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CONSTRUCTION OF MANY d-ALGEBRAS

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Abstract. In this paper we consider constructive function triples on the real numbers $\mathbb{R}$ and on (not necessarily commutative) integral domains D which permit the construction of a multitude of d-algebras via these constructive function triples. At the same time these constructions permit one to consider various conditions on these d-algebras for subsets of solutions of various equations, thereby producing geometric problems and interesting visualizations of some of these subsets of solutions. In particular, one may consider what notions such as "locally BCK" ought to mean, certainly in the setting provided below.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([4, 5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Q. P. Hu and X. Li ([2, 3]) introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. BCK-algebras also have some connections with other areas. D. Mundici [10] proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras, and J. Meng [8] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. J. Neggers and H. S. Kim introduced the notion of d-algebras which is another useful generalization of BCK-algebras, and then investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs ([12]). After that some further aspects were studied ([6, 7, 11]). As a generalization of BCK-algebras (see [9]), d-algebras are obtained by deleting identities. Given one of these deleted identities a related identities are constructed by replacing one of the terms involving the original operation by an identical term involving a second (companion) operation, thus producing the notion of companion d-algebra which also (precisely) generalizes the notion of BCK-algebra and is such that not every d-algebra is

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one of these. Recently, the present author with H. S. Kim and J. Neggers ([1]) developed a theory of companion d-algebras in sufficient detail to demonstrate considerable parallelism with the theory of BCK-algebras as well as to obtain a collection of results of a novel type. In this paper we address the question of the construction of a large class of d-algebras essentially unrelated to other methods of construction of such algebras as derive from the theory of BCK-algebras itself, from the theory of posets, from lattice theory, from the theory of digraphs, each of which imparts a special viewpoint and a special flavor to the subject, which although useful to the intuition when it comes to creating proofs of particular results may also make it difficult to come up with counterexamples to conjectures of a general nature based on observations on more restricted classes of the algebras actually under consideration. The availability of such large classes can be very helpful in the successful application of one's intuitive sense subject to the discipline of counterexamples to the proper formulation of insightful propositions clarifying the subject. It is our hope that not only will the developments in this paper serve this goal, but in addition prove to be interesting in their own right.

2. Preliminaries

A *d-algebra* ([12]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all x, y in X.

A BCK-algebra is a d-algebra $(X; *, 0)$ satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$

for all x, y, z in X.

Example 2.1 ([12])**.** (a) Every BCK-algebra is a d-algebra.

(b) Let $X := \{0, 1, 2\}$ be a set with the following Table 1:

Table 1

$*$	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

Then $(X; *, 0)$ is a d-algebra, but not a BCK-algebra, since $(2 * (2 * 2)) * 2 = (2 * 0) * 2 = 1 * 2 = 2 \neq 0$.

(c) Let $\mathbb{R}$ be the set of all real numbers and define $x * y := x \cdot (x - y)$, $x, y \in \mathbb{R}$, where " $\cdot$ " and " $-$ " are ordinary product and substraction of real numbers.

Then $x * x = 0, 0 * x = 0, x * 0 = x^2$. If $x * y = y * x = 0$, then $x(x - y) = 0$ and $x^2 = xy$, $y(y - x) = 0$, $y^2 = xy$. Thus if $x = 0$, $y^2 = 0$, $y = 0$; if $y = 0$, $x^2 = 0$, $x = 0$ and if $xy \neq 0$, then $x = y$. Hence $(R; *, 0)$ is a d-algebra, but not BCK-algebra, since $(2 * 0) * 2 \neq 0$.

3. Main results

Let $f, g : \mathbb{R} \to \mathbb{R}$ be real valued functions such that $f(t) = 0$ if and only if $t = 0$ and $g(t) = 0$ if and only if $t = 0$. Furthermore, let $h : \mathbb{R}^2 \to \mathbb{R}$ be a real valued function such that $h(u, t) \neq 0$ when $u \neq t$. We say a triple (f, g, h) described above is called a *constructive function triple* on $\mathbb{R}$. For example, $f(t) = g(t) = t$, $h(u, t) = 1$ is such a triple.

Theorem 3.1. *Let (f, g, h) be a constructive function triple on $\mathbb{R}$ and $e \in \mathbb{R}$. If we define*

$$x * y := f(x - y)g(e - x)h(x, y) + e, \tag{1}$$

where $x, y \in \mathbb{R}$. Then $(\mathbb{R}; *, e)$ is a d-algebra.

Proof. For any $x \in \mathbb{R}$, $x * x = f(0)g(e - x)h(x, x) + e = e$ and $e * x = f(e - x)g(0)h(e, x) + e = e$. If $x * y = y * x = e$, then $f(x - y)g(e - x)h(x, y) = 0 = f(y - x)g(e - y)h(y, x)$. Assume $x \neq y$. Then $h(x, y) \neq 0 \neq h(y, x)$ and $f(y - x)g(e - y) = 0$. This means either $x - y = 0$ or $e - x = 0$; either $y - x = 0$ or $e - y = 0$. Since $x \neq y$, we obtain $e - x = 0, e - y = 0$, i.e., $x = e = y$, a contradiction. Hence $(\mathbb{R}; *, e)$ is a d-algebra. $\square$

For example, the functions $f(t) = e^t - 1, g(t) = t^3$ and $h(u, t) = (u - t)^2$ will yield a d-algebra on the reals.

The d-algebra $(\mathbb{R}; *, e)$ described above is called a *constructive function d-algebra on $\mathbb{R}$ determined by (f, g, h)*.

Example 3.2. Let $\mathbb{K}$ be any subring of the real numbers $\mathbb{R}$ and let (f, g, h) be a constructive function triple on $\mathbb{K}$. If we define $x * y$ on $\mathbb{K}$ as in (1), where $e \in \mathbb{K}$, then $(\mathbb{K}; *, e)$ is a d-algebra.

Example 3.3. Let $\mathbb{D}$ be any (not necessarily commutative) integral domain and let (f, g, h) be a constructive function triple on $\mathbb{D}$. If we define $x * y$ on $\mathbb{D}$ as in (1), where $e \in \mathbb{D}$, then $(\mathbb{D}; *, e)$ is a d-algebra.

Proposition 3.4. *Let $(\mathbb{R}; *, e)$ be a constructive function d-algebra determined by (f, g, h) satisfying the condition*:

$$x * e = x \quad \textit{for all} \quad x \in \mathbb{R}. \tag{A}$$

Then $f(t)g(-t)h(t + e, e) = t$ for any t in $\mathbb{R}$.

Proof. Since $x * e = x$, we have $x = x * e = f(x - e)g(e - x)h(x, e) + e$ and thus $f(x - e)g(e - x)h(x, e) = x - e$. If we set $x - e = t$, then $f(t)g(-t)h(t + e, e) = t$. $\square$

For example, if $f(t) = g(t) = \sqrt[3]{t}$, then $h(t+e, e) = \sqrt[3]{t}$, where $e \in \mathbb{R}$. If we take $t := x - e$, then $h(x, e) = \sqrt[3]{x-e}$. Hence $x * y = \sqrt[3]{x-e}\sqrt[3]{x-e}\sqrt[3]{x-y} + e$ satisfies (A).

Theorem 3.5. *Let* $(\mathbb{R}; *, e)$ *be a constructive function* d*-algebra determined by* (f, g, h)*. If it satisfies the condition:*

$$\text{(B)} \qquad (x * (x * y)) * y = e$$

for any $x, t \in \mathbb{R}$*, then it also satisfies* (A).

Proof. Assume (B) holds. Let $u := x * (x * y)$. Then $e = u * y = f(u - y)g(e - u)h(u, y) + e$ and hence $f(u - y)g(e - u)h(u, y) = 0$. If $u \neq y$, then $h(u, y) \neq 0$ and thus either $f(u - y) = 0$ or $g(e - u) = 0$. Hence $e = u = x * (x * y)$ for any $x, y \in \mathbb{R}$. If we take $y := e$, then $e = x*(x*e) = f(x - x*e)g(e - x)h(x, x*e) + e$ and thus $f(x - x * e)g(e - x)h(x, x * e) = 0$. By the definition of constructive functions we obtain either $x = x * e$ or $e - x = 0$, i.e., in any case $x = x * e$ since $e = e * e$ as well. If $u = y$, then $x * (x * y) = u = y$. If we take $y := e$, then $x * (x * e) = x$, which means $x = x * e$ for any $x \in \mathbb{R}$. □

For a d-algebra to be "*commutative*", the required condition is that:

$$\text{(C)} \qquad x * (x * y) = y * (y * x),$$

i.e., $f(x - x * y)g(e - x)h(x, x * y) = f(y - y * x)g(e - y)h(y, y * x)$. If we set $F(x, y) := x * (x * y) - y * (y * x)$, then we obtain a level curve of the function $F : \mathbb{R}^2 \to \mathbb{R}$. If we review condition (B) in the light of condition (C), then we note that if $E(x, y) = (x * (x * y)) * y - e$, then solving the equation $E(x, y) = 0$ and determining properties of this solution set becomes a "geometric problem".

From the algebraic point of view the most interesting case may be the situation $D = \mathbb{C}$, the algebraically closed field of complex numbers.

Theorem 3.6. *Let* $(\mathbb{C}; *, e)$ *be a constructive function* d*-algebra on the algebraically closed field* $\mathbb{C}$ *of complex numbers. If we define* $x*y := (x-y)(e-x)+e$*, then the solution set of* $F(x, y) = x * (x * y) - y * (y * x) = 0$ *is* $\{(x, y) \mid y = x$ *or* $(x - e - \frac{1}{2})^2 + (y - e - \frac{1}{2})^2 = (\frac{1}{\sqrt{2}})^2\}$.

Proof. For any $x, y \in X$, $x * (x * y) = -(1 + x - y)(e - x)^2 + e$. Hence $F(x, y) = x * (x * y) - y * (y * x) = (x - y)[(e - x) + (e - y) - (e - x)^2 - (e - y)^2]$ which produces a solution set $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 : x - y = 0$; $\Gamma_2 : (x - e)^2 + (y - e)^2 - (x - e) - (y - e) = 0$ or $(x - e - \frac{1}{2})^2 + (y - e - \frac{1}{2})^2 = (\frac{1}{\sqrt{2}})^2$. □

This is a description of the *commutativity set* of the d-algebra. Note that this is an algebraic set of the union of two algebraic geometry varieties, viz., the line $x = y$ and the complex circle.

Next, we consider the equation $E(x, y) = 0$. We shall refer to this set as the *implicativity set* of the d-algebra. In Theorem 3.6, since $x*y = (x-y)(e-x)+e$, we have $x * (x * y) = (y - x - 1)(e - x)^2 + e$, and thus $E(x, y) = (y - x - 1)(e - x)^2[y - e - (y - x - 1)(e - x)^2] = 0$, whence $x = e, y = x + 1$, or $y(1 - (e - x)^2) = e - (x + 1)(e - x)^2$. If $(e - x)^2 = 1$, then $x = e \pm 1$, while otherwise, $y = \frac{e-(x+1)(e-x)^2}{1-(e-x)^2}$. If $e = 0$, then $y = x + 1 + \frac{1}{x-1}$, with asymptote $y = x + 1$, which is also on the implicativity set.

Next, consider the condition:

$$G(x, y, z) = ((x * y) * (x * z)) * (z * y) - e = 0.$$

This equation is a surface in $\mathbb{R}^3$ when we are operating under general circumstances. The format will produce a product

$$G(x, y, z) = G_1(x, y, z) \cdots G_s(x, y, z) = 0$$

of "simplest" functions, each of which generates a surface $G_k(x, y, z) = 0$ which may then be analyzed according to the principle outlined above. The resulting structure is the *transitivity set*.

If a point (x, y, z) is on the transitivity set and if (x, y) is on the implicativity set, then (x, y, z) is called a *BCK-point*. Even though (x, y, z) is a BCK-point, it is not true that $\{x, y, z\}$ forms a BCK-algebra, but it is certainly BCK-like in a non-symmetric way.

Let $(\mathbb{R}; *, e_1)$ and $(\mathbb{R}; \circledast, e_2)$ be constructive function d-algebras. A bijective mapping $\varphi : (\mathbb{R}; *, e_1) \to (\mathbb{R}; \circledast, e_2)$ is called an *isomorphism* if $\varphi(x * y) = \varphi(x) \circledast \varphi(y)$ for any $x, y \in \mathbb{R}$.

Proposition 3.7. *Let $(\mathbb{R}; *, e_1)$ and $(\mathbb{R}; \circledast, e_2)$ be constructive function d-algebras determined by $(f, g, h), (\widehat{f}, \widehat{g}, \widehat{h})$ respectively. Then a mapping $\varphi : (\mathbb{R}; *, e_1) \to (\mathbb{R}; \circledast, e_2)$ defined by $\varphi(x) := \lambda(x - e_1) + e_2$, $\lambda \neq 0$, is an isomorphism if $\widehat{f}(\lambda(x - y)) = \lambda f(x - y)$, $\widehat{g}(-\lambda(x - e_1)) = g(e_1 - x)$, $\widehat{h}(\varphi(x), \varphi(y)) = h(x, y)$ for any $x, y \in \mathbb{R}$.*

Proof. $\varphi(x*y) = \varphi(f(x-y)g(e_1-x)h(x, y)+e_1) = \lambda f(x-y)g(e_1-x)h(x, y)+e_2$, and $\varphi(x) \circledast \varphi(y) = \widehat{f}(\varphi(x) - \varphi(y))\widehat{g}(-\lambda(x - e_1))\widehat{h}(\varphi(x), \varphi(y)) + e_2 = \widehat{f}(\lambda(x - y))\widehat{g}(-\lambda(x - e_1))\widehat{h}(\varphi(x), \varphi(y)) + e_2 = \lambda f(x - y)g(e_1 - x)h(x, y) + e_2$, proving $\varphi(x * y) = \varphi(x) \circledast \varphi(y)$. □

Furthermore, if $z := \lambda(x-e_1)+e_2$, then $x = \frac{1}{\lambda}(z-e_2)+e_1$, i.e., $\psi(x) = \frac{1}{\lambda}(z-e_2)+e_1$ has the right form as well as $\varphi^{-1} = \psi$. If we take $\widehat{x} := \frac{1}{\lambda}((x-e_2)+\lambda e_1)$, then $e_1 - \widehat{x} = \frac{1}{\lambda}(e_2 - x)$ and $\widehat{g}(e_2 - x) = \widehat{g}(\lambda(e_1 - \widehat{x}) = g(e_1 - \widehat{x}) = g(\frac{1}{\lambda}(e_2 - x))$. Hence

$$\begin{aligned}
\psi(x \circledast y) &= \frac{1}{\lambda}\widehat{f}(x - y)\widehat{g}(e_2 - x)\widehat{h}(x, y) + e_1 \\
&= \frac{1}{\lambda}\lambda f\left(\frac{x - y}{\lambda}\right) g\left(\frac{e_2 - x}{\lambda}\right)\widehat{h}(x, y) + e_1 \\
&= f(\psi(x) - \psi(y))g(e_1 - \psi(x))h(\psi(x), \psi(y)) + e_1 = \psi(x) * \psi(y).
\end{aligned}$$

Thus the "isomorphism conditions" are therefore for such a linear φ

$$\text{(i) } \widehat{f}(t) = f\left(\frac{t}{\lambda}\right) \qquad \text{(ii) } \widehat{g}(t) = g\left(\frac{t}{\lambda}\right) \qquad \text{(iii) } \widehat{h}(x, y) = h(\psi(x), \psi(y))$$

which is not entirely surprising.

4. Conclusion

The constructions given in this paper provide a large class of examples of d-algebras other than those usually seen when starting from the normal BCK/lattice theory perspectives. At the same time these constructions provide ways to visualize certain interesting subsets or points (such as BCK-points) of these d-algebras as solutions to geometric problems in $\mathbb{R}^3$ (3-space), $\mathbb{C}^3$ (complex 3-space) or elsewhere, in this manner permitting one to visualize a number of concepts in a more geometric setting. It is naturally of interest to provide such a bridge not only to enrich the theory of d-algebras thereby, but also to provide geometers with problems which, other than being of interest to the readers of d-algebras, may be of intrinsic meaning to geometry (mostly in three dimensions perhaps) itself.

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