for x, $y \in K$, $\alpha + \beta = 1$, α, $\beta \geq 0$.

Definition 2.2. A set-valued mapping $W : K \subset X \to 2^Y$ is upper semicontinuous at $x_0 \in K$ if every open set V containing $W(x_0)$ there exists an open set U containing x_0 such that $W(U) \subset V$. W is lower semicontinuous at $x_0 \in K$ if for every open set V intersecting $W(x_0)$ there exists an open set U containing x_0 such that $W(x) \cap V \neq \emptyset$ for every $x \in U$. W is upper semicontinuous (lower semicontinuous) on K if it is upper semicontinuous (lower semicontinuous) at every point of K. W is continuous on K if it is both upper semicontinuous and lower semicontinuous on K.

Lemma 2.2. *Let $W : X \to 2^Y$ be a set-valued mapping and $x_0 \in X$.*

(i) *W is upper semicontinuous at x_0 if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and for any net $\{y_\alpha\}$ in Y with $y_\alpha \in W(x_\alpha)$ such that $y_\alpha \to y_0$ in Y, we have $y_0 \in W(x_0)$.*

(ii) *W is lower semicontinuous at x_0 if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$, and for any $y_0 \in W(x_0)$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in W(x_\alpha)$ and $y_\alpha \to y_0$.*

Lemma 2.3 ([2])**.** *Let $W : X \to 2^Y$ be a set-valued mapping. If for any $x \in X$, $W(x)$ is compact, then W is upper semicontinuous at x_0 if and only if for any net $\{x_\alpha\} \subset X$ such that $x_\alpha \to x_0$ and for every $y_\alpha \in W(x_\alpha)$, there exists $y_0 \in W(x_0)$ and a subnet $\{y_{\alpha_\beta}\}$ of $\{y_\alpha\}$ such that $y_{\alpha_\beta} \to y_0$.*

3. Main results

Unless otherwise specified, we assume that K is a nonempty closed convex cone of a real Banach space X and $\{P(x) : x \in K\}$ is a family of nonempty pointed closed convex cones with the apex at the origin in a real Banach space Y.

Definition 3.1. Let $g, h : K \to K$ be single-valued mappings and $F : K \to 2^Y$, $Q : K \times K \to 2^Y$ set-valued mappings. Let $P : K \to 2^Y$ be a set-valued mapping with nonempty pointed closed convex cones with the apex at the origin in Y. (F, Q) is said to be locally non-positive at $x_0 \in K$ with respect to (g, h) if there exist a neighborhood $N(x_0)$ of x_0 and $z_0 \in K \cap \mathrm{Int}N(x_0)$ such that $a - b + c - d \in -P(x)$ for any $a \in Q(x, g(z_0))$, $b \in Q(x, h(x))$, $c \in F(g(z_0))$ and $d \in F(h(x))$ for $x \in K \cap \partial N(x_0)$, the boundary of $N(x_0)$.

Example 3.1. Let $X = Y = \mathbb{R}$, $K = [0, \infty)$ and $P(x) = [0, \infty)$ for all $x \in K$. Define mappings g, $h : K \to K$ by $g(x) = 2x$ and $h(x) = 2x$, set-valued mappings $F : K \to 2^{\mathbb{R}}$ by $F(x) = \left[\frac{1}{2}x, x\right]$, $Q : K \times K \to 2^{\mathbb{R}}$ by $Q(x, y) = \left[\frac{2}{3}(x + y), x + y\right]$, then (F, Q) is locally non-positive at $x_0 = 0 \in K$ with respect to (g, h). If we take a neighborhood $N(0) = \left(-\frac{1}{2}, \frac{1}{2}\right)$ of $x_0 = 0$ and $z_0 = \frac{1}{4} \in K \cap \mathrm{Int}N(0) = \left[0, \frac{1}{2}\right)$, then for the unique element $x = \frac{1}{2}$ of $K \cap \partial N(0) = \left\{\frac{1}{2}\right\}$, we have for any $a \in Q\left(\frac{1}{2}, g\left(\frac{1}{4}\right)\right)$, $b \in Q\left(\frac{1}{2}, h\left(\frac{1}{2}\right)\right)$, $c \in F\left(g\left(\frac{1}{4}\right)\right)$ and $d \in F\left(h\left(\frac{1}{2}\right)\right)$,

$$a - b + c - d \in -K.$$

In fact, $Q\left(\frac{1}{2}, g\left(\frac{1}{4}\right)\right) = Q\left(\frac{1}{2}, \frac{1}{2}\right) = \left[\frac{2}{3}, 1\right]$, $Q\left(\frac{1}{2}, h\left(\frac{1}{2}\right)\right) = Q\left(\frac{1}{2}, 1\right) = \left[1, \frac{3}{2}\right]$, $F\left(g\left(\frac{1}{4}\right)\right) = F\left(\frac{1}{2}\right) = \left[\frac{1}{4}, \frac{1}{2}\right]$, $F\left(h\left(\frac{1}{2}\right)\right) = F(1) = \left[\frac{1}{2}, 1\right]$, thus

$$1 - 1 + \frac{1}{2} - \frac{1}{2} = 0 \in -K.$$

Theorem 3.1. *Let K be a nonempty closed and convex subset of X. Let $P : K \to 2^Y$ be a set-valued mapping with nonempty pointed closed convex cones with the apex at the origin in Y. Assume that*

(a) *single-valued mappings g, $h : K \to K$ are continuous and set-valued mappings $F : K \to 2^Y$, $Q : K \times K \to 2^Y$ are continuous and P is upper semicontinuous,*

(b) *a single-valued mapping $T : K \times K \to Y$ satisfies*

(b1) *for $x \in K$, $T(x,x) \in P(x)$,*

(b2) *for x, $y \in K$,*

$$a - b + c - d - T(x,y) \in P(x)$$

for any $a \in Q(x, g(y))$, $b \in Q(x, h(x))$, $c \in F(g(y))$ and $d \in F(h(x))$,

(b3) *for $x \in K$ the set $\{y \in K : T(x,y) \notin P(x)\}$ is convex,*

(c) *(F,Q) is locally non-positive at $x_0 \in K$ with respect to (g,h) and there exists a nonempty compact convex subset D of $K \cap N(x_0)$ such that for all $x \in (K \cap N(x_0)) \setminus D$ there exists $y \in D$ satisfying*

$$a - b + c - d \notin P(x)$$

for any $a \in Q(x, g(y))$, $b \in Q(x, h(x))$, $c \in F(g(y))$ and $d \in F(h(x))$,

(d) *g, h and F are linear and Q is linear in the second argument.*

Then (FQ-VI) has a solution in the neighborhood of x_0, that is, there exists $x^* \in K \cap N(x_0)$ such that, for $y \in K$

$$a^* - b^* + c - d^* \in P(x^*)$$

for any $a^* \in Q(x^*, g(y))$, $b^* \in Q(x^*, h(x^*))$, $c \in F(g(y))$ and $d^* \in F(h(x^*))$.

Proof. Since (F,Q) is locally non-positive at $x_0 \in K$ with respect to (g,h), we can assume that $N(x_0)$ is a closed and convex set without loss of generality. Since $K \cap N(x_0)$ is also closed and convex, from Theorem B, (FQ-VI) has a solution $x^* \in K \cap N(x_0)$ such that, for $y \in K \cap N(x_0)$

$$a^* - b^* + c - d^* \in P(x^*) \tag{3.1}$$

for any $a^* \in Q(x^*, g(y))$, $b^* \in Q(x^*, h(x^*))$, $c \in F(g(y))$ and $d^* \in F(h(x^*))$.

Now we show that for $y \in K$, (3.1) also holds.

(i) If $x^* \in K \cap \mathrm{Int}N(x_0)$, then $N(x_0)\backslash\{x^*\}$ is a neighborhood of the origin and so it is absorbing. For any $y \in K$, there exists $t \in (0,1)$ such that $t(y-x^*) \in N(x_0)\backslash\{x^*\}$ and so $y_t := ty + (1-t)x^* \in K \cap N(x_0)$. Hence

$$a_t^* - b^* + c_t - d^* \in P(x^*) \tag{3.2}$$

for any $a_t^* \in Q(x^*, g(y_t))$, $b^* \in Q(x^*, h(x^*))$, $c_t \in F(g(y_t))$ and $d^* \in F(h(x^*))$.

On the other hand, the following set

$$A = \{y \in K : a - b + c - d \in P(x) \text{ for any } a \in Q(x, g(y)), b \in Q(x, h(x)) \\ c \in F(g(y)) \text{ and } d \in F(h(x))\},$$

is convex for all $x \in K$. In fact, if y_1, $y_2 \in A$, then for $x \in K$,

$$a_1 - b + c_1 - d \in P(x)$$

for any $a_1 \in Q(x, g(y_1))$, $b \in Q(x, h(x))$, $c_1 \in F(g(y_1))$ and $d \in F(h(x))$ and

$$a_2 - b + c_2 - d \in P(x)$$

for any $a_2 \in Q(x, g(y_2))$, $b \in Q(x, h(x))$, $c_2 \in F(g(y_2))$ and $d \in F(h(x))$. Hence for $t \in (0,1)$, from the condition (d), we have, for $x \in K$

$$\begin{aligned} &ta_1 + (1-t)a_2 - b + tc_1 + (1-t)c_2 - d \in P(x) \\ \text{for any } &ta_1 + (1-t)a_2 \in tQ(x, g(y_1)) + (1-t)Q(x, g(y_2)) \\ &\qquad = Q(x, g(ty_1 + (1-t)y_2)), \\ &b \in Q(x, h(x)), \\ &tc_1 + (1-t)c_2 \in tF(g(y_1)) + (1-t)F(g(y_2)) \\ &\qquad = F(g(ty_1 + (1-t)y_2)), \text{ and} \\ &d \in F(h(x)). \end{aligned}$$

Hence $ty_1 + (1-t)y_2 \in A$, which shows that A is convex. Thus by the continuities of g, h, F and Q from (3.2) we have for $y \in K$

$$a^* - b^* + c - d^* \in P(x^*)$$

for any $a^* \in Q(x^*, g(y))$, $b^* \in Q(x^*, h(x^*))$, $c \in F(g(y))$ and $d^* \in F(h(x^*))$.

(ii) Since (F, Q) is locally non-positive at $x_0 \in K$ with respect to (g, h), for $x^* \in K \cap \partial N(x_0)$ there exists $z_0 \in K \cap \mathrm{Int}N(x_0)$ such that

$$a_0 - b^* + c_0 - d^* \in -P(x^*) \tag{3.3}$$

for any $a_0 \in Q(x^*, g(z_0))$, $b^* \in Q(x^*, h(x^*))$, $c_0 \in F(g(z_0))$ and $d^* \in F(h(x^*))$.

By a similar method, for any $y \in K$, there exists a $t \in (0,1)$ such that $t(y - z_0) \in N(x_0)\backslash\{z_0\}$, so $z_t := ty + (1-t)z_0 \in K \cap N(x_0)$. Hence it follows from (3.1)

$$a_t - b^* + c_t - d^* \in P(x^*) \tag{3.4}$$

for any $a_t \in Q(x^*, g(z_t))$, $b^* \in Q(x^*, h(x^*))$, $c_t \in F(g(z_t))$ and $d^* \in F(h(x^*))$. Letting $t \to 0$ in (3.4), we obtain

$$a_0 - b^* + c_0 - d^* \in P(x^*) \tag{3.5}$$

for any $a_0 \in Q(x^*, g(z_0))$, $b^* \in Q(x^*, h(x^*))$, $c_0 \in F(g(z_0))$ and $d^* \in F(h(x^*))$. Thus by (3.3) and (3.5),

$$a_0 - b^* + c_0 - d^* = 0 \tag{3.6}$$

for any $a_0 \in Q(x^*, g(z_0))$, $b^* \in Q(x^*, h(x^*))$, $c_0 \in F(g(z_0))$ and $d^* \in F(h(x^*))$. Thus by (3.4) and (3.6), we have

$$ta_t^* + (1-t)b^* - a_0 + tc_t + (1-t)d^* - c_0 \in P(x^*) \tag{3.7}$$

for any $a_t^* \in Q(x^*, g(z_t))$, $b^* \in Q(x^*, h(x^*))$, $a_0 \in Q(x^*, g(z_0))$, $c_t \in F(g(z_t))$, $d^* \in F(h(x^*))$ and $c_0 \in F(g(z_0))$.

Hence by (3.6) and (3.7)

$$a_t^* - b^* + c_t - d^* \in P(x^*) \tag{3.8}$$

for any $a_t^* \in Q(x^*, g(z_t))$, $b^* \in Q(x^*, h(x^*))$, $c_t \in F(g(z_t))$, and $d^* \in F(h(x^*))$. Letting $t \to 1$ in (3.8), by the condition (d) we have

$$a^* - b^* + c - d^* \in P(x^*)$$

for any $a^* \in Q(x^*, g(y))$, $b^* \in Q(x^*, h(x^*))$, $c \in F(g(y))$ and $d^* \in F(h(x^*))$. Hence by (i) and (ii), the proof is completed. □

Letting $D = K$ in the condition (c) of Theorem 3.1, we have the following result as a corollary.

Theorem 3.2. *Let K be a nonempty compact and convex subset of a real Banach space X, and assume that the condition* (a), (b) *and* (d) *of Theorem 3.1 hold with the following condition* (c)′ *instead of* (c) *of Theorem 3.1*;

(c)′ *the mappings (F, Q) is locally non-positive at $x_0 \in K$ with respect to (g, h).*

Then (FQ-VI) has a solution in the neighborhood of x_0, that is, there exists $x^* \in K \cap N(x_0)$ such that, for $y \in K$

$$a^* - b^* + c - d^* \in P(x^*)$$

for any $a^* \in Q(x^*, g(y))$, $b^* \in Q(x^*, h(x^*))$, $c \in F(g(y))$ and $d^* \in F(h(x^*))$.

Theorem 3.3. *Assume that*

(a) *g, $h : K \to K$ are continuous and surjective, set-valued mappings $F : K \to 2^Y$ and $Q : K \times K \to 2^Y$ are continuous and P is upper semicontinuous,*

(b) *a single-valued mapping $T : K \times K \to Y$ satisfies*

  (b1) *for $x \in K$, $T(x, x) \in P(x)$,*

  (b2) *for x, $y \in K$,*

$$a - b + c - d - T(x, y) \in P(x)$$

  for any $a \in Q(x, g(y))$, $b \in Q(x, h(x))$, $c \in F(g(y))$ and $d \in F(h(x))$,

  (b3) *for $x \in K$ the set $\{y \in K : T(x, y) \notin P(x)\}$ is convex,*

(c) *(F, Q) is locally non-positive at $x_0 \in K$ with respect to (g, h), and there exists a nonempty compact and convex subset D of $K \cap N(x_0)$ such that for all $x \in K \cap N(x_0) \setminus D$ there exists $y \in D$ satisfying*

$$a - b + c - d \notin P(x)$$

for any $a \in Q(x, g(y))$, $b \in Q(x, h(x))$, $c \in F(g(y))$ and $d \in F(h(x))$.

(d) *g and F are linear and Q is linear in the second argument.*

Then (FQ-CP) has a solution in the neighborhood of x_0, that is, there exists $x^* \in K \cap N(x_0)$ such that,

$$a^* + b^* = 0 \quad \textit{for any } a^* \in Q(x^*, h(x^*)) \textit{ and } b^* \in F(h(x^*))$$

and for $y \in K$*,*

$$a_y^* + c \in P(x^*) \quad \text{for any } a_y^* \in Q(x^*, g(y)) \text{ and } c \in F(g(y)).$$

Proof. The conclusion follows directly from Theorem A and Theorem 3.1. □

Remark 3.1. Though Theorem A is used to prove Theorem 3.3 and Theorem B is used to prove Theorem 3.1 and Theorem 3.2, Theorem 3.1, 3.2 and 3.3 extend and generalize Theorems A and B.

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