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L^q-ESTIMATES OF MAXIMAL OPERATORS ON THE p-ADIC VECTOR SPACE

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Abstract. For a prime number p, let $\mathbb{Q}_p$ denote the p-adic field and let $\mathbb{Q}_p^d$ denote a vector space over $\mathbb{Q}_p$ which consists of all d-tuples of $\mathbb{Q}_p$. For a function $f \in L^1_{loc}(\mathbb{Q}_p^d)$, we define the Hardy-Littlewood maximal function of f on $\mathbb{Q}_p^d$ by

$$\mathcal{M}_p f(\mathbf{x}) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})|\, d\mathbf{y},$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^d$ and $B_\gamma(\mathbf{x})$ denotes the p-adic ball with center $\mathbf{x} \in \mathbb{Q}_p^d$ and radius p^γ. If $1 < q \le \infty$, then we prove that $\mathcal{M}_p$ is a bounded operator of $L^q(\mathbb{Q}_p^d)$ into $L^q(\mathbb{Q}_p^d)$; moreover, $\mathcal{M}_p$ is of weak type $(1,1)$ on $L^1(\mathbb{Q}_p^d)$, that is to say,

$$|\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{M}_p f(\mathbf{x})| > \lambda\}|_H \le \frac{p^d}{\lambda} \|f\|_{L^1(\mathbb{Q}_p^d)},\ \ \lambda > 0$$

for any $f \in L^1(\mathbb{Q}_p^d)$.

1. Introduction

For a prime number p, let $\mathbb{Q}_p$ denote the p-adic field. From the standard p-adic analysis [8], we see that any non-zero element $x \in \mathbb{Q}_p$ has a unique representation like

$$x = p^\gamma \sum_{j=0}^{\infty} x_j\, p^j, \quad \gamma = \gamma(x) \in \mathbb{Z},$$

where $0 \le x_j \le p-1$ and $x_0 \ne 0$. Here we call $\gamma = \gamma(x)$ the p-adic valuation of x and we write $\gamma = \mathrm{ord_p(x)}$ with convention $\mathrm{ord_p}(0) = \infty$. Then it is well-known [1, 8] that the nonnegative function $|\cdot|_p$ on $\mathbb{Q}_p$ given by $|x|_p = p^{-\mathrm{ord_p(x)}}$ becomes a non-Archimedean norm on $\mathbb{Q}_p$ and $\mathbb{Q}_p$ is defined as the completion of $\mathbb{Q}$ with respect to the norm $|\cdot|_p$. For $d \in \mathbb{N}$, let $\mathbb{Q}_p^d$ denotes a vector space over $\mathbb{Q}_p$ which consists of all points $\mathbf{x} = (x_1, x_2, \dots, x_d)$, $x_1, x_2, \dots, x_d \in \mathbb{Q}_p$. If we define $|\mathbf{x}|_p = \max_{1\le j\le d} |x_j|_p$ for $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{Q}_p^d$, then it is easy

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to see that $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}_p^d$ and moreover $\mathbb{Q}_p^d$ is a locally compact Hausdorff and totally disconnected Banach space with respect to the norm $|\cdot|_p$. For $\gamma \in \mathbb{Z}$, we denote the ball $B_\gamma(\mathbf{a})$ with center $\mathbf{a} \in \mathbb{Q}_p^d$ and radius p^γ and its boundary $S_\gamma(\mathbf{a})$ by

$$B_\gamma(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^d : |\mathbf{x} - \mathbf{a}|_p \le p^\gamma\} \text{ and } S_\gamma(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^d : |\mathbf{x} - \mathbf{a}|_p = p^\gamma\},$$

respectively. Since $\mathbb{Q}_p^d$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a unique Haar measure $d\mathbf{x}$ on $\mathbb{Q}_p^d$ (up to positive constant multiple) which is translation invariant, i.e., $d(\mathbf{x} + \mathbf{a}) = d\mathbf{x}$. We normalize the measure $d\mathbf{x}$ so that

$$\int_{B_0(\mathbf{0})} d\mathbf{x} \fallingdotseq |B_0(\mathbf{0})|_H = 1, \tag{1.1}$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^d$. From this integration theory, it is easy to obtain that $|B_\gamma(\mathbf{a})|_H = p^{\gamma d}$ and $|S_\gamma(\mathbf{a})|_H = p^{\gamma d}(1 - p^{-d})$ for any $\mathbf{a} \in \mathbb{Q}_p^d$.

In what follows, we say that a (real-valued) measurable function f defined on $\mathbb{Q}_p^d$ is in $L^q(\mathbb{Q}_p^d), 1 \le q \le \infty$, if it satisfies

$$\begin{aligned} \|f\|_{L^q(\mathbb{Q}_p^d)} &\fallingdotseq \left(\int_{\mathbb{Q}_p^d} |f(\mathbf{x})|^q \, d\mathbf{x}\right)^{1/q} < \infty, \ 1 \le q < \infty, \\ \|f\|_{L^\infty(\mathbb{Q}_p^d)} &\fallingdotseq \inf\{\alpha : |\{\mathbf{x} \in \mathbb{Q}_p^d : |f(\mathbf{x})| > \alpha\}|_H = 0\}\} < \infty. \end{aligned} \tag{1.2}$$

Here the integral in (1.2) is defined as

$$\int_{\mathbb{Q}_p^d} |f(\mathbf{x})|^q \, d\mathbf{x} = \lim_{n\to\infty} \int_{B_n(\mathbf{0})} |f(\mathbf{x})|^q \, d\mathbf{x} = \lim_{n\to\infty} \sum_{-\infty<\gamma\le n} \int_{S_\gamma(\mathbf{0})} |f(\mathbf{x})|^q \, d\mathbf{x}, \tag{1.3}$$

if the limit exists. We now mention some of the previous works on harmonic analysis on the p-adic field $\mathbb{Q}_p$ as follows; Haran [2, 3] obtained the explicit formula of Riesz potentials on $\mathbb{Q}_p$ and developed an analytical potential theory on the p-adic field $\mathbb{Q}_p$.

For a function $f \in L^1_{loc}(\mathbb{Q}_p^d)$, we define the Hardy-Littlewood maximal function of f on $\mathbb{Q}_p^d$ by

$$\mathcal{M}_p f(\mathbf{x}) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})| \, d\mathbf{y}.$$

The reader can refer to [6] for the definition on the Euclidean case. Then we prove the following theorem.

Theorem 1.1. *If $1 < q \le \infty$, then $\mathcal{M}_p$ is a bounded operator of $L^q(\mathbb{Q}_p^d)$ into $L^q(\mathbb{Q}_p^d)$. Moreover $\mathcal{M}_p$ is of weak type $(1,1)$ on $L^1(\mathbb{Q}_p^d)$; that is to say,*

$$|\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{M}_p f(\mathbf{x})| > \lambda\}|_H \le \frac{p^d}{\lambda} \|f\|_{L^1(\mathbb{Q}_p^d)}, \ \lambda > 0$$

for any $f \in L^1(\mathbb{Q}_p^d)$.

Corollary 1.2. *If* $f \in L^q(\mathbb{Q}_p^d)$ *for* $1 \le q < \infty$, *then we have that*

(a) $\displaystyle \lim_{\gamma\to-\infty} \left\| \frac{1}{|B_\gamma(\cdot)|_H} \int_{B_\gamma(\cdot)} f(\mathbf{y})\,d\mathbf{y} - f \right\|_{L^q(\mathbb{Q}_p^d)} = 0,$

(b) $\displaystyle \left| \left\{ \mathbf{x} \in \mathbb{Q}_p^d : \lim_{\gamma\to-\infty} \left| \frac{1}{|B_\gamma(\mathbf{x})|_H} \int_{B_\gamma(\mathbf{x})} f(\mathbf{y})\,d\mathbf{y} - f(\mathbf{x}) \right| \neq 0 \right\} \right|_H = 0.$

Let $\mathcal{M}(\mathbb{Q}_p^d)$ denote the set of all measurable functions on $\mathbb{Q}_p^d$. For $f, g \in \mathcal{M}(\mathbb{Q}_p^d)$, we define the convolution $f * g$ of f and g by

$$f * g(\mathbf{x}) = \int_{\mathbb{Q}_p^d} f(\mathbf{x}-\mathbf{y})g(\mathbf{y})\,d\mathbf{y},\ \mathbf{x} \in \mathbb{Q}_p^d.$$

Theorem 1.3. *Let* $K(\mathbf{x})$ *be a nonnegative measurable function on* $\mathbb{Q}_p^d$ *such that*

$$K(\mathbf{x}) = \Phi(|\mathbf{x}|_p),$$

where $\Phi(t)$ *is a monotone decreasing function on* $(0,\infty)$ *satisfying*

$$c(p,\Phi) \fallingdotseq \lim_{n\to\infty} \sum_{-\infty<\gamma\le n} p^{\gamma d}\Phi(p^\gamma) < \infty.$$

If we set

$$\mathfrak{M}_p f(\mathbf{x}) = \sup_{\gamma\in\mathbb{Z}} |K_\gamma * f(\mathbf{x})|,\ f \in L^q(\mathbb{Q}_p^d),\ 1 < q \le \infty,$$

where $K_\gamma(\mathbf{x}) = p^{-\gamma d}K(p^\gamma \mathbf{x})$ *for* $\gamma \in \mathbb{Z}$, *then* $\mathfrak{M}_p$ *is a bounded operator of* $L^q(\mathbb{Q}_p^d)$ *into* $L^q(\mathbb{Q}_p^d)$ *for* $1 < q \le \infty$; *moreover,* $\mathfrak{M}_p$ *is of weak type* $(1,1)$ *on* $L^1(\mathbb{Q}_p^d)$.

Corollary 1.4. *Let* $K(\mathbf{x})$ *be a nonnegative measurable function on* $\mathbb{Q}_p^d$ *such that*

$$K(\mathbf{x}) = \Phi(|\mathbf{x}|_p),$$

where $\Phi(t)$ *is a monotone decreasing function on* $(0,\infty)$ *satisfying*

$$c(p,\Phi) \fallingdotseq \lim_{n\to\infty} \sum_{-\infty<\gamma\le n} p^{\gamma d}\Phi(p^\gamma) < \infty. \tag{1.4}$$

If $f \in L^q(\mathbb{Q}_p^d)$ *for* $1 \le q < \infty$, *then we have that*

(*a*) $\displaystyle \lim_{\gamma\to-\infty} \left\| K_\gamma * f - \beta f \right\|_{L^q(\mathbb{Q}_p^d)} = 0,$

(*b*) $\displaystyle \left| \left\{ \mathbf{x} \in \mathbb{Q}_p^d : \lim_{\gamma\to-\infty} \left| K_\gamma * f(\mathbf{x}) - \beta f(\mathbf{x}) \right| \neq 0 \right\} \right|_H = 0,$

where $\int_{\mathbb{Q}_P^d} K(\mathbf{x})\,d\mathbf{x} \fallingdotseq \beta$ *and* $K_\gamma(\mathbf{x}) = p^{-\gamma d}K(p^\gamma \mathbf{x})$ *for* $\gamma \in \mathbb{Z}$.

Remark. We observe that (1.4) and (3.8) imply that $0 \le \beta = (1-p^{-d})\,c(p,\Phi) < \infty$.

Examples. (a) If $K(\mathbf{x}) = \frac{1}{(1+|\mathbf{x}|_p)^\alpha}$, $\mathbf{x} \in \mathbb{Q}_p^d$ for $\alpha > d$, then we have $\Phi(t) = \frac{1}{(1+t)^\alpha}$, and thus we obtain that

$$c(p,\Phi) = \lim_{n\to\infty} \sum_{-\infty<\gamma\le n} \frac{p^{\gamma d}}{(1+p^\gamma)^\alpha} \le \sum_{\gamma=0}^\infty p^{-\gamma d} + \sum_{\gamma=1}^\infty p^{-\gamma(\alpha-d)} < \infty.$$

(b) If $K(\mathbf{x}) = \ln^k(|\mathbf{x}|_p^{-1})\,\chi_{B_0(\mathbf{0})}(\mathbf{x})$, $\mathbf{x} \in \mathbb{Q}_p^d$ for $k \in \mathbb{N}$, then we have that

$$\Phi(t) = \ln^k(t^{-1})\,\chi_{(0,1]}(t).$$

In order to obtain the finiteness of $c(p,\Phi)$, we observe the following inequalities;

$$\frac{k!}{(1-t)^{k+1}} = \sum_{\gamma=0}^\infty (\gamma+1)(\gamma+2)\cdots(\gamma+k)\,t^\gamma \ge \sum_{\gamma=0}^\infty \gamma^k t^\gamma,\ 0<t<1. \tag{1.5}$$

If we set $t = p^{-d}$ in (1.5), then we have that

$$\begin{aligned} c(p,\Phi) &= \sum_{-\infty<\gamma\le 0} p^{\gamma d}\,\ln^k(p^{-\gamma}) = \sum_{\gamma=0}^\infty p^{-\gamma d}\,\ln^k(p^\gamma) \\ &= (\ln p)^k \sum_{\gamma=0}^\infty \gamma^k p^{-\gamma d} \le \frac{k!\,(\ln p)^k}{(1-p^{-d})^{k+1}} < \infty. \end{aligned}$$

(c) If $K(\mathbf{x}) = e^{-|\mathbf{x}|_p}$ for $\mathbf{x} \in \mathbb{Q}_p^d$, then we see that $\Phi(t) = e^{-t}$. We also observe that there exists some constant $c_p > 0$ depending on p such that

$$t^{2d} \le c_p\, e^t$$

whenever $t \ge p$. Thus this implies that

$$\begin{aligned} c(p,\Phi) &= \lim_{n\to\infty} \sum_{-\infty<\gamma\le n} p^{\gamma d} e^{-p^\gamma} \\ &= \sum_{\gamma=0}^\infty p^{-\gamma d} e^{-p^{-\gamma}} + \lim_{n\to\infty} \sum_{\gamma=1}^n p^{\gamma d} e^{-p^\gamma} \\ &\le \sum_{\gamma=0}^\infty p^{-\gamma d} + c_p \sum_{\gamma=1}^\infty p^{-\gamma d} < \infty. \end{aligned}$$

2. The *p*-adic version of the Marcinkiewicz interpolation theorem

First of all, we shall obtain the relation between Riemann-Stieltjes integrals and Haar integrals which we mentioned in (1.3). Let f be a measurable function on $\mathbb{Q}_p^d$ satisfying $f \in L^1(\mathbb{Q}_p^d)$. For $\alpha > 0$, we denote the distribution function $\omega_H(\alpha)$ of $|f|$ on $\mathbb{Q}_p^d$ by

$$\omega_H(\alpha) = |\{\mathbf{x} \in \mathbb{Q}_p^d : |f(\mathbf{x})| > \alpha\}|_H.$$

Then we easily obtain the following proposition as in the Euclidean case.

Proposition 2.1 (Chebyshev's inequality). *If $f \in L^q(\mathbb{Q}_p^d)$ for $q > 0$, then we have that*

$$\omega_H(\alpha) \le \frac{1}{\alpha^q} \int_{\{\mathbf{x}\in\mathbb{Q}_p^d:\, |f(\mathbf{x})|>\alpha\}} |f(\mathbf{x})|^q \, d\mathbf{x}, \;\; \alpha > 0.$$

Lemma 2.2. *If $f \in L^1(\mathbb{Q}_p^d)$, then we have that*

$$\int_{E_{ab}} |f(\mathbf{x})| \, d\mathbf{x} = -\int_a^b \alpha \, d\omega(\alpha),$$

where $E_{ab} = \{\mathbf{x} \in \mathbb{Q}_p^d : a < |f(\mathbf{x})| \le b\}$ for $a, b \in \mathbb{R}$ with $0 < a < b < \infty$.

Proof. Since $f \in L^1(\mathbb{Q}_p^d)$, the distribution function ω_H is of bounded variation on $[a, b]$. So the Riemann-Stieltjes integral on the right exists. Let $\mathcal{P} = \{a = \alpha_0 < \alpha_1 < \cdots < \alpha_k = b\}$ be a partition of $[a, b]$ and let $E_j = \{\mathbf{x} \in \mathbb{Q}_p^d : \alpha_{j-1} < |f(\mathbf{x})| \le \alpha_j\}$ for $j = 1, 2, \ldots, k$. Then we see that $E_{ab} = \cup_{j=1}^k E_j$ is the disjoint union of measurable sets. Thus we have that

$$\int_{E_{ab}} |f(\mathbf{x})| \, d\mathbf{x} = \sum_{j=1}^k \int_{E_j} |f(\mathbf{x})| \, d\mathbf{x}$$

and $|E_j|_H = -[\omega_H(\alpha_j) - \omega_H(\alpha_{j-1})]$, and so we obtain that

$$-\sum_{j=1}^k \alpha_{j-1}[\omega_H(\alpha_j) - \omega_H(\alpha_{j-1})] \le \int_{E_{ab}} |f(\mathbf{x})| \, d\mathbf{x} \le -\sum_{j=1}^k \alpha_j[\omega_H(\alpha_j) - \omega_H(\alpha_{j-1})].$$

Hence we complete the proof by taking $\|\mathcal{P}\| \fallingdotseq \max_{1\le j\le k}(\alpha_j - \alpha_{j-1}) \to 0$. □

Proposition 2.3. *If $f \in L^1(\mathbb{Q}_p^d)$, then we have that*

$$\int_{\mathbb{Q}_p^d} |f(\mathbf{x})| \, d\mathbf{x} = -\int_0^\infty \alpha \, d\omega_H(\alpha).$$

Proof. It easily follows from Lemma 2.2 and the p-adic version [5, 8] of Lebesgue's dominated convergence theorem. □

Lemma 2.4. *If $f \in L^q(\mathbb{Q}_p^d)$ for $q > 0$, then we have that*

$$\int_{\mathbb{Q}_p^d} |f(\mathbf{x})|^q \, d\mathbf{x} = -\int_0^\infty \alpha^q \, d\omega_H(\alpha) = q \int_0^\infty \alpha^{q-1} \, \omega_H(\alpha) \, d\alpha.$$

Proof. It easily follows from the integration by parts on the Riemann-Stieltjes integral, Proposition 2.1 (Chebyshev's inequality), and the p-adic version of Lebesgue's dominated convergence theorem. □

Next we need the p-adic version of the Marcinkiewicz interpolation theorem [3] which is one of powerful tools for $L^q(\mathbb{Q}_p^d)$-estimates of sublinear operators like maximal operators. Indeed its proof can be obtained as in that of the Euclidean case by applying Lemma 2.4.

Theorem 2.5. *For $1 < r \le \infty$, let a mapping $\mathcal{T} : L^1(\mathbb{Q}_p^d) + L^r(\mathbb{Q}_p^d) \to \mathcal{M}(\mathbb{Q}_p^d)$ satisfy*

$$|\mathcal{T}(f+g)(\mathbf{x})| \le |\mathcal{T}f(\mathbf{x})| + |\mathcal{T}g(\mathbf{x})|,\ \mathbf{x} \in \mathbb{Q}_p^d.$$

Suppose that $\mathcal{T}$ is both of weak type $(1,1)$ and of weak type (r,r); that is, there exist some constants $c_1 > 0$ and $c_r > 0$ such that

$$|\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{T}f(\mathbf{x})| > \lambda\}|_H \le \frac{c_1}{\lambda}\,\|f\|_{L^1(\mathbb{Q}_p^d)},\ \lambda > 0,$$
$$|\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{T}f(\mathbf{x})| > \lambda\}|_H \le \frac{c_r^r}{\lambda^r}\,\|f\|^r_{L^r(\mathbb{Q}_p^d)},\ \lambda > 0.$$

Then there exists a constant $C = C(q, r, c_1, c_r) > 0$ such that $\|\mathcal{T}f\|_{L^q(\mathbb{Q}_p^d)} \le C\,\|f\|_{L^q(\mathbb{Q}_p^d)}$ for any $f \in L^q(\mathbb{Q}_p^d)$, $1 < q < r$.

Sketch of the proof. For $\lambda > 0$, we define a function f_1 by

$$f_1(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \text{if } |f(\mathbf{x})| \ge \lambda/2, \\ 0, & \text{if } |f(\mathbf{x})| < \lambda/2. \end{cases}$$

In case that $r = \infty$, we may assume that $\|\mathcal{T}f\|_{L^\infty(\mathbb{Q}_p^d)} \le \|f\|_{L^\infty(\mathbb{Q}_p^d)}$ by dividing $\mathcal{T}$ by the constant c_∞. From the assumption we can easily obtain that

$$\begin{aligned} |\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{T}f(\mathbf{x})| > \lambda\}|_H &\le |\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{T}f_1(\mathbf{x})| > \lambda/2\}|_H \\ &\le \frac{2c_1}{\lambda} \int_{|f|>\lambda/2} |f(\mathbf{x})|\, d\mathbf{x}. \end{aligned}$$

Applying Lemma 2.4 and changing the order of integration, try to derive that

$$\int_{\mathbb{Q}_p^d} |\mathcal{T}f(\mathbf{x})|^q\, d\mathbf{x} \le \frac{2^q q\, c_1}{q-1} \int_{\mathbb{Q}_p^d} |f(\mathbf{x})|^q\, d\mathbf{x}.$$

We now consider the case $1 < r < \infty$. If we set $f_2 = f - f_1$, then it easily follow from the above assumptions that

$$\begin{aligned} &|\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{T}f(\mathbf{x})| > \lambda\}|_H \\ &\le |\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{T}f_1(\mathbf{x})| > \lambda\}|_H + |\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{T}f_2(\mathbf{x})| > \lambda\}|_H \\ &\le \frac{2c_1}{\lambda} \int_{|f|>\lambda/2} |f(\mathbf{x})|\, d\mathbf{x} + \frac{2^r c_r^r}{\lambda^r} \int_{|f|\le\lambda/2} |f(\mathbf{x})|^r\, d\mathbf{x}. \end{aligned}$$

Then apply Lemma 2.4 and changing the order of integration to obtain that

$$\int_{\mathbb{Q}_p^d} |\mathcal{T}f(\mathbf{x})|^q\, d\mathbf{x} \le 2^q q \left(\frac{c_1}{q-1} + \frac{c_r^r}{r-q} \right) \int_{\mathbb{Q}_p^d} |f(\mathbf{x})|^q\, d\mathbf{x}.$$

Therefore we complete the proof. □

3. $L^q(\mathbb{Q}_p^d)$-estimates of maximal operators

First of all we observe several interesting properties on the family

$$\mathfrak{F}_p = \{B_\gamma(\mathbf{x}) : \gamma \in \mathbb{Z}, \mathbf{x} \in \mathbb{Q}_p^d\}$$

of all the p-adic balls, which differ from those of the Euclidean case.

Lemma 3.1. *The family $\mathfrak{F}_p$ has the following properties*:
(a) *If $\gamma \le \gamma'$, then either $B_\gamma(\mathbf{x}) \cap B_{\gamma'}(\mathbf{y}) = \phi$ or $B_\gamma(\mathbf{x}) \subset B_{\gamma'}(\mathbf{y})$.*
(b) *$B_\gamma(\mathbf{x}) = B_\gamma(\mathbf{y})$ if and only if $\mathbf{y} \in B_\gamma(\mathbf{x})$.*

Proof. The first part (a) can easily be derived from the non-Archimedean property of the p-adic norm $|\cdot|_p$. Also the second part (b) is a natural by-product of (a). □

Lemma 3.2. *Let $\mathcal{C} = \{B_\alpha\}_{\alpha\in\mathcal{A}}$ be a subfamily of $\mathfrak{F}_p$ with $\sup_{\alpha\in\mathcal{A}} \mathfrak{r}(B_\alpha) = c_0 < \infty$, where $\mathfrak{r}(B_\alpha)$ denotes the radius of such p-adic ball B_α. If there exists some ball $B_0 \in \mathcal{C}$ with $\mathfrak{r}(B_0) \ge c_0/p$ such that*

$$\mathcal{C}_0 \fallingdotseq \{B_\alpha \in \mathcal{C} : B_\alpha \cap B_0 = \phi\} = \phi,$$

then the subfamily $\mathcal{C}$ is a partially ordered set by inclusion which has a unique maximal element.

Proof. By Lemma 3.1, it is trivial that $\mathcal{C}$ is a partially ordered set by inclusion. From the uniform boundedness of the radii of balls in $\mathcal{C}$, we see that every linearly ordered subset of $\mathcal{C}$ has an upper bound. Thus the subfamily $\mathcal{C}$ has a maximal element by Zorn's lemma. So it suffices to show the uniqueness of maximal element. To see this, we have only to show that if $B_\alpha, B_{\alpha'} \in \mathcal{C}$ with $B_\alpha \supset B_0$ and $B_{\alpha'} \supset B_0$, then either $B_\alpha \subset B_{\alpha'}$ or $B_{\alpha'} \subset B_\alpha$. Indeed, this can easily be derived from Lemma 3.1. Hence we complete the proof. □

We now state a covering lemma which will be useful in proving Theorem 1.1.

Lemma 3.3 (Covering Lemma)**.** *Let E be a measurable subset of $\mathbb{Q}_p^d$ and let $\mathcal{C} = \{B_\alpha\}_{\alpha\in\mathcal{A}}$ be a covering of E which consists of p-adic balls with*

$$\sup_{\alpha\in\mathcal{A}} \mathfrak{r}(B_\alpha) < \infty.$$

Then there exists a pairwise disjoint countable subcovering $\mathcal{C}_0 = \{B_k\}_{k=1}^\infty$ of $\mathcal{C}$ such that

$$|E|_H \le p^d \sum_{k=1}^\infty |B_k|_H.$$

Proof. We see that $\sup_{\alpha\in\mathcal{A}} \mathfrak{r}(B_\alpha) = p^{\gamma_0}$ for some $\gamma_0 \in \mathbb{Z}$. First we choose a ball $B_1 \in \mathcal{C}$ with $\mathfrak{r}(B_1) \ge p^{\gamma_0 - 1}$. We set $\mathcal{C}_1 = \{B_\alpha \in \mathcal{C} : B_\alpha \cap B_1 = \phi\}$. If $\mathcal{C}_1 = \phi$, then by Lemma 3.2 the covering $\mathcal{C}$ of E must be a partially ordered set by inclusion whose unique maximal element with radius p^{γ_0} contains E, and so we are done. So we may assume that $\mathcal{C}_1 \neq \phi$. Then we choose $B_2 \in \mathcal{C}_1$ so that

$$p\ \mathfrak{r}(B_2) \geq \sup_{B_\alpha \in \mathcal{C}_1} \mathfrak{r}(B_\alpha).$$

We set $\mathcal{C}_2 = \{B_\alpha \in \mathcal{C} : B_\alpha \cap (B_1 \cup B_2) = \phi\}$. If $\mathcal{C}_2 = \phi$, then by Lemma 3.2 the covering $\mathcal{C}$ must be the union of two disjoint partially ordered sets by inclusion the union of whose two distinct unique maximal elements with radius less than p^{γ_0} contains E, and thus we are done. Thus we may assume that $\mathcal{C}_2 \neq \phi$. Next we choose $B_3 \in \mathcal{C}_2$ so that

$$p\ \mathfrak{r}(B_3) \geq \sup_{B_\alpha \in \mathcal{C}_2} \mathfrak{r}(B_\alpha).$$

Assume that $B_1, B_2, \ldots, B_k$ have been selected likewise. We now set

$$\mathcal{C}_k = \{B_\alpha \in \mathcal{C} : B_\alpha \cap (\cup_{i=1}^k B_i) = \phi\}.$$

If $\mathcal{C}_k = \phi$, then applying Lemma 3.2 again the covering $\mathcal{C}$ should be the union of k pairwise disjoint partially ordered sets by inclusion the union of whose k distinct unique maximal elements with radius less than p^{γ_0} contains E, and so we are done. Thus we may assume that $\mathcal{C}_k \neq \phi$. Next we choose $B_{k+1} \in \mathcal{C}_k$ so that

$$p\ \mathfrak{r}(B_{k+1}) \geq \sup_{B_\alpha \in \mathcal{C}_k} \mathfrak{r}(B_\alpha). \tag{3.1}$$

Continuing this process, we obtain a countable collection $\mathcal{C}_0 = \{B_k\}_{k=1}^\infty$ of pairwise disjoint p-adic balls. If $\sum_{k=1}^\infty |B_k|_H = \infty$, then there is nothing to prove. So we may assume that

$$\sum_{k=1}^\infty |B_k|_H < \infty. \tag{3.2}$$

If B_k^* denotes the p-adic concentric ball of B_k with $\mathfrak{r}(B_k^*) = p\ \mathfrak{r}(B_k)$, then we claim that

$$E \subset \bigcup_{k=1}^\infty B_k^*. \tag{3.3}$$

To show the claim (3.3), it suffices to prove that $B_\alpha \subset \cup_{k=1}^\infty B_k^*$ for any $B_\alpha \in \mathcal{C}$. If $B_\alpha \in \mathcal{C}_0$, then we are done. So we assume that $B_\alpha \notin \mathcal{C}_0$. Since $\lim_{k\to\infty} |B_k|_H = 0$ by (3.2), the number $k_0 \in \mathbb{N}$ given by

$$k_0 = \min\{k \in \mathbb{N} : p\ \mathfrak{r}(B_{k+1}) < \mathfrak{r}(B_\alpha)\} \tag{3.4}$$

is well-defined. Then the ball B_α must intersect one of the balls $B_1, B_2, \ldots, B_{k_0}$; which otherwise contradicts (3.1). If $B_\alpha \cap B_{i_0} \neq \phi$ for some i_0 with $1 \leq i_0 \leq k_0$, then it follows from Lemma 3.1 that $B_\alpha \subset B_{i_0}^*$ because $\mathfrak{r}(B_{i_0}^*) = p\ \mathfrak{r}(B_{i_0}) \geq \mathfrak{r}(B_\alpha)$ by (3.4). Therefore the claim (3.3) implies that

$$|E|_H \leq \sum_{k=1}^\infty |B_k^*|_H = p^d \sum_{k=1}^\infty |B_k|_H.$$

Hence we complete the proof. $\square$

Proof of Theorem 1.1. Since it is easy to see that $\mathcal{M}_p$ is bounded on $L^\infty(\mathbb{Q}_p^d)$, by Theorem 2.5 it suffices to show that $\mathcal{M}_p$ is of weak type $(1,1)$ on $L^1(\mathbb{Q}_p^d)$. For $\lambda > 0$, we set $E_\lambda = \{\mathbf{x} \in \mathbb{Q}_p^d : \mathcal{M}_p f(\mathbf{x}) > \lambda\}$. We take any $\mathbf{x} \in E_\lambda$. Then there exists a p-adic ball $B_{\gamma_\mathbf{x}}(\mathbf{x})$ such that

$$\int_{B_{\gamma_\mathbf{x}}(\mathbf{x})} |f(\mathbf{y})|\, d\mathbf{y} > \lambda\, |B_{\gamma_\mathbf{x}}(\mathbf{x})|_H. \tag{3.5}$$

By Lemma 3.3, we may choose a sequence $\{\mathbf{x}_k\}_{k=1}^\infty \subset E_\lambda$ such that the collection $\{B_{\gamma_{\mathbf{x}_k}}(\mathbf{x}_k)\}_{k=1}^\infty$ of such p-adic balls is pairwise disjoint and

$$|E_\lambda|_H \le p^d \sum_{k=1}^\infty |B_{\gamma_{\mathbf{x}_k}}(\mathbf{x}_k)|_H.$$

Hence by (3.5) we conclude that

$$|E_\lambda|_H \le p^d \sum_{k=1}^\infty |B_{\gamma_{\mathbf{x}_k}}(\mathbf{x}_k)|_H \le \frac{p^d}{\lambda} \int_{\cup_{k=1}^\infty B_{\gamma_{\mathbf{x}_k}}(\mathbf{x}_k)} |f(\mathbf{y})|\, d\mathbf{y} \le \frac{p^d}{\lambda}\, \|f\|_{L^1(\mathbb{Q}_p^d)}.$$

Therefore we complete the proof. $\square$

Proof of Theorem 1.3. From Theorem 1.1, it suffices to prove that

$$\mathfrak{M}_p f(\mathbf{x}) \le (1-p^{-d})\, c(p,\Phi)\, \mathcal{M}_p f(\mathbf{x}), \;\; \mathbf{x} \in \mathbb{Q}_p^d$$

for any $f \in L^q(\mathbb{Q}_p^d)$, $1 < q \le \infty$. For $\gamma \in \mathbb{Z}$, we set $B = \{\, (\mathbf{y}, t) \in \mathbb{Q}_p^d \times \mathbb{R}_+ : K_\gamma(\mathbf{y}) > t \,\}$. We observe that

$$K_\gamma(\mathbf{y}) = \int_0^{K_\gamma(\mathbf{y})} dt = \int_0^\infty \chi_B(\mathbf{y}, t)\, dt.$$

Then it follows from the translation invariance of the Haar measure and changing the order of integration that

$$\begin{aligned} |K_\gamma * f(\mathbf{x})| &= \left| \int_{\mathbb{Q}_p^d} f(\mathbf{x}-\mathbf{y}) K_\gamma(\mathbf{y})\, d\mathbf{y} \right| \\ &\le \int_{\mathbb{Q}_p^d} |f(\mathbf{x}-\mathbf{y})| \left(\int_0^\infty \chi_B(\mathbf{y},t)\, dt \right) d\mathbf{y} \\ &= \int_0^\infty \left(\int_{\mathbb{Q}_p^d} |f(\mathbf{x}-\mathbf{y})|\, \chi_B(\mathbf{y},t)\, d\mathbf{y} \right) dt \\ &= \int_0^\infty \left(\int_{B_t} |f(\mathbf{x}-\mathbf{y})|\, d\mathbf{y} \right) dt, \end{aligned} \tag{3.6}$$

where $B_t = \{\mathbf{y} \in \mathbb{Q}_p^d : K_\gamma(\mathbf{y}) > t\,\}$ for $t > 0$. Here we note that B_t is a p-adic ball because $K(\mathbf{y}) = \Phi(|\mathbf{y}|_p)$ and $\Phi(t)$ is a nonnegative monotone decreasing function on $(0,\infty)$. Thus by (3.6) we have that

$$\begin{aligned}|K_\gamma * f(\mathbf{x})| &\le \int_0^\infty |B_t|_H \left(\frac{1}{|B_t|_H}\int_{B_t} |f(\mathbf{x}-\mathbf{y})|\,d\mathbf{y}\right) dt \\ &\le \left(\int_0^\infty |B_t|_H\,dt\right) \mathcal{M}_p f(\mathbf{x}),\ \mathbf{x}\in\mathbb{Q}_p^d \end{aligned} \tag{3.7}$$

for any $\gamma\in\mathbb{Z}$. It also follows from Lemma 2.4 and simple calculation on the integration on $\mathbb{Q}_p^d$ that

$$\begin{aligned}\int_0^\infty |B_t|_H\,dt = \|K_\gamma\|_{L^1(\mathbb{Q}_p^d)} &= \|K\|_{L^1(\mathbb{Q}_p^d)} \\ &= \lim_{n\to\infty} \sum_{-\infty<\gamma\le n} \int_{S_\gamma(\mathbf{0})} \Phi(|\mathbf{x}|_p)\,d\mathbf{x} \\ &= \lim_{n\to\infty} \sum_{-\infty<\gamma\le n} \Phi(p^\gamma)\,|S_\gamma(\mathbf{0})|_H = (1-p^{-d})\,c(p,\Phi).\end{aligned} \tag{3.8}$$

Therefore by (3.7) and (3.8) we conclude that

$$\mathfrak{M}_p f(\mathbf{x}) \le (1-p^{-d})\,c(p,\Phi)\,\mathcal{M}_p f(\mathbf{x}),\ \mathbf{x}\in\mathbb{Q}_p^d$$

for any $f\in L^q(\mathbb{Q}_p^d)$, $1<q\le\infty$. Hence this complete the proof by Theorem 1.1. □

4. Several convergence of convolution means with kernel integrable on $\mathbb{Q}_p^d$

In this section, we prove Corollary 1.2 and Corollary 1.4. Since Corollary 1.2 is a special case of Corollary 1.4 with kernel $K(\mathbf{x}) = \frac{1}{|B_\gamma(\mathbf{0})|_H}\chi_{B_\gamma(\mathbf{0})}(\mathbf{x})$, it suffices to show Corollary 1.4.

Lemma 4.1. *If $K\in L^1(\mathbb{Q}_p^d)$ and $K_\gamma(\mathbf{x}) = p^{-\gamma d}K(p^\gamma\mathbf{x})$ for $\gamma\in\mathbb{Z}$, then we have the following properties*:

(a) $\displaystyle\int_{\mathbb{Q}_p^d} |K_\gamma(\mathbf{x})|\,d\mathbf{x} = \int_{\mathbb{Q}_p^d} |K(\mathbf{x})|\,d\mathbf{x}$ *for all* $\gamma\in\mathbb{Z}$.

(b) $\displaystyle\lim_{\gamma\to-\infty}\int_{\{\mathbf{x}\in\mathbb{Q}_p^d:\,|\mathbf{x}|_p>\delta\}} |K_\gamma(\mathbf{x})|\,d\mathbf{x} = 0$ *for any fixed* $\delta>0$.

Proof. (a) It easily follows from the change of variable and the fact that $d(x\mathbf{x}) = |x|_p^d\,d\mathbf{x}$ for any $x\in\mathbb{Q}_p\setminus\{0\}$.

(b) By the change of variable and the p-adic version of Lebesgue dominated convergence theorem, we obtain that

$$\int_{\{\mathbf{x}\in\mathbb{Q}_p^d:\,|\mathbf{x}|_p>\delta\}} |K_\gamma(\mathbf{x})|\,d\mathbf{x} = \int_{\{\mathbf{x}\in\mathbb{Q}_p^d:\,|\mathbf{x}|_p>\delta p^{-\gamma}\}} |K(\mathbf{x})|\,d\mathbf{x} \to 0$$

as $\gamma\to-\infty$. Hence we complete the proof. □

Lemma 4.2. *For* $\mathbf{y} \in \mathbb{Q}_p^d$ *and* $f \in L^q(\mathbb{Q}_p^d), 1 \le q < \infty$, *we define the translation operator* $\tau_{\mathbf{y}}$ *by* $\tau_{\mathbf{y}} f(\mathbf{x}) = f(\mathbf{x} - \mathbf{y})$. *Then the mapping* $\mathbf{y} \mapsto \tau_{\mathbf{y}} f$ *is a (vector-valued) uniformly continuous function of* $\mathbb{Q}_p^d$ *into* $L^q(\mathbb{Q}_p^d)$ *for* $1 \le q < \infty$.

Proof. We observe that the space $C_c(\mathbb{Q}_p^d)$ is dense in $L^q(\mathbb{Q}_p^d)$, because $\mathbb{Q}_p^d$ is a locally compact Hausdorff space. It thus follows from the uniform continuity of a function in $C_c(\mathbb{Q}_p^d)$ on its compact support. □

Lemma 4.3. *For* $\gamma \in \mathbb{Z}$ *and* $K \in L^1(\mathbb{Q}_p^d)$ *with* $\int_{\mathbb{Q}_p^d} K(\mathbf{x})\, d\mathbf{x} \risingdotseq \beta$, *we set* $K_\gamma(\mathbf{x}) = p^{-\gamma d} K(p^\gamma \mathbf{x})$. *If* $f \in C_c(\mathbb{Q}_p^d)$, *then the convolution means* $K_\gamma * f$ *converge to* βf *uniformly on* $\mathbb{Q}_p^d$ *as* $\gamma \to -\infty$.

Proof. Fix any $\varepsilon > 0$. Since $K \in L^1(\mathbb{Q}_p^d)$, there is some constant $c_1 > 0$ such that $\|K\|_{L^1(\mathbb{Q}_p^d)} \le c_1$. By the uniform continuity of f, there exists some $\delta > 0$ such that

$$\sup_{\mathbf{x} \in \mathbb{Q}_p^d} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})| < \frac{\varepsilon}{2\, c_1} \tag{4.1}$$

whenever $\mathbf{y} \in \mathbb{Q}_p^d$ and $|\mathbf{y}|_p \le \delta$. Since f is uniformly bounded on $\mathbb{Q}_p^d$, there is some constant $c_0 > 0$ such that

$$\sup_{\mathbf{x} \in \mathbb{Q}_p^d} |f(\mathbf{x})| \le c_0. \tag{4.2}$$

From (b) of Lemma 4.1, we see that there is some constant $M > 0$ so large that

$$\int_{\{\mathbf{x} \in \mathbb{Q}_p^d :\, |\mathbf{x}|_p > \delta p^{-\gamma}\}} |K(\mathbf{x})|\, d\mathbf{x} < \frac{\varepsilon}{2\, c_0} \tag{4.3}$$

whenever $\gamma < -M$ and $\gamma \in \mathbb{Z}$. Then it follows from (4.1), (4.2), and (4.3) that

$$\begin{aligned} &\sup_{\mathbf{x} \in \mathbb{Q}_p^d} |K_\gamma * f(\mathbf{x}) - \beta f(\mathbf{x})| \\ &\le \int_{\{\mathbf{y} \in \mathbb{Q}_p^d : |\mathbf{y}|_p \le \delta\}} \left(\sup_{\mathbf{x} \in \mathbb{Q}_p^d} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})| \right) |K_\gamma(\mathbf{y})|\, d\mathbf{y} \\ &\quad + \int_{\{\mathbf{y} \in \mathbb{Q}_p^d : |\mathbf{y}|_p > \delta\}} \left(\sup_{\mathbf{x} \in \mathbb{Q}_p^d} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})| \right) |K_\gamma(\mathbf{y})|\, d\mathbf{y} \\ &\le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \end{aligned}$$

whenever $\gamma < -M$ and $\gamma \in \mathbb{Z}$. Hence we complete the proof. □

Proof of Corollary 1.4. (a) Take any $f \in L^q(\mathbb{Q}_p^d)$ for $1 \le q < \infty$. Then there is some constant $c_2 > 0$ such that $\|f\|_{L^q(\mathbb{Q}_p^d)} \le c_2$. Since we see that $K \in L^1(\mathbb{Q}_p^d)$

from (1.4), there exists some constant $c_1 > 0$ such that $\|K\|_{L^1(\mathbb{Q}_p^d)} \le c_1$. Fix any $\varepsilon > 0$. By Lemma 4.2, there exists some $\delta > 0$ such that

$$\|\tau_{\mathbf{y}} f - f\|_{L^q(\mathbb{Q}_p^d)} < \frac{\varepsilon}{2\,c_1} \tag{4.4}$$

whenever $\mathbf{y} \in \mathbb{Q}_p^d$ and $|\mathbf{y}|_p \le \delta$. From (b) of Lemma 4.1, we see that there is some constant $M > 0$ so large that

$$\int_{\{\mathbf{x}\in\mathbb{Q}_p^d:\, |\mathbf{x}|_p > \delta\}} |K_\gamma(\mathbf{x})|\, d\mathbf{x} < \frac{\varepsilon}{4\,c_2} \tag{4.5}$$

whenever $\gamma < -M$ and $\gamma \in \mathbb{Z}$. Then it follows from the p-adic version of the integral Minkowski's inequality and Minkowski's inequality, (4.4), and (4.5) that

$$\begin{aligned}
\|K_\gamma * f - \beta f\|_{L^q(\mathbb{Q}_p^d)} &\le \int_{\mathbb{Q}_p^d} \|\tau_{\mathbf{y}} f - f\|_{L^q(\mathbb{Q}_p^d)}\, |K_\gamma(\mathbf{y})|\, d\mathbf{y} \\
&= \int_{\{\mathbf{y}\in\mathbb{Q}_p^d:\, |\mathbf{y}|_p \le \delta\}} \|\tau_{\mathbf{y}} f - f\|_{L^q(\mathbb{Q}_p^d)}\, |K_\gamma(\mathbf{y})|\, d\mathbf{y} \\
&\quad + 2\,\|f\|_{L^q(\mathbb{Q}_p^d)} \int_{\{\mathbf{y}\in\mathbb{Q}_p^d:\, |\mathbf{y}|_p > \delta\}} |K_\gamma(\mathbf{y})|\, d\mathbf{y} \\
&\le \frac{1}{2}\,\varepsilon + \frac{1}{2}\,\varepsilon = \varepsilon,
\end{aligned}$$

whenever $\gamma < -M$ and $\gamma \in \mathbb{Z}$.

(b) Take any $f \in L^q(\mathbb{Q}_p^d)$ for $1 \le q < \infty$ and fix any $\varepsilon > 0$. Since the space $C_c(\mathbb{Q}_p^d)$ is dense in $L^q(\mathbb{Q}_p^d)$ for each $n \in \mathbb{N}$ there exists some $g_n \in C_c(\mathbb{Q}_p^d)$ such that

$$\|f - g_n\|_{L^q(\mathbb{Q}_p^d)} < \frac{\varepsilon^{1/q}}{2\,c_3\,n}, \tag{4.6}$$

where $c_3 > 0$ is some constant with $c_3 > c_{pq}$ for the operator norm c_{pq} of $\mathfrak{M}_p$ in Theorem 1.3 which is given by

$$c_{pq} = \begin{cases} \|\mathfrak{M}_p\|_{L^q(\mathbb{Q}_p^d)\to L^q(\mathbb{Q}_p^d)}, & 1 < q < \infty, \\ \|\mathfrak{M}_p\|_{L^1(\mathbb{Q}_p^d)\to L^{1,\infty}(\mathbb{Q}_p^d)}, & q = 1. \end{cases}$$

Here, we note that $L^{1,\infty}(\mathbb{Q}_p^d)$ denotes the weak $L^1(\mathbb{Q}_p^d)$ space. For $\mathbf{x} \in \mathbb{Q}_p^d$ and $h \in L^q(\mathbb{Q}_p^d)$, $1 \le q < \infty$, we define the operator Ω by

$$\Omega(h)(\mathbf{x}) = \limsup_{\gamma\to-\infty} K_\gamma * h(\mathbf{x}) - \liminf_{\gamma\to-\infty} K_\gamma * h(\mathbf{x}) \ge 0.$$

Then we see that $\Omega(h)(\mathbf{x}) \le 2\,\mathfrak{M}_p\, h(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{Q}_p^d$, and also $\Omega(g_n) = 0$ for all $n \in \mathbb{N}$ by Lemma 4.3. Since $\Omega(f) \le \Omega(f - g_n)$ for all $n \in \mathbb{N}$, by Theorem 1.3 and (4.6) we obtain the following estimate

$$\begin{aligned}
|\{\mathbf{x} \in \mathbb{Q}_p^d : \Omega(f)(\mathbf{x}) > 0\}|_H &= \lim_{n\to\infty} |\{\mathbf{x} \in \mathbb{Q}_p^d : \Omega(f)(\mathbf{x}) > 1/n\}|_H \\
&= \lim_{n\to\infty} |\{\mathbf{x} \in \mathbb{Q}_p^d : \Omega(f-g_n)(\mathbf{x}) > 1/n\}|_H \\
&\le \lim_{n\to\infty} |\{\mathbf{x} \in \mathbb{Q}_p^d : 2\,\mathfrak{M}_p(f-g_n)(\mathbf{x}) > 1/n\}|_H \\
&\le \lim_{n\to\infty} 2^q\, n^q\, c_3^q\, \|f-g_n\|^q_{L^q(\mathbb{Q}_p^d)} < \varepsilon.
\end{aligned}$$

Taking $\varepsilon \downarrow 0$, we have that $|\{\mathbf{x} \in \mathbb{Q}_p^d : \Omega(f)(\mathbf{x}) > 0\}|_H = 0$. This implies the required result. Hence we complete the proof. □

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