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ON SOME CLASSES OF GENERALIZED QUASI-EINSTEIN MANIFOLDS

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Abstract. The object of the present paper is to study the generalized quasi-Einstein manifolds satisfying some conditions. Finally the existence of such manifolds is ensured by several interesting examples.

1. Introduction

The notion of quasi-Einstein manifolds was introduced by M. C. Chaki and R. K. Maity [1]. A Riemannian manifold $(M^n, g)(n > 2)$ is said to be quasi-Einstein manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following:

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y), \tag{1.1}$$

where α, β are scalars of which $\beta \neq 0$ and A is a nowhere vanishing 1-form defined by $g(X, \rho) = A(X)$ for all vector fields X; ρ being a unit vector field, called the generator of the manifold. An n -dimensional manifold of this kind is denoted by $(QE)_n$. The scalars α, β are known as the associated scalars.

As a generalization of quasi-Einstein manifold, in [2], U. C. De and G. C. Ghosh introduced the notion of generalized quasi-Einstein manifold. A Riemannian manifold $(M^n, g)(n \geq 3)$ is said to be generalized quasi-Einstein manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following:

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma B(X)B(Y), \tag{1.2}$$

where α, β, γ are scalars of which $\beta \neq 0, \gamma \neq 0$ and A, B are nowhere vanishing 1-forms such that $g(X, \rho) = A(X)$, $g(X, \mu) = B(X)$ for all vector fields X. The unit vectors ρ and μ corresponding to the 1-forms A and B are orthogonal to each other. Also ρ and μ are known as the generators of the manifold. Such an n-dimensional manifold is denoted by $G(QE)_n$.

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The present paper deals with $G(QE)_n$ satisfying some conditions. The paper is organized as follows. Section 2 is concerned with the preliminaries. Section 3 is devoted to the study of Ricci-pseudosymmetric $G(QE)_n$. In Sections 4–7, we investigate the $G(QE)_n (n > 3)$ satisfying the conditions $C \cdot S = 0$, $\tilde{C} \cdot S = 0$, $W \cdot S = 0$ and $P \cdot S = 0$, where C, $\tilde{C}$, W and P respectively denote the conformal curvature tensor, concircular curvature tensor, quasi-conformal curvature tensor and projective curvature tensor. Then it is proved that in each of the case, either the associated scalars β and γ are equal or the curvature tensor R satisfies a definite condition.

In Section 8, we study conformally flat Ricci-semisymmetric $G(QE)_n (n > 3)$ and it is shown that if in a conformally flat Ricci-semisymmetric $G(QE)_n (n > 3)$, $\frac{r}{n-1}$ is not an eigenvalue of the Ricci-operator, then either the associated scalars β and γ are equal or the vector fields ρ and μ corresponding to the 1-forms A and B are co-directional. The last section provides the existence of proper $G(QE)_n$.

2. Preliminaries

In this section we will obtain some formulas of $G(QE)_n$, which will be required in the sequel. Let $\{e_i : i = 1, 2, \ldots, n\}$ be an orthonormal frame field at any point of $G(QE)_n$. Then setting $X = Y = e_i$ in (1.2) and taking summation over $i, 1 \leq i \leq n$, we obtain

$$r = n\alpha + \beta + \gamma, \tag{2.1}$$

where r is the scalar curvature of the manifold.

Also, from (1.2), we have

$$S(\rho, \rho) = \alpha + \beta, \tag{2.2}$$

$$S(\mu, \mu) = \alpha + \gamma, \tag{2.3}$$

and

$$S(\rho, \mu) = 0. \tag{2.4}$$

Let Q be the Ricci-operator, i.e., $g(QX, Y) = S(X, Y)$ for all X, Y.

3. Ricci-pseudosymmetric $G(QE)_n$

An n-dimensional Riemannian manifold (M^n, g) is called Ricci-pseudosymmetric [4] if the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent, where

$$(R(X, Y) \cdot S)(Z, U) = -S(R(X, Y)Z, U) - S(Z, R(X, Y)U), \tag{3.1}$$

$$Q(g, S)(Z, U; X, Y) = -S((X \wedge_g Y)Z, U) - S(Z, (X \wedge_g Y)U), \tag{3.2}$$

and

$$(X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y \tag{3.3}$$

for all vector fields X, Y, Z, U of M, R denotes the curvature tensor of M.

Then (M^n, g) is Ricci-pseudosymmetric if and only if

$$(R(X,Y)\cdot S)(Z,U) = L_S Q(g,S)(Z,U;X,Y) \tag{3.4}$$

holds on $U_s = \{x \in M : S \neq \frac{r}{n} g \text{ at } x\}$, where L_S is some function on U_S. If $R \cdot S = 0$, then M^n is called Ricci-semisymmetric. Every Ricci-semisymmetric manifold is Ricci-pseudosymmetric but the converse is not true [4].

In [2] De and Ghosh studied a Ricci-semisymmetric $G(QE)_n$. We now consider a Ricci-pseudosymmetric $G(QE)_n$. Then, from (3.1)–(3.4), we can write

$$\begin{aligned} & S(R(X,Y)Z,U) + S(Z,R(X,Y)U) \\ = \; & L_S[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) \\ & + g(Y,U)S(X,Z) - g(X,U)S(Y,Z)]. \end{aligned} \tag{3.5}$$

Using (1.2) in (3.5), we get

$$\begin{aligned} & \beta[A(R(X,Y)Z)A(U) + A(Z)A(R(X,Y)U)] \\ & + \gamma[B(R(X,Y)Z)B(U) + B(Z)B(R(X,Y)U)] \\ = \; & L_S[\beta\{g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) + g(Y,U)A(X)A(Z) \\ & - g(X,U)A(Y)A(Z)\} + \gamma\{g(Y,Z)B(X)B(U) \\ & - g(X,Z)B(Y)B(U) + g(Y,U)B(X)B(Z) - g(X,U)B(Y)B(Z)\}]. \end{aligned} \tag{3.6}$$

Setting $Z = \rho$ and $U = \mu$ in (3.6), we get

$$(\gamma - \beta)[R(X,Y,\rho,\mu) - L_S\{A(Y)B(X) - A(X)B(Y)\}] = 0, \tag{3.7}$$

which yields either $\beta = \gamma$ or

$$R(X,Y,\rho,\mu) = L_S\{A(Y)B(X) - A(X)B(Y)\}. \tag{3.8}$$

Hence we can state the following:

Theorem 3.1. *In a Ricci-pseudosymmetric $G(QE)_n (n > 3)$, either the associated scalars β and γ are equal or the curvature tensor R of the manifold satisfies the relation* (3.8).

Corollary 3.1. *In a Ricci-semisymmetric $G(QE)_n (n > 3)$, the associated scalars β and γ are equal* [2].

4. $G(QE)_n (n > 3)$ satisfying the condition $C \cdot S = 0$

The Weyl conformal curvature tensor C of type (1,3) of an n-dimensional Riemannian manifold $(M^n, g)(n > 3)$ is defined by [3]

$$\begin{aligned} C(X,Y)Z = \; & R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ & + g(Y,Z)QX - g(X,Z)QY] \\ & + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{4.1}$$

We now consider a $G(QE)_n(n > 3)$ satisfying the condition $C \cdot S = 0$. Then we have

$$S(C(X,Y)Z,U) + S(Z,C(X,Y)U) = 0. \tag{4.2}$$

Using (1.2) in (4.2), we obtain

$$\begin{aligned}\beta[A(C(X,Y)Z)A(U) + A(Z)A(C(X,Y)U)]\\ +\gamma[B(C(X,Y)Z)B(U) + B(Z)B(C(X,Y)U)] = 0.\end{aligned} \tag{4.3}$$

Setting $Z = \rho$ and $U = \mu$ in (4.3), we get

$$(\gamma - \beta)C(X,Y,\rho,\mu) = 0. \tag{4.4}$$

From (4.4), it follows that either $\beta = \gamma$ or

$$C(X,Y,\rho,\mu) = 0,$$

which further yields

$$\begin{aligned} R(X,Y,\rho,\mu) = & \frac{1}{n-2}[A(QY)B(X) - A(X)B(QY) \\ & +A(Y)B(QX) - A(QX)B(Y)] \\ & -\frac{r}{(n-1)(n-2)}\{A(Y)B(X) - A(X)B(Y)\}.\end{aligned} \tag{4.5}$$

Hence we can state the following:

Theorem 4.1. *If a $G(QE)_n(n > 3)$ satisfies the condition $C \cdot S = 0$, then either the associated scalars β and γ are equal or the curvature tensor R of the manifold satisfies the property* (4.5).

5. $G(QE)_n(n > 3)$ satisfying the condition $\tilde{C} \cdot S = 0$

The concircular curvature tensor $\tilde{C}$ of type (1,3) of an n-dimensional Riemannian manifold $(M^n, g)(n > 3)$ is defined by [3]

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{5.1}$$

for any vector fields $X, Y, Z \in \chi(M)$. Let us consider a $G(QE)_n(n > 3)$ satisfying the condition $\tilde{C} \cdot S = 0$. Then we have

$$S(\tilde{C}(X,Y)Z,U) + S(Z,\tilde{C}(X,Y)U) = 0. \tag{5.2}$$

By virtue of (1.2), it follows from (5.2) that

$$\begin{aligned}\beta[A(\tilde{C}(X,Y)Z)A(U) + A(Z)A(\tilde{C}(X,Y)U)]\\ +\gamma[B(\tilde{C}(X,Y)Z)B(U) + B(Z)B(\tilde{C}(X,Y)U)] = 0.\end{aligned} \tag{5.3}$$

Putting $Z = \rho$ and $U = \mu$ in (5.3), we get

$$(\gamma - \beta)[R(X,Y,\rho,\mu) - \frac{r}{n(n-1)}\{A(Y)B(X) - A(X)B(Y)\}] = 0. \tag{5.4}$$

This leads to the following:

Theorem 5.1. *In a $G(QE)_n(n > 3)$ with $\tilde{C} \cdot S = 0$, either the associated scalars β and γ are equal or the curvature tensor R of the manifold satisfies the following property*

$$R(X, Y, \rho, \mu) = \frac{r}{n(n-1)}\{A(Y)B(X) - A(X)B(Y)\}. \tag{5.5}$$

6. $G(QE)_n(n > 3)$ satisfying the condition $W \cdot S = 0$

In 1968, Yano and Sawaki [5] defined and studied a curvature tensor W of type (1,3) which includes both the conformal curvature tensor C and the concircular curvature tensor $\tilde{C}$ as special cases and is called quasi-conformal curvature tensor. The quasi-conformal curvature tensor W of type (1, 3) of a manifold $(M^n, g)(n > 3)$ is defined by

$$W(X, Y)Z = -(n-2)bC(X, Y)Z + [a + (n-2)b]\tilde{C}(X, Y)Z, \tag{6.1}$$

where a and b are arbitrary constants not simultaneously zero. In particular, if $a = 1$, $b = 0$, then W reduces to the concircular curvature tensor and if $a = 1$ and $b = -\frac{1}{(n-2)}$, then W reduces to the conformal curvature tensor. Using the expression of the conformal and the concircular curvature tensor in (6.1), the quasi-conformal curvature tensor W of type (1, 3) can be written as

$$\begin{aligned} W(X, Y)Z \;=\; & aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y \\ & + g(Y, Z)QX - g(X, Z)QY] \\ & - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\{g(Y, Z)X - g(X, Z)Y\}. \end{aligned} \tag{6.2}$$

We now consider a $G(QE)_n(n > 3)$ satisfying the condition $W \cdot S = 0$. Then we have

$$S(W(X, Y)Z, U) + S(Z, W(X, Y)U) = 0. \tag{6.3}$$

In view of (1.2), (6.3) yields

$$\begin{aligned} & \beta[A(W(X, Y)Z)A(U) + A(Z)A(W(X, Y)U)] \\ & + \gamma[B(W(X, Y)Z)B(U) + B(Z)B(W(X, Y)U)] = 0. \end{aligned} \tag{6.4}$$

Substituting $Z = \rho$ and $U = \mu$ in (6.4), we obtain

$$(\gamma - \beta)W(X, Y, \rho, \mu) = 0. \tag{6.5}$$

From (6.5), it follows that either $\beta = \gamma$ or

$$W(X, Y, \rho, \mu) = 0,$$

which implies that

$$\begin{aligned} aR(X, Y, \rho, \mu) \;=\; & -b[A(QY)B(X) - A(QX)B(Y) \\ & + A(Y)B(QX) - A(X)B(QY)] \\ & + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\{A(Y)B(X) - A(X)B(Y)\}. \end{aligned} \tag{6.6}$$

Thus we can state the following:

Theorem 6.1. *If a $G(QE)_n(n > 3)$ satisfies the condition $W \cdot S = 0$, then either the associated scalars β and γ are equal or the curvature tensor R of the manifold satisfies the property* (6.6).

7. $G(QE)_n(n > 3)$ satisfying the condition $P \cdot S = 0$

The Weyl projective curvature tensor P of type (1,3) of an n-dimensional Riemannian manifold $(M^n, g)(n > 3)$ is defined by [3]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] \tag{7.1}$$

for any vector fields X, Y, $Z \in \chi(M)$. Let us take a $G(QE)_n(n > 3)$ with $P \cdot S = 0$. Then we get

$$S(P(X,Y)Z,U) + S(Z,P(X,Y)U) = 0. \tag{7.2}$$

Using (1.2) in (7.2), we get

$$\begin{aligned} &\alpha[\tilde{P}(X,Y,Z,U) + \tilde{P}(X,Y,U,Z)] + \beta[A(P(X,Y)Z)A(U) \\ &+A(Z)A(P(X,Y)U)] + \gamma[B(P(X,Y)Z)B(U) \\ &+B(Z)B(P(X,Y)U)] = 0, \end{aligned} \tag{7.3}$$

where $\tilde{P}(X,Y,Z,U) = g(P(X,Y)Z,U)$. Setting $Z = \rho$ and $U = \mu$ in (7.3), we get

$$(\alpha+\gamma)\tilde{P}(X,Y,\rho,\mu) + (\alpha+\beta)\tilde{P}(X,Y,\mu,\rho) = 0. \tag{7.4}$$

In view of (7.1), we have from (7.4) that

$$\begin{aligned} & (n-1)(\gamma-\beta)R(X,Y,\rho,\mu) \\ = \; & (\alpha+\gamma)\{A(QY)B(X) - A(QX)B(Y)\} \\ & +(\alpha+\beta)\{A(X)B(QY) - A(Y)B(QX)\}, \end{aligned} \tag{7.5}$$

provided $\gamma - \beta \neq 0$. This leads to the following:

Theorem 7.1. *If a $G(QE)_n(n > 3)$ satisfies the condition $P \cdot S = 0$, then the curvature tensor R of the manifold satisfies the property* (7.5)*, provided $\beta \neq \gamma$.*

8. Conformally flat $G(QE)_n(n > 3)$ with $R(X,Y) \cdot S = 0$

Let us consider a conformally flat $G(QE)_n(n > 3)$. Then, from (4.1), we get

$$\begin{aligned} R(X,Y)Z \;=\; & \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ & +g(Y,Z)QX - g(X,Z)QY] \\ & -\frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{8.1}$$

Since the manifold satisfies $R(X,Y) \cdot S = 0$, we get

$$S(R(X,Y)Z,U) + S(Z,R(X,Y)U) = 0. \tag{8.2}$$

Using (8.1) in (8.2), we get

$$\begin{aligned} & g(Y,Z)S(QX,U) - g(X,Z)S(QY,U) \\ & +g(Y,U)S(QX,Z) - g(X,U)S(QY,Z) \\ = & \frac{r}{n-1}[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) \\ & +g(Y,U)S(X,Z) - g(X,U)S(Y,Z)]. \end{aligned} \tag{8.3}$$

Let λ be the eigenvalue of the endomorphism Q corresponding to an eigenvector X. Then $QX = \lambda X$, i.e., $S(X,U) = \lambda g(X,U)$ and hence

$$S(QX,U) = \lambda S(X,U). \tag{8.4}$$

By virtue of (8.4), it follows from (8.3) that

$$\left(\lambda - \frac{r}{n-1}\right)[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) \\ +g(Y,U)S(X,Z) - g(X,U)S(Y,Z)] = 0,$$

which yields

$$g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + g(Y,U)S(X,Z) - g(X,U)S(Y,Z) = 0, \tag{8.5}$$

provided $\lambda \neq \frac{r}{n-1}$. Again using (1.2) in (8.5), we get

$$\begin{aligned} & \beta[g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) + g(Y,U)A(X)A(Z) \\ & -g(X,U)A(Y)A(Z)] + \gamma[g(Y,Z)B(X)B(U) - g(X,Z)B(Y)B(U) \\ & +g(Y,U)B(X)B(Z) - g(X,U)B(Y)B(Z)] = 0, \quad \text{provided } \lambda \neq \frac{r}{n-1}. \end{aligned} \tag{8.6}$$

Setting $Z = \rho$ and $U = \mu$, we get

$$(\beta - \gamma)\{A(X)B(Y) - A(Y)B(X)\} = 0. \tag{8.7}$$

From (8.7), we get either $\beta = \gamma$ or

$$A(X)B(Y) = A(Y)B(X),$$

that is, the vector fields ρ and μ are co-directional. Thus we can state the following:

Theorem 8.1. *If, in a conformally flat Ricci-semisymmetric $G(QE)_n (n > 3)$, $\frac{r}{n-1}$ is not an eigenvalue of the Ricci-operator Q, then either the associated scalars β and γ of the manifold are equal or the vector fields ρ and μ corresponding to the 1-forms A and B respectively are co-directional.*

9. Some Examples of $G(QE)_n$

This section deals with several non-trivial examples of $G(QE)_n$.

Example 9.1. We define a Riemannian metric g on $\mathbb{R}^4$ by the formula

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], (i,j = 1,2,3,4), \tag{9.1}$$

where $p = \frac{e^{x^1}}{k^2}$ and k is a non-zero constant. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor and scalar curvature are given by

$$\Gamma^1_{22} = -\frac{p}{(1+2p)} = \Gamma^1_{33} = \Gamma^1_{44} = -\Gamma^1_{11} = -\Gamma^2_{12} = -\Gamma^3_{13} = -\Gamma^4_{14},$$

$$R_{1221} = R_{1331} = R_{1441} = \frac{p}{(1+2p)}, S_{11} = \frac{3p}{(1+2p)^2},$$

$$S_{22} = S_{33} = S_{44} = \frac{p}{(1+2p)^2}, r = \frac{6p}{(1+2p)^3} \neq 0$$

and the components which can be obtained from these by the symmetry properties.
Therefore $\mathbb{R}^4$ is a Riemannian manifold (M^4, g) of non-vanishing scalar curvature. We shall now show that M^4 is a $G(QE)_4$, i.e., it satisfies (1.2). Let us now consider the associated scalars as follows:

$$\alpha = \frac{p}{(1+2p)^3},\ \beta = 3p,\ \gamma = -\frac{1}{(1+2p)^2}. \tag{9.2}$$

In terms of local coordinate system, let us consider the 1-forms A and B as follows:

$$A_i(x) = \begin{cases} \frac{1}{1+2p} \text{ for } i = 1, \\ 0 \text{ otherwise,} \end{cases}$$
$$B_i(x) = \begin{cases} \sqrt{p} \text{ for } i = 1, \\ 0 \text{ otherwise.} \end{cases} \tag{9.3}$$

In terms of local coordinate system, the defining condition (1.2) of a $G(QE)_n$ can be written as

$$S_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j, \quad i, j = 1, 2, 3, 4. \tag{9.4}$$

By virtue of (9.2) and (9.3), it can be easily shown that (9.4) holds for $i, j = 1, 2, 3, 4$. Therefore (M^4, g) is a $G(QE)_4$, which is not quasi-Einstein. Hence we can state the following:

Theorem 9.1. *Let (M^4, g) be a Riemannian manifold endowed with the metric given in* (9.1). *Then (M^4, g) is a $G(QE)_4$ with non-vanishing scalar curvature which is not quasi-Einstein.*

Example 9.2. We define a Riemannian metric g on $\mathbb{R}^4$ by the formula

$$ds^2 = e^{2x^1}(dx^1)^2 + \sin^2 x^1[(dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{9.5}$$

where $0 < x^1 < \frac{\pi}{2}$ but $x^1 \neq \frac{\pi}{4}$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor and scalar curvature are

$$\begin{aligned}
&\Gamma^1_{11} = 1, \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \cot x^1,\\
&\Gamma^1_{22} = -\frac{\sin 2x^1}{2e^{2x^1}} = \Gamma^1_{33} = \Gamma^1_{44},\\
&R_{1221} = -\sin^2 x^1(1+\cot x^1) = R_{1331} = R_{1441},\\
&R_{2332} = \frac{\sin^2 x^1 \cos^2 x^1}{e^{2x^1}} = R_{2442} = R_{3443},\\
&S_{22} = \frac{2\cos^2 x^1 - \sin^2 x^1(1+\cot x^1)}{e^{2x^1}} = S_{33} = S_{44},\\
&S_{11} = -3(1+\cot x^1), r = \frac{6(\cot^2 x^1 - \cot x^1 - 1)}{e^{2x^1}} \neq 0,
\end{aligned}$$

provided $(\cot^2 x^1 - \cot x^1 - 1) \neq 0$ and the components which can be obtained from these by the symmetry properties. Therefore $\mathbb{R}^4$ with the considered metric is a Riemannian manifold (M^4, g) of non-vanishing scalar curvature. We shall now show that this M^4 is a $G(QE)_4$, i.e., it satisfies (1.2). Let us now consider the associated scalars as follows:

$$\alpha = \frac{2\cot^2 x^1 - \cot x^1 - 1}{e^{2x^1}}, \ \beta = -(1+\cot x^1), \ \gamma = -2\cot x^1. \tag{9.6}$$

In terms of local coordinate system, let us consider the 1-forms A and B as follows:

$$A_i(x) = \begin{cases} \sqrt{2} \ \text{ for } \ i=1, \\ 0 \ \text{ otherwise,} \end{cases} \qquad B_i(x) = \begin{cases} \sqrt{\cot x^1} \ \text{ for } \ i=1, \\ 0 \ \text{ otherwise.} \end{cases} \tag{9.7}$$

In terms of local coordinate system, the defining condition (1.2) of a $G(QE)_n$ can be written as (9.4). By virtue of (9.6) and (9.7), it can be easily shown that (9.4) holds for $i, j = 1, 2, 3, 4$. Therefore (M^4, g) is a $G(QE)_4$, which is not quasi-Einstein. Hence we can state the following:

Theorem 9.2. *Let (M^4, g) be a Riemannian manifold endowed with the metric given in* (9.5)*. Then (M^4, g) is a $G(QE)_4$ with non-vanishing scalar curvature which is not quasi-Einstein.*

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