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FUZZY TRANSLATIONS AND FUZZY MULTIPLICATIONS OF BCK/BCI-ALGEBRAS

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Abstract. Fuzzy translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications of fuzzy subalgebras in BCK/BCI-algebras are discussed. Relations among fuzzy translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications are investigated.

1. Introduction

The study of BCK-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. For the general development of BCK/BCI-algebras, the ideal theory and its fuzzification play an important role. Jun (together with Kim, Meng, Song, and Xin) studied fuzzy trends of several notions in BCK/BCI-algebras (see [2, 3, 4, 6]). In this paper, we discuss fuzzy translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications of fuzzy subalgebras in BCK/BCI-algebras. We investigate relations among fuzzy translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications.

2. Preliminaries

A BCK-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, \theta)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)$ $(((x * y) * (x * z)) * (z * y) = \theta)$,
(II) $(\forall x, y \in X)$ $((x * (x * y)) * y = \theta)$,
(III) $(\forall x \in X)$ $(x * x = \theta)$,
(IV) $(\forall x, y \in X)$ $(x * y = \theta, y * x = \theta \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

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(V) $(\forall x \in X)\ (\theta * x = \theta)$,

then X is called a *BCK-algebra*. Any BCK-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * \theta = x)$,
(a2) $(\forall x, y, z \in X)\ (x * y = \theta \Rightarrow (x * z) * (y * z) = \theta,\ (z * y) * (z * x) = \theta)$,
(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X)\ (((x * z) * (y * z)) * (x * y) = \theta)$.

A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [1] and [5] for further information regarding BCK/BCI-algebras.

A fuzzy subset μ of a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X if it satisfies:

$$(\forall x, y \in X)(\mu(x * y) \geq \min\{\mu(x), \mu(y)\}).$$

3. Fuzzy translations and fuzzy multiplications of fuzzy subalgebras

In what follows let $X = (X, *, \theta)$ denote a BCK/BCI-algebra, and for any fuzzy set μ of X, we denote $\top := 1 - \sup\{\mu(x) \mid x \in X\}$ unless otherwise specified.

Definition 3.1. Let μ be a fuzzy subset of X and let $\alpha \in [0, \top]$. A mapping $\mu_\alpha^T : X \to [0, 1]$ is called a *fuzzy α-translation* of μ if it satisfies:

$$(\forall x \in X)(\mu_\alpha^T(x) = \mu(x) + \alpha).$$

Theorem 3.2. *Let μ be a fuzzy subalgebra of X and $\alpha \in [0, \top]$. Then the fuzzy α-translation μ_α^T of μ is a fuzzy subalgebra of X.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} \mu_\alpha^T(x * y) &= \mu(x * y) + \alpha \ \geq\ \min\{\mu(x), \mu(y)\} + \alpha \\ &= \min\{\mu(x) + \alpha, \mu(y) + \alpha\} \ =\ \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}. \end{aligned}$$

Hence μ_α^T is a fuzzy subalgebra of X. □

Theorem 3.3. *Let μ be a fuzzy subset of X such that the fuzzy α-translation μ_α^T of μ is a fuzzy subalgebra of X for some $\alpha \in [0, \top]$. Then μ is a fuzzy subalgebra of X.*

Proof. Assume that μ_α^T is a fuzzy subalgebra of X for some $\alpha \in [0, \top]$. Let $x, y \in X$, we have

$$\begin{aligned} \mu(x * y) + \alpha &= \mu_\alpha^T(x * y) \ \geq\ \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\} \\ &= \min\{\mu(x) + \alpha, \mu(y) + \alpha\} \ =\ \min\{\mu(x), \mu(y)\} + \alpha \end{aligned}$$

which implies that $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. Hence μ is a fuzzy subalgebra of X. □

Definition 3.4. Let μ_1 and μ_2 be fuzzy subsets of X. If $\mu_1(x) \leq \mu_2(x)$ for all $x \in X$, then we say that μ_2 is a *fuzzy extension* of μ_1.

Definition 3.5. Let μ_1 and μ_2 be fuzzy subsets of X. Then μ_2 is called a *fuzzy S-extension* of μ_1 if the following assertions are valid:

(i) μ_2 is a fuzzy extension of μ_1.
(ii) If μ_1 is a fuzzy subalgebra of X, then μ_2 is a fuzzy subalgebra of X.

By means of the definition of fuzzy α-translation, we know that $\mu_\alpha^T(x) \geq \mu(x)$ for all $x \in X$. Hence we have the following theorem.

Theorem 3.6. *Let μ be a fuzzy subalgebra of X and $\alpha \in [0, \top]$. Then the fuzzy α-translation μ_α^T of μ is a fuzzy S-extension of μ.*

The converse of Theorem 3.6 is not true in general as seen in the following example.

Example 3.7. Consider a BCK-algebra $X = \{\theta, a, b, c, d\}$ with the following Cayley table:

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	a	θ	θ
b	b	b	θ	b	θ
c	c	a	c	θ	a
d	d	d	d	d	θ

Define a fuzzy subset μ of X by

X	θ	a	b	c	d
μ	0.8	0.5	0.3	0.6	0.2

Then μ is a fuzzy subalgebra of X. Let ν be a fuzzy subset of X given by

X	θ	a	b	c	d
ν	0.84	0.56	0.38	0.67	0.21

Then ν is a fuzzy S-extension of μ. But it is not the fuzzy α-translation μ_α^T of μ for all $\alpha \in [0, \top]$.

Clearly, the intersection of fuzzy S-extensions of a fuzzy subalgebra μ is a fuzzy S-extension of μ. But the union of fuzzy S-extensions of a fuzzy subalgebra μ is not a fuzzy S-extension of μ as seen in the following example.

Example 3.8. Consider a BCK-algebra $X = \{\theta, a, b, c, d\}$ with the following Cayley table:

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	θ	θ	θ
b	b	a	θ	θ	θ
c	c	a	a	θ	θ
d	d	c	c	a	θ

Define a fuzzy subset μ of X by

X	θ	a	b	c	d
μ	0.7	0.4	0.6	0.3	0.3

Then μ is a fuzzy subalgebra of X. Let ν and δ be fuzzy subsets of X given by

X	θ	a	b	c	d
ν	0.8	0.6	0.8	0.4	0.4
δ	0.9	0.6	0.6	0.6	0.7

Then ν and δ are fuzzy S-extensions of μ. But the union $\nu \cup \delta$ is not a fuzzy S-extension of μ since $(\nu \cup \delta)(d * b) = 0.6 \ngeq 0.7 = \min\{(\nu \cup \delta)(d), (\nu \cup \delta)(b)\}$.

For a fuzzy subset μ of X, $\alpha \in [0, \top]$ and $t \in [0, 1]$ with $t \geq \alpha$, let

$$U_\alpha(\mu; t) := \{x \in X \mid \mu(x) \geq t - \alpha\}.$$

If μ is a fuzzy subalgebra of X, then it is clear that $U_\alpha(\mu; t)$ is a subalgebra of X for all $t \in \mathrm{Im}(\mu)$ with $t \geq \alpha$. But, if we do not give a condition that μ is a fuzzy subalgebra of X, then $U_\alpha(\mu; t)$ is not a subalgebra of X as seen in the following example.

Example 3.9. Let $X = \{\theta, a, b, c, d\}$ be a BCK-algebra which is given in Example 3.8. Define a fuzzy subset μ of X by

X	θ	a	b	c	d
μ	0.7	0.4	0.6	0.3	0.5

Then μ is not a fuzzy subalgebra of X since $\mu(d*b) = 0.3 \ngeq 0.5 = \min\{\mu(d), \mu(b)\}$. For $\alpha = 0.1$ and $t = 0.5$, we obtain $U_\alpha(\mu; t) = \{\theta, a, b, d\}$ which is not a subalgebra of X since $d * b = c \notin U_\alpha(\mu; t)$.

Theorem 3.10. *Let μ be a fuzzy subset of X and $\alpha \in [0, \top]$. Then the fuzzy α-translation μ_α^T of μ is a fuzzy subalgebra of X if and only if $U_\alpha(\mu; t)$ is a subalgebra of X for all $t \in \mathrm{Im}(\mu)$ with $t \geq \alpha$.*

Proof. Necessity is clear. To prove the sufficiency, assume that there exist $a, b \in X$ such that $\mu_\alpha^T(a * b) < \beta \leq \min\{\mu_\alpha^T(a), \mu_\alpha^T(b)\}$. Then $\mu(a) \geq \beta - \alpha$ and $\mu(b) \geq \beta - \alpha$, but $\mu(a * b) < \beta - \alpha$. This shows that $a, b \in U_\alpha(\mu; \beta)$ and $a * b \notin U_\alpha(\mu; \beta)$. This is a contradiction, and so $\mu_\alpha^T(x * y) \geq \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$ for all $x, y \in X$. Hence μ_α^T is a fuzzy subalgebra of X. □

Theorem 3.11. *Let μ be a fuzzy subalgebra of X and let $\alpha, \beta \in [0, \top]$. If $\alpha \geq \beta$, then the fuzzy α-translation μ_α^T of μ is a fuzzy S-extension of the fuzzy β-translation μ_β^T of μ.*

Proof. Straightforward. □

For every fuzzy subalgebra μ of X and $\beta \in [0, \top]$, the fuzzy β-translation μ_β^T of μ is a fuzzy subalgebra of X. If ν is a fuzzy S-extension of μ_β^T, then there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and $\nu(x) \geq \mu_\alpha^T(x)$ for all $x \in X$. Thus we have the following theorem.

Theorem 3.12. *Let μ be a fuzzy subalgebra of X and $\beta \in [0, \top]$. For every fuzzy S-extension ν of the fuzzy β-translation μ_β^T of μ, there exists $\alpha \in [0, \top]$ such that $\alpha \geq \beta$ and ν is a fuzzy S-extension of the fuzzy α-translation μ_α^T of μ.*

The following example illustrates Theorem 3.12.

Example 3.13. Consider a BCK-algebra $X = \{\theta, a, b, c, d\}$ with the following Cayley table:

$*$	θ	a	b	c	d
θ	θ	θ	θ	θ	θ
a	a	θ	θ	a	a
b	b	b	θ	b	b
c	c	c	c	θ	c
d	d	d	d	d	θ

Define a fuzzy subset μ of X by

X	θ	a	b	c	d
μ	0.7	0.4	0.2	0.5	0.1

Then μ is a fuzzy subalgebra of X, and $\top = 0.3$. If we take $\beta = 0.2$, then the fuzzy β-translation μ_β^T of μ is given by

X	θ	a	b	c	d
μ_β^T	0.9	0.6	0.4	0.7	0.3

Let ν be a fuzzy subset of X defined by

X	θ	a	b	c	d
ν	0.94	0.63	0.55	0.88	0.37

Then ν is clearly a fuzzy subalgebra of X which is fuzzy extension of μ_β^T, and hence ν is a fuzzy S-extension of the fuzzy β-translation μ_β^T of μ. But ν is a not a fuzzy α-translation of μ for all $\alpha \in [0, \top]$. Take $\alpha = 0.23$. Then $\alpha = 0.23 > 0.2 = \beta$, and the fuzzy α-translation μ_α^T of μ is given as follows:

X	θ	a	b	c	d
μ_α^T	0.93	0.63	0.43	0.73	0.33

Note that $\nu(x) \geq \mu_\alpha^T(x)$ for all $x \in X$, and hence ν is a fuzzy S-extension of the fuzzy α-translation μ_α^T of μ.

A fuzzy S-extension ν of a fuzzy subalgebra μ of X is said to be *normalized* if there exists $x_0 \in X$ such that $\nu(x_0) = 1$. Let μ be a fuzzy subalgebra of X. A fuzzy subset ν of X is called a *maximal fuzzy S-extension* of μ if it satisfies:

(i) ν is a fuzzy S-extension of μ,
(ii) there does not exist another fuzzy subalgebra of X which is a fuzzy extension of ν.

Example 3.14. Let $\mathbb{N}$ be the set of all natural numbers and let $*$ be a binary operation on $\mathbb{N}$ defined by

$$(\forall a, b \in \mathbb{N})\left(a * b = \tfrac{a}{(a,b)}\right),$$

where (a, b) is the greatest common divisor of a and b. Then $(\mathbb{N}; *, 1)$ is a BCK-algebra. Let μ and ν be fuzzy subsets of $\mathbb{N}$ which are defined by $\mu(x) = \frac{1}{3}$ and $\nu(x) = 1$ for all $x \in \mathbb{N}$. Clearly μ and ν are fuzzy subalgebras of $\mathbb{N}$. It is easy to verify that ν is a maximal fuzzy S-extension of μ.

Proposition 3.15. *If a fuzzy subset ν of X is a normalized fuzzy S-extension of a fuzzy subalgebra μ of X, then $\nu(\theta) = 1$.*

Proof. It is clear because $\nu(\theta) \geq \nu(x)$ for all $x \in X$. □

Theorem 3.16. *Let μ be a fuzzy subalgebra of X. Then every maximal fuzzy S-extension of μ is normalized.*

Proof. This follows from the definitions of the maximal and normalized fuzzy S-extensions. □

Definition 3.17. Let μ be a fuzzy subset of X and $\gamma \in [0, 1]$. A *fuzzy γ-multiplication* of μ, denoted by μ_γ^m, is defined to be a mapping

$$\mu_\gamma^m : X \to [0, 1], \; x \mapsto \mu(x) \cdot \gamma.$$

For any fuzzy subset μ of X, a fuzzy 0-multiplication μ_0^m of μ is clearly a fuzzy subalgebra of X.

Theorem 3.18. *If μ is a fuzzy subalgebra of X, then the fuzzy γ-multiplication of μ is a fuzzy subalgebra of X for all $\gamma \in [0, 1]$.*

Proof. Straightforward. □

Theorem 3.19. *For any fuzzy subset μ of X, the following are equivalent:*

(i) *μ is a fuzzy subalgebra of X.*
(ii) $(\forall\gamma\in(0,1])$ *(μ_γ^m is a fuzzy subalgebra of X).*

Proof. Necessity follows from Theorem 3.18. Let $\gamma\in(0,1]$ be such that μ_γ^m is a fuzzy subalgebra of X. Then

$$\begin{aligned}\mu(x*y)\cdot\gamma &= \mu_\gamma^m(x*y) \;\geq\; \min\{\mu_\gamma^m(x),\mu_\gamma^m(y)\}\\ &= \min\{\mu(x)\cdot\gamma,\ \mu(y)\cdot\gamma\} \;=\; \min\{\mu(x),\mu(y)\}\cdot\gamma\end{aligned}$$

for all $x,y\in X$, and so $\mu(x*y)\geq\min\{\mu(x),\mu(y)\}$ for all $x,y\in X$ since $\gamma\neq 0$. Hence μ is a fuzzy subalgebra of X. □

Theorem 3.20. *Let μ be a fuzzy subset of X, $\alpha\in[0,\top]$ and $\gamma\in(0,1]$. Then every fuzzy α-translation μ_α^T of μ is a fuzzy S-extension of the fuzzy γ-multiplication μ_γ^m of μ.*

Proof. For every $x\in X$, we have $\mu_\alpha^T(x)=\mu(x)+\alpha\geq\mu(x)\geq\mu(x)\cdot\gamma=\mu_\gamma^m(x)$, and so μ_α^T is a fuzzy extension of μ_γ^m. Assume that μ_γ^m is a fuzzy subalgebra of X. Then μ is a fuzzy subalgebra of X by Theorem 3.19. It follows from Theorem 3.2 that μ_α^T is a fuzzy subalgebra of X for all $\alpha\in[0,\top]$. Hence every fuzzy α-translation μ_α^T is a fuzzy S-extension of the fuzzy γ-multiplication μ_γ^m. □

The following example shows that Theorem 3.20 is not valid for $\gamma=0$.

Example 3.21. Consider a BCI-algebra $(\mathbb{Z},*,0)$ where $\mathbb{Z}$ is the set of all integers and $*$ is the minus operation. Define a fuzzy set $\mu:\mathbb{Z}\to[0,1]$ by

$$\mu(x):=\begin{cases}0 & \text{if } x>0,\\ \frac{1}{2} & \text{if } x\leq 0.\end{cases}$$

Taking $\gamma=0$, we see that $\mu_0^m(x*y)=0=\min\{\mu_0^m(x),\mu_0^m(y)\}$ for all $x,y\in\mathbb{Z}$, that is, μ_0^m is a fuzzy subalgebra of $\mathbb{Z}$. But if we take $x=-3$ and $y=-5$, then $\mu_\alpha^T(x*y)=\mu_\alpha^T(2)=\mu(2)+\alpha=\alpha<\frac{1}{2}+\alpha=\min\{\mu_\alpha^T(x),\mu_\alpha^T(y)\}$ for all $\alpha\in[0,\frac{1}{2}]$. Hence μ_α^T is not a fuzzy S-extension of μ_0^m for all $\alpha\in[0,\frac{1}{2}]$.

The following example illustrates Theorem 3.20.

Example 3.22. Let $X=\{\theta,a,b,c,d\}$ be a BCK-algebra which is given in Example 3.13, and consider a fuzzy subalgebra μ of X that is defined in Example 3.13. If we take $\gamma=0.1$, then the fuzzy γ-multiplication $\mu_{0.1}^m$ of μ is given by

X	θ	a	b	c	d
$\mu_{0.1}^m$	0.07	0.04	0.02	0.05	0.01

Clearly $\mu_{0.1}^m$ is a fuzzy subalgebra of X. Also, for any $\alpha \in [0, 0.3]$, the fuzzy α-translation μ_α^T of μ is given by

X	θ	a	b	c	d
μ_α^T	$0.7+\alpha$	$0.4+\alpha$	$0.2+\alpha$	$0.5+\alpha$	$0.1+\alpha$

Then μ_α^T is a fuzzy extension of $\mu_{0.1}^m$ and μ_α^T is always a fuzzy subalgebra of X for all $\alpha \in [0, 0.3]$. Hence μ_α^T is a fuzzy S-extension of $\mu_{0.1}^m$ for all $\alpha \in [0, 0.3]$.

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