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STRONG CONVERGENCE OF AN ITERATIVE METHOD FOR FINDING COMMON ZEROS OF A FINITE FAMILY OF ACCRETIVE OPERATORS

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Abstract. Strong convergence theorems on viscosity approximation methods for finding a common zero of a finite family accretive operators are established in a reflexive and strictly Banach space having a uniformly Gâteaux differentiable norm. The main theorems supplement the recent corresponding results of Wong et al. [29] and Zegeye and Shahzad [32] to the viscosity method together with different control conditions. Our results also improve the corresponding results of [9, 16, 18, 19, 25] for finite nonexpansive mappings to the case of finite pseudocontractive mappings.

1. Introduction

Let E be a real Banach space and C be a nonempty closed convex subset of E. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0, 1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|,\ x,\ y \in C$. We use Σ_C to denote the collection of mappings f verifying the above inequality. That is, $\Sigma_C = \{f : C \to C \mid f \text{ is a contraction with constant } k\}$. Note that each $f \in \Sigma_C$ has a unique fixed point in C.

Now let $T : C \to C$ be a nonexpansive mapping (recall that a mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|,\ \ x,\ y \in C$), and $F(T)$ denote the set of fixed points of T; that is, $F(T) = \{x \in C : x = Tx\}$. T is called *pseudocontractive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 \text{ for all } x,\ y \in C,$$

where J is the normalized duality mapping from E to 2^{E^*}. Clearly the class of nonexpansive mappings is a subset of the class of pseudocontractive mappings.

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Closely related to the class of pseudocontractive mappings is the class of accretive operators. Recall that a (possibly multivalued) operator $A \subset E \times E$ with the domain $D(A)$ and the range $R(A)$ in E is *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1,\ 2)$, there exists a $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$. (Here J is the duality mapping.) An accretive operator A is said to satisfy the *range condition* if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$. An accretive operator A is m-accretive if $R(I + rA) = E$ for each $r > 0$. If A is an accretive operator which satisfies the range condition, then we can define, for each $r > 0$ a mapping $J_r : R(I + rA) \to D(A)$ defined by $J_r = (I + rA)^{-1}$, which is called the resolvent of A. We know that J_r is nonexpansive single-valued mapping and $F(J_r) = A^{-1}0$ for all $r > 0$. The set of zero of A is denoted by $N(A)$, that is,

$$N(A) := \{x \in D(A) : 0 \in Ax\} = A^{-1}0.$$

If $A^{-1}0 \neq \emptyset$, then the inclusion $0 \in Ax$ is solvable. We also observe that x is a zero of the accretive operator A if and only if it is a fixed point of the pseudocontractive mapping $T = I - A$. It is well known that if A is accretive, then the solutions of the equation $0 \in Ax$ correspond to the equilibrium points of some evolution systems. For this reason, iterative methods for approximating the zeros of accretive operator A have extensively been studies over the last forty years (see, e.g., [1, 2, 3, 4, 5, 6, 13, 14, 15, 20, 22, 23, 24, 31]).

Let C be a closed convex subset of E and $T : C \to C$ a nonexpansive mapping. In [16], Kirk studied the iterative scheme given by

$$x_{n+1} = a_0x_n + a_1Tx_n + a_2T^2x_n + \cdots + a_kT^kx_n, \quad n \geq 0,$$

where $x_0 \in C$, $a_i \geq 0$, $a_0 > 0$ and $\sum_{i=0}^{k} a_i = 1$ for approximating fixed points of nonexpansive mappings. Liu et al. [18] introduced the following iterative scheme for finite nonexpansive mappings $T_i : C \to C$ $(i = 1, \ldots, k)$:

$$x_{n+1} = a_0x_n + a_1T_1x_n + a_2T_2x_n + \cdots + a_kT_kx_n, \quad n \geq 0, \tag{1.1}$$

where $x_0 \in C$, $a_i \geq 0$, $a_0 > 0$ and $\sum_{i=0}^{k} a_i = 1$, and showed that $\{x_n\}$ generated by (1.1) converges to a common fixed point of T_i $(i = 1, 2, \ldots, k)$, in a Banach space with a certain property, say, condition A. The result improved the corresponding result of Kirk [16], Maiti and Saha [19] and Senter and Doston [25]. In 2002, Jung [9] established the weak convergence of $\{x_n\}$ generated by (1.1) in a reflexive and strictly convex Banach space having a uniformly Gâteaux differentiable norm.

Recently, Zegeye and Shahzad [32] considered the following iterative scheme for a finite family of m-accretive operators $A_i : C \to E$ $(i = 1, \ldots, k)$:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)S_kx_n,\ n \geq 0, \tag{1.2}$$

where $S_k := a_0I + a_1J_{A_1} + a_2J_{A_2} + \cdots + a_kJ_{A_k}$ with $J_{A_i} : (I + A_i)^{-1}$ for $0 < a_i < 1$ $(i = 0, 1, \ldots, k)$, $\sum_{i=0}^{k} a_i = 1$, and under the control conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, or, equivalently, $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$,

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, or (iii)* $\lim_{n\to\infty} \frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}} = 0$,

showed that the sequence $\{x_n\}$ generated by (1.2) converges strongly to a common solution of the equation $A_i x \ni 0$ for $i = 1, \dots, k$ in a reflexive and strictly convex Banach space having a uniformly Gâteaux differentiable norm and satisfying that every weakly compact convex subset of E has the fixed point property for nonexpansive mapping. On the other hand, as the viscosity approximation method, Moudafi [21] and Xu [30] considered the iterative scheme: for T a nonexpansive mapping, $f \in \Sigma_C$ and $\alpha_n \in (0,1)$,

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 0. \tag{1.3}$$

Under the conditions (i), (ii) and (iii) on $\{\alpha_n\}$, Xu [30] showed in a uniformly smooth Banach space that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a fixed point of T, which solves a certain variational inequality. The results of Xu [30] extended the results of Moudafi [21] to a Banach space setting. In 2006, Jung [10] considered the iterative scheme: for $N > 1$, $T_1, T_2, \dots, T_k$ nonexpansive mappings, $f \in \Sigma_C$ and $\alpha_n \in (0,1)$,

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_{n+1}x_n, \quad n \geq 0, \tag{1.4}$$

where $T_n := T_{n \bmod k}$, and extended results of Xu [30] (and Moudafi [21]) to the case of a family of finite nonexpansive mappings. In particular, under the conditions (i), (ii) and the perturbed control condition on $\{\alpha_n\}$

(iv) $|\alpha_{n+k} - \alpha_n| \leq o(\alpha_{n+k}) + \sigma_n, \quad \sum_{n=0}^{\infty} \sigma_n < \infty$,

he obtained the strong convergence of the sequence $\{x_n\}$ generated by (1.4) to a solution in $\bigcap_{i=1}^{k} \mathrm{Fix}(T_i)$ of a certain variational inequality in a reflexive Banach space having a uniformly Gâteaux differentiable norm with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mapping, and gave an example which satisfies the conditions (i), (ii) and (iv), but fails to satisfy the condition (iii) for $k > 1$; $\sum_{n=0}^{\infty} |\alpha_{n+k} - \alpha_n| < \infty$.

In this paper, motivated by above-mentioned results, we introduce the viscosity approximation method for a finite family of accretive operators: for resolvent J_{r_i} of accretive operator A_i such that $\bigcap_{i=1}^{k} N(A_i) \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA_i)$ $(i = 1, \dots, k)$, $f \in \Sigma_C$ and $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$,

$$\text{(IS)} \qquad \begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n)S_k x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \quad n \geq 0, \end{cases}$$

where $S_k := a_0 I + a_1 J_{r_1}^{A_1} + a_2 J_{r_2}^{A_2} + \cdots + a_k J_{r_k}^{A_k}$ with $J_{r_i}^{A_i} := (I + r_i A_i)^{-1}$ for $r_i > 0$, and $0 < a_i < 1$ $(i = 0, 1, \dots, k)$ and $\sum_{i=0}^{k} a_i = 1$, and establish the strong convergence of the sequence $\{x_n\}$ generated by (IS) to a common solution of the equations $A_i x \ni 0$ for $i = 1, \dots, k$, in a reflexive and strictly convex Banach space having a uniformly Gâteaux differentiable norm under certain different control conditions on sequences $\{\alpha_n\}$ and $\{\beta_n\}$. The main results improve the recent results of Wong et al. [29] and Zegeye and Shahzad [32]. Our results also improve the corresponding results of [9, 16, 18, 19, 25] for finite nonexpansive mappings to the case of finite pseudocontractive mappings.

2. Preliminaries and lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to x.

The norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. Such an E is called a *smooth* Banach space. The norm is said to be *uniformly Gâteaux differentiable* if for $y \in U$, the limit is attained uniformly for $x \in U$. The space E is said to have a *uniformly Fréchet differentiable norm* (and E is said to be *uniformly smooth*) if the limit in (2.1) is attained uniformly for $(x, y) \in U \times U$. It is well known that if E has a uniformly Gâteaux differentiable norm, J is uniformly norm to weak* continuous on each bounded subsets of E ([7, 28]).

The (*normalized*) *duality mapping* J from E into the family of nonempty (by Hahn-Banach theorem) weak* compact subsets of its dual E^* is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

for each $x \in E$. It is single valued if and only if E is smooth.

A Banach space E is said to be *strictly convex* if $\|a_1x_1+a_2x_2+\cdots+a_kx_k\| < 1$ for $x_i \in E$ $(i = 1, 2, \ldots, k)$ with $\|x_i\| = 1$ $(i = 1, 2, \ldots, k)$ and $x_i \neq x_j$ for some $i \neq j$, and for $a_i \in (0, 1)$ $(i = 1, 2, \ldots, k)$ such that $\sum_{i=1}^k a_i = 1$.

Let D be a subset of C. Then $Q : C \to D$ is called a *retraction* from C onto D if $Qx = x$ for all $x \in D$. A retraction $Q : C \to D$ is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Qx) \in C$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D for more details, see [8]. In a smooth Banach space E, it is known [8, p. 48]) that $Q : C \to D$ is a sunny nonexpansive retraction if and only if the following condition holds:

$$\langle x - Qx, J(z - Qx)\rangle \leq 0, \quad x \in C, \quad z \in D. \tag{2.2}$$

We need the following lemmas for the proof of our main results. Lemma 2.1 was also given in [11]. Lemma 2.2 is Lemma 2 of [27] and Lemma 2.3 is essentially Lemma 2 of [17].

Lemma 2.1. *Let X be a real Banach space and J be the duality mapping. Then, for any given $x,\ y \in X$, one has*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $j(x+y) \in J(x+y)$.

Lemma 2.2. *Let $\{x_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space E and let $\{\gamma_n\}$ be a sequence in $[0,1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1-\gamma_n)w_n$, $n \ge 0$, and

$$\limsup_{n\to\infty}(\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|w_n - x_n\| = 0$.

Lemma 2.3. *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \le (1-\lambda_n)s_n + \lambda_n\delta_n + \gamma_n, \quad n \ge 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} \lambda_n\delta_n < \infty$;
(iii) $\gamma_n \ge 0$ $(n \ge 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

By using the same method as Lemma 3.1 in [32], we can prove the following lemma. So we omit its proof.

Lemma 2.4. *Let E be a strictly convex Banach space. Let C be a nonempty closed convex subset of E and $A_i \subset E \times E$ $(i = 1,\dots,k)$ accretive operators in E such that $\bigcap_{i=1}^{k} N(A_i) \ne \emptyset$ and $\overline{D(A_i)} \subset C \subset \bigcap_{r>0} R(I + rA_i)$. Let $S_k := a_0 I + a_1 J_{r_1}^{A_1} + \dots + a_k J_{r_k}^{A_k}$ with $J_{r_i}^{A_i} := (I + r_i A_i)^{-1}$ for $r_i > 0$ $(i = 1,\dots,k)$, $0 < a_i < 1$ $(i = 0,1,\dots,k)$ and $\sum_{i=0}^{k} a_i = 1$. Then S_k is nonexpansive and $F(S_k) = \bigcap_{i=1}^{k} N(A_i)$.*

3. Main results

Now, we study the strong convergence results for the iterative scheme (IS) in Banach spaces.

We need the following result for the existence of a solution of the variational inequality

$$\langle (I-f)(q), J(q-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F(T),$$

which Jung and Sahu [12] established recently.

Theorem JS. ([12, Theorem 2]) *Let E be a reflexive and strictly convex Banach space having a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E, $A : C \to C$ a continuous strongly pseudocontractive* *mapping with constant* $k \in [0,1)$ *and* $T : C \to E$ *a continuous pseudocontractive mapping satisfying the weakly inward condition. If* T *has a fixed point in* C*, then the path* $\{x_t\}$ *defined by*

$$x_t = tAx_t + (1-t)Tx_t, \quad t \in (0,1)$$

converges strongly to a fixed point q *of* T*, which is the unique solution of a variational inequality*:

$$\langle (I-A)q, J(q-p)\rangle \le 0 \quad \text{for all} \quad p \in F(T).$$

Remark 3.1. (1) Theorem JS generalizes Theorem 3.1 of Song and Chen [26] to a more general class of mappings. In fact, in Theorem 3.1 of [26], $T : C \to C$ is a nonexpansive self-mapping and $A = f$ is a contraction.

(2) In Theorem JS, if $A(x) = u$, $x \in C$, is a constant and $Qu = q = \lim_{t\to 0} x_t$, then it follows from (2.2) that Q is reduced to the sunny nonexpansive retraction from C onto $F(T)$,

$$\langle Qu - u, J(Qu - p)\rangle \le 0, \quad u \in C, \quad p \in F(T).$$

Using Theorem JS, we establish the following main result.

Theorem 3.1. *Let* E *be a reflexive and strictly convex Banach space having a uniformly Gâteaux differentiable norm. Let* C *be a nonempty closed convex subset of* E *and* $A_i \subset E \times E$ $(i = 1, \ldots, k)$ *accretive operators in* E *such that* $\bigcap_{i=1}^k N(A_i) \ne \emptyset$ *and* $\overline{D(A_i)} \subset C \subset \bigcap_{r>0} R(I + rA_i)$*. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $(0,1)$ *which satisfy the conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Let $f \in \Sigma_C$ *and* $x_0 \in C$ *be chosen arbitrarily. Let* $\{x_n\}$ *be a sequence generated by*

$$\text{(IS)} \qquad \begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n) S_k x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \end{cases}$$

where $S_k := a_0 I + a_1 J_{r_1}^{A_1} + \cdots + a_k J_{r_k}^{A_k}$ *with* $J_{r_i}^{A_i} := (I + r_i A_i)^{-1}$ *for* $r_i > 0$ $(i = 1, \ldots, k)$*,* $0 < a_i < 1$ $(i = 0, 1, \ldots, k)$ *and* $\sum_{i=0}^k a_i = 1$*. Then* $\{x_n\}$ *converges strongly to* $q \in F(S_k) = \bigcap_{i=1}^k N(A_i)$*, where* q *is the unique solution of the variational inequality*

$$\langle (I-f)(q), J(q-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F(S_k).$$

Proof. First, we note that by Theorem JS and Lemma 2.4, there exists the unique solution $q \in F(S_k) = \bigcap_{i=1}^k N(A_i)$ of the variational inequality

$$\langle (I - f)(q), J(q - p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F(S_k),$$

where $q = \lim_{t\to 0} z_t$ and z_t is defined by $z_t = tf(z_t) + (1-t)S_k z_t$ for $0 < t < 1$.

We proceed with the following steps:

Step 1. We show that $\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}$ for all $n \ge 0$ and all $p \in F$ and so $\{x_n\}$ is bounded. Indeed, let $p \in F(S_k)$ and $d = \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}$. Noting that

$$\|y_n - p\| \le \beta_n\|x_n - p\| + (1-\beta_n)\|S_k x_n - p\| \le \|x_n - p\|,$$

we have

$$\begin{aligned}
\|x_1 - p\| &\le (1-\alpha_0)\|y_0 - p\| + \alpha_0\|f(x_0) - p\| \\
&\le (1-\alpha_0)\|x_0 - p\| + \alpha_0(\|f(x_0) - f(p)\| + \|f(p) - p\|) \\
&\le (1-(1-k)\alpha_0)\|x_0 - p\| + \alpha_0\|f(p) - p\| \\
&\le (1-(1-k)\alpha_0)d + \alpha_0(1-k)d = d.
\end{aligned}$$

Using an induction, we obtain $\|x_{n+1} - p\| \le d$. Hence $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{S_k x_n\}$ and $\{f(x_n)\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\|$. To this end, set $\gamma_n = (1-\alpha_n)\beta_n, \quad n \ge 0$. Then it follow from (C1)and (C3) that

$$0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1. \tag{3.1}$$

Define

$$x_{n+1} = \gamma_n x_n + (1-\gamma_n)w_n. \tag{3.2}$$

Observe that

$$\begin{aligned}
w_{n+1} - w_n &= \frac{x_{n+2} - \gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} - \frac{x_{n+1} - \gamma_n x_n}{1-\gamma_n} \\
&= \frac{\alpha_{n+1}f(x_{n+1}) + (1-\alpha_{n+1})y_{n+1} - \gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} \qquad (3.3)\\
&\quad - \frac{\alpha_n f(x_n) + (1-\alpha_n)y_n - \gamma_n x_n}{1-\gamma_n} \\
&= \left(\frac{\alpha_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\gamma_n}\right) \\
&\quad - \frac{(1-\alpha_n)[\beta_n x_n + (1-\beta_n)S_k x_n] - \gamma_n x_n}{1-\gamma_n} \\
&\quad + \frac{(1-\alpha_{n+1})[\beta_{n+1}x_{n+1} + (1-\beta_{n+1})S_k x_{n+1}] - \gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} \\
&= \left(\frac{\alpha_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\gamma_n}\right) + \frac{(1-\alpha_{n+1})(1-\beta_{n+1})S_k x_{n+1}}{1-\gamma_{n+1}}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{(1-\alpha_n)(1-\beta_n)S_k x_n}{1-\gamma_n} \\
&= \left(\frac{\alpha_{n+1} f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\gamma_n}\right) + (S_k x_{n+1} - S_k x_n) \\
&\quad - \frac{\alpha_{n+1}}{1-\gamma_{n+1}} S_k x_{n+1} + \frac{\alpha_n}{1-\gamma_n} S_k x_n.
\end{aligned}
$$

It follows from (3.3) that

$$
\begin{aligned}
&\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| \\
&\le \frac{\alpha_{n+1}}{1-\gamma_{n+1}}(\|f(x_{n+1})\| + \|S_k x_{n+1}\|) + \frac{\alpha_n}{1-\gamma_n}(\|f(x_n)\| + \|S_k x_n\|).
\end{aligned}
\tag{3.4}
$$

Since $\{f(x_n)\}$ and $\{S_k x_n\}$ are bounded, by (C1), (3.1) and (3.4) we obtain that

$$
\limsup_{n\to\infty}(\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\|) \le 0.
$$

Hence by Lemma 2.2, we have

$$
\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.5}
$$

It then follows from (3.1) and (3.2) that

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.
$$

Step 3. We show that $\lim_{n\to\infty} \|x_n - S_k x_n\| = 0$. Indeed, as a consequence with the control condition (C1), by Step 1, we get

$$
\|x_{n+1} - y_n\| \le \alpha_n(\|f(x_n)\| + \|y_n\|) \to 0 \quad (n \to \infty). \tag{3.6}
$$

Combining Step 2 and (3.6), we get

$$
\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.7}
$$

Observe that

$$
y_n - x_n = (1-\beta_n)(S_k x_n - x_n). \tag{3.8}
$$

It follows from (C3), (3.7) and (3.8)

$$
\lim_{n\to\infty} \|x_n - S_k x_n\| = 0.
$$

Step 4. We show that $\limsup_{n\to\infty}\langle (I-f)(q), J(q - x_n)\rangle \le 0$. To prove this, let a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ be such that

$$
\limsup_{n\to\infty}\langle (I-f)(q), J(q - x_n)\rangle = \lim_{j\to\infty}\langle (I-f)(q), J(q - x_{n_j})\rangle
$$

and

$$
x_{n_j} \rightharpoonup p \quad \text{for some } p \in E.
$$

Now let z_t be defined by $z_t = tf(z_t) + (1-t)S_k z_t$ for $0 < t < 1$. Then

$$
z_t - x_n = (1-t)(S_k z_t - x_n) + t(f(z_t) - x_n).
$$

Applying Lemma 2.1, we have

$$\|z_t - x_n\|^2 \leq (1-t)^2\|S_k z_t - x_n\|^2 + 2t\langle f(z_t) - x_n, J(z_t - x_n)\rangle.$$

Putting

$$a_j(t) = (1-t)^2\|S_k x_{n_j} - x_{n_j}\|(2\|z_t - x_{n_j}\| + \|S_k x_{n_j} - x_{n_j}\|) \to 0 \ (j \to \infty)$$

by Step 3 and using Lemma 2.1, we obtain

$$\begin{aligned}\|z_t - x_{n_j}\|^2 &\leq (1-t)^2\|S_k z_t - x_{n_j}\|^2 + 2t\langle f(z_t) - x_{n_j}, J(z_t - x_{n_j})\rangle \\ &\leq (1-t)^2(\|S_k z_t - S_k x_{n_j}\| + \|S_k x_{n_j} - x_{n_j}\|)^2 \\ &\quad + 2t\langle f(z_t) - z_t, J(z_t - x_{n_j})\rangle + 2t\|z_t - x_{n_j}\|^2 \\ &\leq (1-t)^2\|z_t - x_{n_j}\|^2 + a_j(t) \\ &\quad + 2t\langle f(z_t) - z_t, J(z_t - x_{n_j})\rangle + 2t\|z_t - x_{n_j}\|^2.\end{aligned}$$

The last inequality implies

$$\langle z_t - f(z_t), J(z_t - x_{n_j})\rangle \leq \frac{t}{2}\|z_t - x_{n_j}\|^2 + \frac{1}{2t}a_j(t).$$

It follows that

$$\lim_{j\to\infty}\langle z_t - f(z_t), J(z_t - x_{n_j})\rangle \leq \frac{t}{2}M, \tag{3.9}$$

where $M > 0$ is a constant such that $M \geq \|z_t - x_n\|^2$ for all $n \geq 0$ and $t \in (0,1)$. Taking the lim sup as $t \to 0$ in (3.9) and noticing the fact that the two limits are interchangeable due to the fact that the duality mapping J is norm to weak* uniformly continuous on bounded subset of E, we have

$$\limsup_{j\to\infty}\langle (I-f)(q), J(q - x_{n_j})\rangle \leq 0.$$

Indeed, letting $t \to 0$, from (3.9) we have

$$\limsup_{t\to 0}\limsup_{j\to\infty}\langle z_t - f(z_t), J(z_t - x_{n_j})\rangle \leq 0.$$

So, for any $\varepsilon > 0$, there exists a positive number δ_1 such that for any $t \in (0,\delta_1)$,

$$\limsup_{j\to\infty}\langle z_t - f(z_t), J(z_t - x_{n_j})\rangle \leq \frac{\varepsilon}{2}.$$

Moreover, since $z_t \to q$ as $t \to 0$, the set $\{z_t - x_{n_j}\}$ is bounded and the duality mapping J is norm to weak* uniformly continuous on bounded subset of E, there exists $\delta_2 > 0$ such that, for any $t \in (0,\delta_2)$,

$$\begin{aligned}&|\langle q - f(q), J(q - x_{n_j})\rangle - \langle z_t - f(z_t), J(z_t - x_{n_j})\rangle| \\ &= |\langle q - f(q), J(q - x_{n_j}) - J(z_t - x_{n_j})\rangle + \langle q - f(q) - (z_t - f(z_t)), J(z_t - x_{n_j})\rangle| \\ &\leq |\langle q - f(q), J(z_t - x_{n_j}) - J(q - x_{n_j})\rangle| + \|q - f(q) - (z_t - f(z_t))\|\|z_t - x_{n_j}\| \\ &< \frac{\varepsilon}{2}.\end{aligned}$$

Choose $\delta = \min\{\delta_1, \delta_2\}$, we have for all $t \in (0, \delta)$ and $j \in \mathbb{N}$,

$$\langle q - f(q), J(q - x_{n_j})\rangle < \langle z_t - f(z_t), J(z_t - x_{n_j})\rangle + \frac{\varepsilon}{2},$$

which implies that

$$\limsup_{j\to\infty}\langle q - f(q), J(q - x_{n_j})\rangle \le \limsup_{j\to\infty}\langle z_t - f(z_t), J(z_t - x_{n_j})\rangle + \frac{\varepsilon}{2}.$$

Since $\limsup_{j\to\infty}\langle z_t - f(z_t), J(z_t - x_{n_j})\rangle \le \frac{\varepsilon}{2}$, we have

$$\limsup_{j\to\infty}\langle q - f(q), J(q - x_{n_j})\rangle \le \varepsilon.$$

Since ε is arbitrary, we obtain that

$$\limsup_{j\to\infty}\langle (I - f)(q), J(q - x_{n_j})\rangle \le 0.$$

Step 5. We show that $\lim_{n\to\infty}\|x_n - q\| = 0$. By using (IS), we have

$$x_{n+1} - q = \alpha_n(f(x_n) - q) + (1 - \alpha_n)(y_n - q).$$

Applying Lemma 2.1, we obtain

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le (1 - \alpha_n)^2\|y_n - q\|^2 + 2\alpha_n\langle f(x_n) - q, J(x_{n+1} - q)\rangle \\ &\le (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle f(x_n) - f(q), J(x_{n+1} - q)\rangle \\ &\quad + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\ &\le (1 - \alpha_n)^2\|x_n - q\|^2 + 2k\alpha_n\|x_n - q\|\|x_{n+1} - q\| \\ &\quad + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\ &\le (1 - \alpha_n)^2\|x_n - q\|^2 + k\alpha_n(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) \\ &\quad + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle.\end{aligned}$$

It then follows that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le \frac{1 - (2 - k)\alpha_n + \alpha_n^2}{1 - k\alpha_n}\|x_n - q\|^2 \\ &\quad + \frac{2\alpha_n}{1 - k\alpha_n}\langle f(q) - q, J(x_{n+1} - q)\rangle \\ &\le \frac{1 - (2 - k)\alpha_n}{1 - k\alpha_n}\|x_n - q\|^2 + \frac{\alpha_n^2}{1 - k\alpha_n}M \\ &\quad + \frac{2\alpha_n}{1 - k\alpha_n}\langle (I - f)(q), J(q - x_{n+1})\rangle,\end{aligned} \tag{3.10}$$

where $M = \sup_{n\ge 0}\|x_n - q\|^2$. Put

$$\lambda_n = \frac{2(1 - k)\alpha_n}{1 - k\alpha_n} \quad \text{and}$$

$$\delta_n = \frac{M\alpha_n}{2(1 - k)} + \frac{1}{1 - k}\langle (I - f)(q), J(q - x_{n+1})\rangle.$$

From (C1), (C2) and Step 4, it follows that $\lambda_n \to 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \delta_n \le 0$. Since (3.10) reduces to

$$\|x_{n+1} - q\|^2 \le (1 - \lambda_n)\|x_n - q\|^2 + \lambda_n \delta_n,$$

from Lemma 2.3, we conclude that $\lim_{n\to\infty} \|x_n - q\| = 0$. □

Corollary 3.1. *Let E be a uniformly convex and uniformly smooth Banach space. Let A_i $(i = 1, \dots, k)$ be m-accretive operators in E such that $C = \overline{D(A_i)}$ is convex and $\bigcap_{i=1}^{k} N(A_i) \neq \emptyset$. Let $J_{r_i}^{A_i}$ $(i = 1, \dots, k)$, S_k, $\{\alpha_n\}$, $\{\beta_n\}$, f, x_0 and $\{x_n\}$ be as in Theorem 3.1. Then the conclusion of Theorem 3.1 still holds.*

Remark 3.2. (1) Theorem 3.1 supplements Theorem 3.3 of Zegeye and Shahzad [32] in several aspects. In particular, Theorem 3.1 develops Theorem 3.3 of Zegeye and Shahzad [32] to the viscosity method and removes the assumption imposed in Theorem 3.3 of Zegeye and Shahzad [32] that every nonempty closed bounded convex subset of E has the fixed point property for nonexpansive mappings. Moreover, by using the iterative scheme (IS), Theorem 3.1 removes the condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}}$ imposed on sequence $\{\alpha_n\}$ in Theorem 3.3 of Zegeye and Shahzad [32].

(2) Using the iterative scheme (IS), Theorem 3.1 also develops Theorem 6.3 of Wong et al. [29] without the condition $\lim_{n\to\infty} \frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}}$.

(3) In general, the conditions (C3) in Theorem 3.1 and the condition

$$\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$$

are not comparable; neither of them implies other.

As a direct consequence of Theorem 3.1, we obtain strong convergence to a common fixed point of a family of pseudocontractive mappings.

Theorem 3.2. *Let E be a reflexive and strictly convex Banach space having a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E and $T_i : C \to E$ $(i = 1, \dots, k)$ pseudocontractive mappings such that $(I - T_i)$ is m-accretive on C with $\bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. Let $J_{T_i} := (I + (I - T_i))^{-1} = (2I - T_i)^{-1}$ for $i = 1, \dots, k$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ which satisfy the conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Let $f \in \Sigma_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) S_k x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \quad n \ge 0 \end{cases}$$

where $S_k := a_0 I + a_1 J_{T_1} + \cdots + a_k J_{T_k}$ *for* $0 < a_i < 1$ $(i = 0, 1, \ldots, k)$ *and* $\sum_{i=0}^{k} a_i = 1$*. Then* $\{x_n\}$ *converges strongly to* $q \in F(S_k) = \bigcap_{i=1}^{k} F(T_i)$*, where* q *is the unique solution of the variational inequality*

$$\langle (I - f)(q), J(q - p) \rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F(S_k).$$

Proof. Let $A_i := (I - T_i)$ for each $i = 1, \ldots, k$. Then clearly, $F(T_i) = N(A_i)$ and hence $\bigcap_{i=1}^{k} N(A_i) = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. Moreover, each A_i for $i = 1, \ldots, k$ is m-accretive. Thus the results follows from Theorem 3.2. □

Remark 3.3. (1) Theorem 3.2 complements Theorem 3.9 of Zegeye and Shahzad [32] to the viscosity method together with certain different control conditions in more general Banach space.

(2) Theorem 3.2 also develops the corresponding results of [9, 16, 18, 19, 25] for finite nonexpansive mappings to the case of finite pseudocontractive mappings.

(3) We point out that our results are applicable to, in particular, in all L^p spaces, $1 < p < \infty$.

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References

[1] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in Banach spaces*, Nonlinear Anal. **67** (2007), 2350–2360.

[2] T. D. Benavides, G. L. Acedo, and H. K. Xu, *Iterative solutions for zeros of accretive operators*, Math. Nach. **248-249** (2003), 62–71.

[3] H. Bréziz and P. L. Lions, *Products infinis de resolvents*, Israel J. Math. **29** (1978), 329–345.

[4] R. E. Bruck and G. B. Passty, *Almost convergence of the infinite product of resolvents in Banach spaces*, Nonlinear Anal. **3** (1979), 279–282.

[5] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents in Banach spaces*, Houston J. Math. **3** (1977), 459–470.

[6] R. Chen and Z. Zhu, *Viscosity approximation fixed points for nonexpansive and m-acctrive operators*, Fixed Point Theory Appl. **2006** (2006), 1–10.

[7] J. Diestel, *Geometry of Banach Spaces*, Lectures Notes in Math. 485, Springer-Verlag, Berlin, Heidelberg, 1975.

[8] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.

[9] J. S. Jung, *Convergence of nonexpansive iteration process in Banach spaces*, J. Math. Anal. Appl. **273** (2002), 153–159.

[10] ———, *Viscosity approximation methods for a family of finite nonexpansive mappings in Banach spaces*, Nonlinear Anal. **64** (2006), 2536–2552.

[11] J. S. Jung and C. Morales, *The Mann process for perturbed m-accretive operators in Banach spaces*, Nonlinear Anal. **46** (2001), 231–243.

[12] J. S. Jung and D. R. Sahu, *Convergence of approximating paths to solutions of variational inequalities involving non-Lipschitzian mappings*, J. Korean Math. Soc. **45** (2008), no. 2, 377–392.

[13] J. S. Jung and W. Takahashi, *Dual convergence theorems for the nfinite products of resolvents in Banach spaces*, Kodai Math. J. **14** (1991), 358–365.

[14] ———, *On the asymptotic behavior of infinite products of resolvents in Banach spaces*, Nonlinear Anal. **20** (1993), 469–479.

[15] T. H. Kim and H. K. Xu, *Strong convergence of modified Mann iterations*, Nonlinear Anal. **61** (2005), 51–60.

[16] W. A. Kirk, *On successive approximations for nonexpansive mappings in Banach spaces*, Glasgow Math. J. **12** (1971), 6–9.

[17] L. S. Liu, *Iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194** (1995), 114–125.

[18] G. Liu, D. Lei, and S. Li, *Approximating fixed points of nonexpansive mappings*, Internat. J. Math. Math. Sci. **24** (2000), 173–177.

[19] M. Maiti and B. Saha, *Approximating fixed points of nonexpansive and generalized nonexpansive mappings*, Internat. J. Math. Math. Sci. **16** (1993), 81–86.

[20] H. Miyake and W. Takahashi, *Approximating zero points of accretive operators with compact domains in general Banach spaces*, Fixed Point Theory Appl. **2005** (2005), no. 1, 93–102.

[21] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl. **241** (2000), 46–55.

[22] X. Qin and Y. Su, *Approximation of a zero point of accretive operator in Banach spaces*, J. Math. Anal. Appl. **329** (2007), 415–424.

[23] S. Reich, *On infinite products of resolvents*, Atti. Accad. Naz. Lincei **63** (1977), 338–340.

[24] ———, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.

[25] H. F. Senter and W. G. Dotson Jr, *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc. **44** (1974), 375–380.

[26] Y. Song and R. Chen, *Strong convergence theorems on an iterative method for a family of finite nonexpansive mappings*, Applied. Math. Comput. **180** (2006), 275–287.

[27] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one parameter nonexpansive semigroups without Bochner integral*, J. Math. Anal. Appl. **305** (2005), 227–239.

[28] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[29] N. C. Wong, D. R. Sahu, and J. C. Yao, *Solving variational inequalities involving nonexpansive type mappings*, Nonlinear Anal. **69** (2008), 4732–4753.

[30] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298** (2004), 279–291.

[31] ———, *Strong convergence of an iterative method for nonexpansive and accretive operators*, J. Math. Anal. Appl. **314** (2006), 631–643.

[32] H. Zegeye and N. Shahzad, *Strong convergence theorems for a common zero of a finite family of accretive operators*, Nonlinear Anal. **66** (2007), 1161–1169.

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