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ATOMIC HYPER BCK-ALGEBRAS

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Abstract. In this manuscript, we introduce the concept of an atomic subset of the hyper BCK-algebra and study its properties. Also, we give a characterization of the atomic hyper BCK-algebra and show that there are exactly (up to isomorphism) n atomic hyper BCK-algebras H with $|H| = n$ for any natural number n.

1. Introduction and preliminaries

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [2] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. Since then a great deal of literature has been produced on the theory of BCK-algebras. The hyper structure theory (called also multi algebras) was introduced in 1934 by F. Marty at the 8th congress of Scandinavian Mathematicians. In [4], Y. B. Jun et al. applied the hyper structures to BCK-algebras, and introduced the notion of a hyper BCK-algebra which is a generalization of BCK-algebra, and investigated some related properties. Now, we follow [1] and [4] and introduce the concept of an atomic subset of the hyper BCK-algebra and study its properties.

Let H be a non-empty set endowed with a hyper operation " $\circ$ ", that is, $\circ$ is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(\mathcal{H})\backslash\{\phi\}$. For two subset A and B of H, denote by $A \circ B$ the set $\bigcup_{a\in A, b\in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}, \{x\} \circ y$, or $\{x\} \circ \{y\}$.

Definition 1.1 ([4]). By a *hyper BCK-algebra* we mean a non-empty set H endowed with a hyper operation "$\circ$" and a constant 0 satisfying the following axioms:
(H1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(H2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(H3) $x \circ H \ll \{x\}$,

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(H4) $x \ll y$ and $y \ll x$ imply $x = y$
for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyper order* in H.

Theorem 1.2 ([4]). *In any hyper BCK-algebra H, the following hold*:
(a1) $0 \circ 0 = \{0\}$,
(a2) $0 \ll x$,
(a3) $x \ll x$,
(a4) $A \ll A$,
(a5) $A \ll 0$ *implies* $A = \{0\}$,
(a6) $A \subseteq B$ *implies* $A \ll B$,
(a7) $0 \circ x = \{0\}$,
(a8) $x \circ y \ll x$,
(a9) $x \circ 0 = \{x\}$,
(a10) $y \ll z$ *implies* $x \circ z \ll x \circ y$,
(a11) $x \circ y = \{0\}$ *implies* $(x \circ z) \circ (y \circ z) = \{0\}$ *and* $x \circ z \ll y \circ z$,
(a12) $A \circ \{0\} = \{0\}$ *implies* $A = \{0\}$
for all $x, y, z \in H$ *and* $A, B \subseteq H$.

Definition 1.3 ([4]). Let H be a hyper BCK-algebra. Then
(i) A non-empty subset S of H is called a *hyper subalgebra* of H if S is a hyper BCK-algebra with respect to the hyper operation "$\circ$" on H.

Definition 1.4 ([4]). Let H be a hyper BCK-algebra. Then, a non-empty subset I of H with $0 \in I$ is called a weak hyper BCK-ideal of H if it satisfies: $(\forall x, y \in H)(x \circ y \subseteq I$, and $y \in I \Longrightarrow x \in I)$; hyper BCK-ideal of H if it satisfies: $(\forall x, y \in H)(x \circ y \ll I$ and $y \in I \Longrightarrow x \in I)$; reflexive hyper BCK-ideal of H if it is a hyper BCK-ideal of H and satisfies: $(\forall x \in H)$ $x \circ x \subseteq I$; strong hyper BCK-ideal of H if it satisfies: $(\forall x, y \in H)(x \circ y \cap I \neq \phi$ and $y \in I \Longrightarrow x \in I)$.

Theorem 1.5 ([4]). *Let S be a non-empty subset of a hyper BCK-algebra H. Then S is a hyper subalgebra of H if and only if $x \circ y \subseteq S$ for all $x, y \in S$.*

Theorem 1.6 ([4]). *Let H be a hyper BCK-algebra. Then the set $S(H) := \{x \in H | x \circ x = \{0\}\}$ is a hyper subalgebra of H, which is called BCK-part of H.*

Theorem 1.7 ([3]). *Let A be a subset of a hyper BCK-algebra H. If I is a hyper BCK-ideal of H such that $A \ll I$, then A is contained in I.*

Theorem 1.8 ([1]). *Let Θ be a regular congruence relation on H and $\frac{H}{\Theta} = \{[x]_\Theta \mid x \in H\}$. Then $\frac{H}{\Theta}$ with hyperoperation "$\circ$" and hyperorder "$<$" which is defined as follows, is a hyper BCK-algebra which is called quotient hyper BCK-algebra,*

$$[x]_\Theta \circ [y]_\Theta = \{[z]_\Theta : z \in x \circ y\},\ [x]_\Theta < [y]_\Theta \Longleftrightarrow [0]_\Theta \in [x]_\Theta \circ [y]_\Theta.$$

Theorem 1.9 ([1]). *Let Θ be a regular congruence relation on H. Then*

$$[0]_\Theta \text{ is a reflexive hyper } BCK\text{-ideal of } H \Longleftrightarrow \frac{H}{\Theta} \text{ is a } BCK\text{-algebra.}$$

2. Main results

Definition 2.1. Let H be a hyper BCK-algebra. Then

(i) An element a belong to H is said to be an *atom* if it satisfies:

$$(\forall\, x \in H)(x \ll a \Longrightarrow x = 0 \text{ or } x = a).$$

(ii) A subset A of H is said to be *atomic* if each element of A is an atom.

Example 2.2. Consider a hyper BCK-algebra $H = \{0, 1, 2, 3\}$ with the following Cayley table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,3\}$

It is easy to verify that the element 1 is an atom and the subset $A := \{0, 1\}$ is atomic but the subset $B := \{0, 2\}$ is not atomic.

Now, we consider the property of atomic subsets. The following lemma shows that the concept of weak hyper BCK-ideal and hyper BCK-ideal is the same when the subset is atomic.

Lemma 2.3. *Let H be a hyper BCK-algebra and let $A \subseteq H$ be atomic. Then*

$$A \text{ is a weak hyper BCK-ideal of } H \iff A \text{ is a hyper BCK-ideal of } H.$$

Proof. ($\Longrightarrow$) It is clear that $0 \in A$. Suppose that $x, y \in H$ such that $x \circ y \ll A$ and $y \in A$. Let $t \in x \circ y$. Then, by Theorem 1.2(a8), there exists $a \in A$ such that $t \ll a$. Since A is atomic, it follows from definition 2.1 that $t = 0$ or $t = a$. Hence $x \circ y \subseteq A$. Since A is a weak hyper BCK-ideal of H and $y \in A$, it follows from Definition 1.4 that $x \in A$. Therefore A is a hyper BCK-ideal of H.

($\Longleftarrow$) It follows from Theorem 3.21 [4]. □

Lemma 2.4. *Let H be a hyper BCK-algebra. If $A \subseteq H$ is atomic, then $a \circ x \subseteq \{0, a\}$ for all $a \in A$ and $x \in H$.*

Proof. Let $a \in A$ and $x \in H$. It follows from Theorem 1.2(a8) that $t \ll a$ for any $t \in a \circ x$. Since A is atomic, it follows that $t = 0$ or $t = a$. Hence $a \circ x \subseteq \{0, a\}$. □

By using Lemma 2.4, we have the following corollary.

Corollary 2.5. *Every atomic subset of a hyper BCK-algebra is a hyper subalgebra.*

Lemma 2.6. *If a hyper BCK-algebra H is atomic, then*
(i) $x \circ y \subseteq \{0, x\}$ *for all* $x, y \in H$,
(ii) $x \circ y = \{x\}$ *for all* $x, y \in H$ *with* $x \neq y$.

Proof. (i) By using Lemma 2.4, the result holds.

(ii) Let $x, y \in H$ be such that $x \neq y$. If $x = 0$, then $x \circ y = 0 \circ y = \{0\} = \{x\}$ by Theorem 1.2(a7). If $x \neq 0$, then by (i) it is enough to show that $0 \notin x \circ y$. If not, then $x \ll y$ and so $x = 0$ or $x = y$, which contracts with hypothesis. Therefore $x \circ y = \{x\}$. □

We suppose that the hyper BCK-algebra is atomic and consider its properties.

Theorem 2.7. *Every subset containing* 0 *of an atomic hyper BCK-algebra is a strong hyper BCK-ideal.*

Proof. Let H be an atomic hyper BCK-algebra, and let $0 \in A \subseteq H$. Suppose $x \in H$ and $y \in A$ be such that $x \circ y \cap A \neq \phi$. Then we may assume that $a \in A$ and $a \in x \circ y$. Since $x \circ y \ll x$ by Theorem 1.2(a8), it follows that $a \ll x$. Since H is atomic, we get $a = 0$ or $a = x$. If $a = 0$, then $x \ll y$ and so $x = 0$ or $x = y$, which implies $x \in A$. If $a = x$, then $x \in A$. Therefore A is a strong hyper BCK-ideal of H. □

Theorem 2.8. *Let H be an atomic hyper BCK-algebra and let K be an arbitrary hyper BCK-algebra. If $f : H \longrightarrow K$ is an epimorphism, then $f(A)$ is atomic for all $(\phi \neq) A \subseteq H$. In particular, K is atomic.*

Proof. Assume that $f(x) \ll f(y)$ for some $x, y \in A$. Then $0 \in f(x) \circ f(y) = f(x \circ y)$. Hence there exists $t \in x \circ y$ such that $f(t) = 0$. By Corollary 2.5, we have $x \circ y \subseteq A$. Thus $t \in A$. Since $x \circ y \ll x$, it follows that $t \ll x$ and so $t = 0$ or $t = x$. If $t = 0$, then $x \ll y$, which implies that $x = 0$ or $x = y$. Hence $f(x) = 0$ or $f(x) = f(y)$. If $t = x$, then $f(x) = f(t) = 0$. Therefore $f(A)$ is an atomic subset of K. Since f is surjective, $f(H) = K$ and so K is atomic. □

Proposition 2.9. *Let H and K be two atomic hyper BCK-algebras. If there exists a bijection mapping $f : H \longrightarrow K$ such that $f(0) = 0$, then*

$$f \text{ is an isomorphism} \iff f(S(H)) = S(K).$$

Proof. Let $f(S(H)) = S(K)$. We will show that f is an isomorphism. By the hypothesis, it is enough to show that f is a homomorphism, that is, $f(x \circ y) = f(x) \circ f(y)$ for all $x, y \in H$. Let $x, y \in H$. We consider the following cases:

(i) $x, y \in S(H)$.

This implies $f(x), f(y) \in S(K)$ by hypothesis. If $x = y$, then $x \circ y = \{0\}$ and so $f(x \circ y) = \{0\}$. Since f is a bijection mapping and $x = y$, we get $f(x) = f(y)$ and so $f(x) \circ f(y) = \{0\}$ because $f(x) = f(y) \in S(K)$. If $x \neq y$, then $f(x) \neq f(y)$. It follows from Lemma 2.6(ii) that $x \circ y = \{x\}$ and so $f(x) \circ f(y) = \{f(x)\}$. Hence $f(x \circ y) = f(x) \circ f(y)$.

(ii) $x, y \notin S(H)$.

This implies $f(x), f(y) \notin S(K)$ by hypothesis. If $x = y$, then $x \circ y = \{0, x\}$ by Lemma 2.6 and assumption (ii). Hence $f(x \circ y) = \{0, f(x)\}$. Since f is a bijection mapping and $x = y$, we get $f(x) = f(y)$ and so $f(x) \circ f(y) = \{0, f(x)\}$ because $f(x), f(y) \notin S(K)$. If $x \neq y$, then $f(x) \neq f(y)$. It follows from Lemma 2.6(ii) that $x \circ y = \{x\}$ and $f(x) \circ f(y) = \{f(x)\}$. Hence $f(x \circ y) = f(x) \circ f(y)$.

(iii) $x \in S(H),\ y \notin S(H)$ or $x \notin S(H),\ y \in S(H)$.

This implies $x \neq y$ and similar to proof of (i), we get $f(x \circ y) = \{f(x)\} = f(x) \circ f(y)$. Therefore f is an isomorphism.

Conversely, let f be an isomorphism. For any $x \in S(H)$, we have

$$f(x) \circ f(x) = f(x \circ x) = f(\{0\}) = \{0\}.$$

This shows that $f(x) \in S(K)$. Hence $f(S(H)) \subseteq S(K)$. Let $y \in S(K)$. Since f is surjective, we have $y = f(x)$ for some $x \in H$. It follows that $\{0\} = y \circ y = f(x) \circ f(x) = f(x \circ x)$ and so $f(t) = 0$ for all $t \in x \circ x$. Since f is one-to-one, $t = 0$. Hence $x \circ x = \{0\}$ and so $x \in S(H)$. Therefore $y = f(x) \in f(S(H)$ and so $S(K) \subseteq f(S(H))$, which completes the proof. □

Proposition 2.10. *Let H be a set, and let $0 \in A \subseteq H$. Define a hyper operation "$\circ$" on H as follows*:

$$x \circ y = \begin{cases} \{0\}, & \text{if } x = 0 \text{ or } x = y \in A \\ \{0, x\}, & \text{if } x = y \notin A \\ \{x\}, & x \neq y \end{cases} \tag{2.1}$$

Then $(H, \circ, 0)$ is an atomic hyper BCK-algebra and $S(H) = A$.

Proof. Let $x, y, z \in H$. We show that H satisfies the axioms of the hyper BCK-algebra.

(H1) $(x \circ z) \circ (y \circ z) \ll x \circ y$.

By the definition of hyper operation "$\circ$", it is enough to show that if whenever $x \circ y = \{0\}$, then $(x \circ z) \circ (y \circ z) = \{0\}$. It follows from $x \circ y = \{0\}$ that $x \ll y$, and so $x = 0$ or $x = y \in A$ by Definition 1.1. It is easy to check that $(x \circ z) \circ (y \circ z) \ll x \circ y = \{0\}$, that is, $(x \circ z) \circ (y \circ z) = \{0\}$. Therefore (H1) holds.

(H2) $(x \circ y) \circ z = (x \circ z) \circ y$.

We consider the following cases:

(i) $y = z$ and $x \neq y$.

In this case, by using (2.1), we have $(x \circ y) \circ z = x \circ z = \{x\} = x \circ y = (x \circ z) \circ y$.

(ii) $y \neq z$ and $x = y \in A$.

In this case, we have $(x \circ y) \circ z = \{0\} = x \circ y = (x \circ z) \circ y$ by (2.1).

(iii) $y \neq z$ and $x = y \notin A$.

In this case, it follow from (2.1) that $(x \circ y) \circ z = \{0, x\} \circ z = \{0, x\} = x \circ y = (x \circ z) \circ y$.

(iv) $y \neq z$, $x \neq y$ and $x \neq z(y \neq z$ and $x = z)$.

In this case, we have $(x \circ y) \circ z = \{x\} \circ z = \{x\} = x \circ z = (x \circ z) \circ y$. Therefore (H2) holds.

(H3) $x \circ H \ll x$.

It is clear.

(H4) If $x \ll y$ and $y \ll x$, then $x = y$.

Since $x \ll y$ and $y \ll x$, it follows from definition of " $\ll$ " that $0 \in x \circ y$ and $0 \in y \circ x$. If $x = 0$, then $y \circ x = \{y\}$ by (2.1), and so $y = 0$. Hence $x = y$. If $x \neq 0$ and $x \circ y = \{0\}$, then $x = y \in A$ by (2.1). If $x \neq 0$ and $x \circ y = \{0, x\}$, then $x = y \notin A$. Therefore (H4) holds and so H is a hyper BCK-algebra. Now, let $x, y \in H$ be such that $x \ll y$. Thus $0 \in x \circ y$. It follows from (2.1) that $x = 0$ or $x = y$. Therefore H is atomic. □

Theorem 2.11. *For any natural number n, there are exactly (up to isomorphism) n atomic hyper BCK-algebras such H, where $|H| = n$.*

Proof. Let n be a natural number, and let H be a set with $|H| = n$. By Proposition 2.10, there exists an atomic hyper BCK-algebra H such that $|S(H)| = |A|$, where $(\phi \neq)A$ be an arbitrary subset of H. Since $|A|$ has n choice, that is, $|A| = 1, 2, \ldots, n$, the assertion is immediate consequence of Proposition 2.9. □

Now, we give some properties of quotient of atomic hyper BCK-algebra.

Lemma 2.12. *Let H be an atomic hyper BCK-algebra with $S(H) = \{0\}$, and let Θ be a regular congruence relation on H. If the hyper BCK-ideal $[0]_\Theta$ is reflexive, then $[0]_\Theta = H$.*

Proof. By Lemma 2.6, we have $x \circ x \subseteq \{0, x\}$ for every $x \in H$. It follows from $S(H) = \{0\}$ that $x \circ x = \{x\}$ for all $x \in H$. Since $[0]_\Theta$ is reflexive, we have $x \circ x \subseteq [0]_\Theta$ and so $x \in [0]_\Theta$. Therefore $H = [0]_\Theta$. □

Corollary 2.13. *Let H be an atomic hyper BCK-algebra such that $S(H) = \{0\}$, and let Θ be a regular congruence relation on H. Then*

$$\frac{H}{\Theta} \text{ is } BCK\text{-algebra if and only if } \frac{H}{\Theta} = \{0\}.$$

Proof. If $\frac{H}{\Theta}$ is a BCK-algebra, then $[0]_\Theta$ is reflexive by Theorem 1.9. It follows from Lemma 2.12 that $[0]_\Theta = H$ and so $\frac{H}{\Theta} = \{0\}$. Conversely, it is clear. □

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References

[1] R. A. Borzooei and H. Harizavi, *Regular congruence relations on Hyper BCK-algebras*, Scientiae Mathematicae Japonicae **61** (2005), no. 1, 83–97.
[2] Y. Imai and K. Iséki, *On axiom systems of propositional calculi XIV*, Proc. Japan Academy **42** (1966), 19–22.
[3] Y. B. Jun, X. L. Xin, E. H. Roh, and M. M. Zahedi, *Strong hyper BCK-ideals of hyper BCK-algebra*, Mathematicae Japonicae, Vol. **51** (2000), no. 3, 493–498.
[4] Y. B. Jun, M. M. Zahedi, X. L. Xin, and R. A. Borzooei, *On hyper BCK-algebra*, Italian Journal of Pure and Applied Mathematics, Oxford Ser. **10** (2000), 127–136.

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