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A NOTE ON f-DERIVATIONS OF BCI-ALGEBRAS

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Abstract. In this paper, we investigate some fundamental properties and establish some results of f-derivations of BCI-algebras. Also, we prove Der(X), the collection of all f-derivations, form a semigroup under certain binary operation.

1. Introduction and preliminaries

BCI-algebra has been developed from BCI-logic on the similar way as Boolean algebra was developed from Boolean logic which have a lot of application in computer sciences ([14]). Recently greater interest has been developed in the derivation of BCI-algebras, introduced by Y. B. Jun and X. L. Xin [8], which was motivated from a lot of work done on derivations of rings and Near rings (see [9, 11]). The notion was further explored in the form of f-derivations of BCI-algebras by J. M. Zhan and Y. L. Liu [15]. In this paper, we prove some results on f-derivations of BCI-algebras. First, we show that an f-derivation of BCK-algebra is regular. However, we are able to show that under certain conditions namely, for $a \in X, f(a) * d_f(x) = 0$ or $d_f(x) * f(a) = 0$, for all $x \in X$ the f-derivation, d_f, of a BCI-algebra X is regular and X is a BCK-algebra. Also, we study derivations in a p-semisimple BCI-algebra and show that if $d_f, d_f^{'}$ are f-derivations in X, then $d_f \circ d_f^{'}$ is also a f-derivation and $d_f \circ d_f^{'} = d_f^{'} \circ d_f$. Consequently it is shown that $(f \circ d_f^{'}) \bullet (d_f \circ f) = (d_f \circ f) \bullet (f \circ d_f^{'})$. Now, we include necessary preliminaries required for the sequel. $(X, *, 0)$ with a binary operation $*$ and distinguished element 0 is called a BCI-algebra, if it satisfies the following axioms for all $x, y, z \in X$.

(BCI-1) $((x * y) * (x * z)) \leq (z * y)$.
(BCI-2) $(x * (x * y)) \leq y$.
(BCI-3) $x \leq x$.
(BCI-4) $x \leq y$ and $y \leq x$ imply $x = y$,

where $\leq$ is defined as $x \leq y$ if and only if $x * y = 0$.

Also, $(X, \leq)$ is a partially ordered set. A BCI-algebra X satisfying $0 \leq x$, for all $x \in X$, is called a BCK-algebra. If A is a branch of X, then X is said to

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be commutative on A if $x \wedge y = y \wedge x$ for all $x, y \in A$, where $x \wedge y = y * (y * x)$. In any BCI-algebra X, the following properties are valid (see [1, 7]) for all $x, y, z \in X$:

(1) $x * 0 = x$.
(2) $(x * y) * z = (x * z) * y$.
(3) $x \leq y$ implies that $x * z \leq y * z$, $z * y \leq z * x$.
(4) $(x * z) * (y * z) \leq x * y$.
(5) $x * (x * (x * y)) = x * y$.
(6) $0 * (x * y) = (0 * x) * (0 * y)$.
(7) $x * 0 = 0$ implies $x = 0$.

For a BCI-algebra X, we define $X_+ = \{x \in X : 0 \leq X\}$, the BCK-part of X, $G(X) = \{x \in X : 0 * x = x\}$, the BCI-G part of X. If $X_+ = \{0\}$, then X is called a p-semisimple BCI-algebra. If X is a p-semisimple BCI-algebra, then the following properties are valid for all $x, y, z \in X$ [3, 4, 5, 11].

(8) $(x * z) * (y * z) = x * y$.
(9) $(0 * (0 * x)) = x$ for all $x \in X$.
(10) $(x * (0 * y)) = y * (0 * x)$.
(11) $x * y = 0$ implies $x = y$.
(12) $x * a = x * b$ implies $a = b$.
(13) $a * x = b * x$ implies $a = b$.
(14) $a * (a * x) = x$.

Theorem 1.1 ([8, Theorem 3.4])**.** *Let X be a BCI-algebra. X is commutative if and only if it is branch wise commutative.*

On commutative BCI-algebras, we refer to [2, 7, 9, 12, 13].

Definition 1.2 ([8])**.** Let X be a BCI-algebra and $f \in \text{Hom}(X)$. By a (l, r)-f-derivation of X, we mean a self map d_f of X satisfying the identity $d_f(x * y) = d_f(x) * f(y) \wedge f(x) * d_f(y)$ for all $x, y \in X$.

If X satisfies the identity $d_f(x * y) = f(x) * d_f(y) \wedge d_f(x) * f(y)$ for all $x, y \in X$, then we say that d_f is a (r, l)-f-derivation of X. Moreover, if d_f is both a (l, r) and a (r, l)-f-derivation, we say that d_f is an f-derivation of X.

Definition 1.3 ([15])**.** A self map d_f of a BCI-algebra X is said to be regular if $d_f(0) = 0$.

Proposition 1.4 ([15])**.** *Let d_f be a regular derivation of a BCI-algebra X. Then the following hold.*

(i) $d_f(x) \leq f(x) \ \ \forall\, x \in X$.
(ii) $d_f(x) * f(y) \leq f(x) * d_f(y) \ \ \forall\, x, y \in X$.
(iii) $d_f(x * y) = d_f(x) * f(y) \leq d_f(x) * d_f(y) \ \ \forall\, x, y \in X$.
(iv) $\ker d_f$ *is a subalgebra of X. Especially, if f is monic, then* $\ker d_f \subseteq X_+$.

2. Some results on derivations

First, we study f-derivations on BCK-algebras.

Proposition 2.1. *Every (r,l)-f-derivation ((l,r)-f-derivation) of a BCK-algebra is regular.*

Proof. Let X be a BCK-algebra and d_f a (r,l)-f-derivation of X. Then for all $x \in X$, we have:

$$\begin{aligned} d_f(0) &= d_f(0 * x) = f(0) * d_f(x) \wedge d_f(0) * f(x) \\ &= 0 * d_f(x) \wedge d_f(0) * f(x) = 0 \wedge d_f(0) * f(x) = 0. \end{aligned}$$

Let d_f be a (l,r)-f-derivation of X. Then for all $x \in X$, we have:

$$\begin{aligned} d_f(0) &= d_f(0 * x) = d_f(0) * f(x) \wedge f(0) * d_f(x) \\ &= d_f(0) * f(x) \wedge 0 * d_f(x) = d_f(0) * f(x) \wedge 0 = 0. \end{aligned}$$ □

Proposition 2.2. *Let $f \in \mathrm{Epi}(X)$, d_f be a f-derivation of a BCI-algebra X and $a \in X$ such that $d_f(x) * a = 0$ and $d_f(x) * f(a) = 0$ for all $x \in X$. Then d_f is a regular f-derivation of X. Moreover, X is a BCK algebra.*

Proof. Let d_f be a f-derivation of a BCI-algebra X and let $a \in X$ such that $d_f(x) * a = 0$ and $d_f(x) * f(a) = 0$ for all $x \in X$. Since d_f is (l,r)-f-derivation, we have:

$$\begin{aligned} 0 &= d_f(x * a) * a = (d_f(x) * f(a) \wedge f(x) * d_f(a)) * a \\ &= (0 \wedge f(x) * d_f(a)) * a = 0 * a, \end{aligned}$$

this implies that $0 \leq a$, and therefore, $a \in X_+$. This shows that

$$\begin{aligned} d_f(0) &= d_f(0 * a) = d_f(0) * f(a) \wedge f(0) * d_f(a) \\ &= d_f(0) * f(a) \wedge 0 * d_f(a) = 0 \wedge 0 * d_f(a) = 0. \end{aligned}$$

Hence d_f is a regular f-derivation of X. So by Proposition 1.4 [15], we have $d_f(x) \leq f(x)$ for all $x \in X$ and so

$$0 * f(x) \leq 0 * d_f(x) = (d_f(x) * a) * d_f(x) = (d_f(x) * d_f(x)) * a = 0 * a = 0.$$

Thus $0 * f(x) \leq 0$ for all $x \in X$ and so $0 = (0 * f(x)) * 0 = 0 * f(x)$. Then we have $0 \leq f(x)$ for all $x \in X$. Which implies that $f(X)$ is a BCK-algebra. As $f \in \mathrm{Epi}(X)$, therefore, $f(X) = X$. □

Similarly, we can prove:

Proposition 2.3. *Let d_f be a f-derivation of a BCI-algebra X and $a \in X$ such that $a * d_f(x) = 0$ and $f(a) * d_f(x) = 0$ for all $x \in X$. Then d_f is a regular f-derivation of X. Moreover, X is a BCK-algebra.*

Example 2.4 ([3, page 8])**.** Let X be the set of natural number. For any element $x, y \in X$ define

$$(x * y) = \begin{cases} 0 & \text{if} \quad x \leq y \\ x - y & \text{if} \quad x > y, \end{cases}$$

then X is a BCI-algebra.

Define $f : X \to X$ by $f(x) = 2x$ then $f \in \mathrm{Epi}(X)$, indeed f is an BCI-isomorphism. Consider $x, y \in X$. If $x \leq y$, then $f(x * y) = f(0) = 2(0) = 0$, $f(x) * f(y) = 2x * 2y = 0$. Hence $f(x * y) = f(x) * f(y)$. If $x > y$, then $f(x*y) = f(x-y) = 2(x-y) = f(x)*f(y)$. This shows that $f(x*y) = f(x)*f(y)$ and hence $f \in \mathrm{Hom}(X)$. Obviously f is bijective, therefore $f \in \mathrm{Epi}(X)$.

Define $d_f(x) = 0$ for all $x \in X$. Then

$$\begin{aligned} d_f(x * y) &= 0, \\ d_f(x * y) &= d_f(x) * f(y) \wedge f(x) * d_f(y) = f(x) * d_f(y) \wedge d_f(x) * f(y) \\ &= 0 * 2y \wedge 2x * 0 \\ &= 0 \wedge 2x = 0. \end{aligned}$$

This shows that d_f is (l, r)-derivation. Similarly we can show that d_f is (r, l)-derivation.

Also note that $d_f(x)*a = 0*a = 0$. This implies that $0 \leq a$ for all $a \in X$ and hence X is a BCK-algebra. And $d_f(x)*f(a) = 0*f(a) = 0$, where $f \in \mathrm{Epi}(X)$. Moreover, observe that X is commutative BCK-algebra (see [3]).

Finally, we study f-derivations of a p-semisimple BCI-algebras, $\mathrm{Der}(X)$ denotes the set of all f-derivations of X.

Definition 2.5. Let X be a BCI-algebra and d_f , d_f' be two self maps of X.

We define $d_f \circ d_f' : X \to X$ as: $(d_f \circ d_f')(x) = d_f(d_f'(x))$ for all $x \in X$.

Proposition 2.6. *Let X be a p-semisimple BCI-algebra, d_f' and d_f are the (l, r)-f-derivations of X. Then $d_f \circ d_f'$ is also a (l, r)-f-derivation of X.*

Proof. Let X be a p-semisimple BCI-algebra and d_f' and d_f are (l, r)-f-derivations of X. Then by (14) and Proposition 1.4(ii), we get for all $x, y \in X$:

$$\begin{aligned} (d_f \circ d_f')(x * y) &= d_f(d_f'(x) * f(y) \wedge f(x) * d_f'(y)) = d_f(d_f'(x) * f(y)) \\ &= d_f(d_f'(x)) * f(y) \wedge f(d_f'(x)) * d_f(f(y)) = d_f(d_f'(x)) * f(y) \\ &= (f(x) * d_f(d_f'(y))) * (f(x) * d_f(d_f'(y))) * (d_f(d_f'(x)) * f(y))) \\ &= (d_f \circ d_f')(x) * f(y) \wedge f(x) * (d_f \circ d_f')(y). \end{aligned}$$

Which implies that $d_f \circ d_f'$ is a (l, r)-f-derivation of X. □

Similarly, we can prove:

Proposition 2.7. *Let X be a p-semisimple BCI-algebra and d_f, d are (r, l)-f-derivations of X. Then $d_f \circ d_f'$ is also a (r, l)-f-derivation of X.*

Combining Propositions 2.6 and 2.7, we get:

Theorem 2.8. *Let X be a p-semisimple BCI-algebra and d_f' and d_f be f-derivations of X. Then $d_f \circ d_f'$ is also a f-derivation of X.*

Proposition 2.9. *Let X be a p-semisimple BCI-algebra and $d_f, d_f^{'}$ be f-derivations of X such that $f \circ d_f = d_f \circ f$, $d_f^{'} \circ f = f \circ d_f^{'}$. Then $d_f \circ d_f^{'} = d_f^{'} \circ d_f$.*

Proof. Let X be a p-semisimple BCI-algebra and $d_f, d_f^{'}$ the f-derivations of X. Since $d_f^{'}$ is a (l, r)-f-derivation of X, then for all $x, y \in X$:

$$\begin{aligned} d_f \circ d_f^{'}(x * y) &= d_f(d_f^{'}(x * y)) = d_f(d_f^{'}(x) * f(y) \wedge f(x) * d_f^{'}(y)) \\ &= d_f(d_f^{'}(x) * f(y). \end{aligned}$$

But d_f is a (r, l)-f-derivation of X, so

$$\begin{aligned} (d_f \circ d_f^{'})(x * y) &= d_f(d_f^{'}(x) * f(y)) \\ &= f(d_f^{'}(x)) * d_f(f(y)) \wedge d_f(d_f^{'}(x)) * f(y) \\ &= f(d_f^{'}(x)) * d_f(f(y)) \\ &= f \circ d_f^{'}(x) * d_f \circ f(y) \end{aligned}$$

thus we have for all $x, y \in X$:

$$(d_f \circ d_f^{'})(x * y) = f \circ d_f^{'}(x) * d_f \circ f(y). \tag{2.1}$$

Also, since d_f is a (r, l)-f-derivation of X, then for all $x, y \in X$:

$$\begin{aligned} (d_f^{'} \circ d_f)(x * y) &= d_f^{'}(f(x) * d_f(y) \wedge d_f(x) * f(y)) \\ &= d_f^{'}(f(x) * d_f(y)). \end{aligned}$$

But $d_f^{'}$ is a (l, r)-f-derivation of X, so

$$\begin{aligned} (d_f^{'} \circ d_f)(x * y) &= d_f^{'}(f(x) * d_f(y)) \\ &= d_f^{'}(f(x)) * f(d_f(y)) \wedge f^2(x) * d_f^{'}(d_f(y)) \\ &= d_f^{'}(f(x)) * f(d_f(y)) \\ &= d_f^{'} \circ f(x) * f \circ d_f(y) \\ &= f \circ d_f^{'}(x) * d_f \circ f(y). \end{aligned}$$

Thus we have for all $x, y \in X$:

$$(d_f^{'} \circ d_f)(x * y) = f \circ d_f^{'}(x) * d_f \circ f(y). \tag{2.2}$$

From (2.1) and (2.2) we get for all $x, y \in X$:

$$(d_f \circ d_f^{'})(x * y) = (d_f^{'} \circ d_f)(x * y).$$

By putting $y = 0$ we get for all $x \in X$:

$$(d_f \circ d_f^{'})(x) = (d_f^{'} \circ d_f)(x).$$

Which implies that $d_f \circ d_f^{'} = d_f^{'} \circ d_f$. □

Definition 2.10. Let X be a BCI-algebra and d_f, d_f', two self maps of X. We define $d_f \bullet d_f' : X \to X$ as:

$$(d_f \bullet d_f')(x) = d_f(x) \bullet d_f'(x) \text{ for all } x \in X.$$

Proposition 2.11. *Let X be a p-semisimple BCI-algebra, $f \in \mathrm{Epi}(X)$ and d_f, d_f' are f-derivations of X. Then $(f \circ d_f') \bullet (d_f \circ f) = (d_f \circ f) \bullet (f \circ d_f')$.*

Proof. Let X is a p-semisimple BCI-algebra and d_f, d_f' be f-derivations of X. Since d_f' is a (l, r)-f-derivation of X, then for all $x, y \in X$:

$$(d_f \circ d_f')(x \bullet y) = d_f(d_f'(x) \bullet f(y) \wedge f(x) \bullet d_f'(y) = d_f(d_f'(x) \bullet f(y)).$$

But d_f is a (r, l)-f-derivation of X, so

$$\begin{aligned} d_f(d_f'(x) \bullet f(y)) &= f(d_f'(x)) \bullet d_f(f(y)) \wedge d_f(d_f'(x)) \bullet f(y) \\ &= f(d_f'(x)) \bullet d_f(f(y)) \\ &= f \circ d_f'(x)) \bullet d_f \circ f(y)) \end{aligned}$$

and hence

$$(d_f \circ d_f')(x \bullet y) = f \circ d_f'(x)) \bullet d_f \circ f(y)) \text{ for all } x, y \in X. \tag{2.3}$$

Also, we have that d_f' is a (r, l)-f-derivation of X, then for all $x, y \in X$:

$$(d_f \circ d_f')(x \bullet y) = d_f(f(x) \bullet d_f'(y) \wedge d_f'(x) \bullet f(y)) = d_f(f(x) \bullet d_f'(y)).$$

But d_f is a (l, r)-f-derivation of X, so

$$\begin{aligned} d_f(f(x) \bullet d_f'(y)) &= d_f(f(x)) \bullet f(d_f'(y)) \wedge f^2(x) \bullet d_f(d_f'(y)) \\ &= d_f(f(x)) \bullet f(d_f'(y)). \end{aligned}$$

Thus

$$(d_f \circ d_f')(x \bullet y) = d_f \circ f(x)) \bullet f \circ d_f'(y)) \text{ for all } x, y \in X. \tag{2.4}$$

From (2.3) and (2.4) we get:

$$f \circ d_f'(x) \bullet d_f \circ f(y) = d_f \circ f(x) \bullet f \circ d_f'(y) \text{ for all } x, y \in X.$$

By putting $x = y$ we get for all $x \in X$:

$$\begin{aligned} f \circ d_f'(x) \bullet d_f \circ f(x) &= d_f \circ f(x) \bullet f \circ d_f'(x), \\ (f \circ d_f' \bullet d_f \circ f)(x) &= (d_f \circ f \bullet f \circ d_f')(x). \end{aligned}$$

Which implies that $(f \circ d_f') \bullet (d_f \circ f) = (d_f \circ f) \bullet (f \circ d_f')$. □

3. Der(X), set of all f-Derivations

Der(X) denotes the set of all f-derivations on X.

Definition 3.1. Let $d_f, d_f^{'} \in \mathrm{Der}(X)$. Define the binary operation $\wedge$ as:

$$(d_f \wedge d_f^{'})(x) = d_f(x) \wedge d_f^{'}(x).$$

Proposition 3.2. *Let X be a p-semisimple BCI algebra and $d_f, d_f^{'}$ are (l,r)-f-derivations of X. Then $d_f\wedge\, d_f^{'}$ is also a (l,r)-f-derivation of X.*

Proof. Let X be a p-semisimple BCI-algebra and $d_f, d_f^{'}$ are (l,r)-f-derivations of X. Then by (14) and Proposition 1.4, we have

$$\begin{aligned}
&(d_f \wedge d_f^{'})(x*y)\\
&= d_f(x*y) \wedge d_f^{'}(x*y)\\
&= \{((d_f(x)*f(y)) \wedge (f(x)*d_f(y))\} \wedge \{(d_f^{'}(x)*f(y)) \wedge (f(x)*d_f^{'}(y))\},
\end{aligned}$$

$$\begin{aligned}
&(d_f \wedge d_f^{'})(x*y)\\
&= (d_f(x)*f(y)) \wedge (d_f^{'}(x)*f(y))\\
&= d_f(x)*f(y)\\
&= (d_f^{'}(x)*(d_f^{'}(x)*d_f(x)))*f(y)\\
&= (d_f(x) \wedge d_f^{'}(x))*f(y)\\
&= (d_f \wedge d_f^{'})(x)*f(y)\\
&= (f(x)*(d_f \wedge d_f^{'})(y))*\{(f(x)*(d_f \wedge d_f^{'})(y))*((d_f \wedge d_f^{'})(x))*f(y)\}\\
&= ((d_f \wedge d_f^{'})(x)*f(y)) \wedge (f(x)*(d_f \wedge d_f^{'})(y)) = (d_f \wedge d_f^{'})(x*y).
\end{aligned}$$

This shows that $(d_f\wedge\, d_f^{'})$ is a (l,r)-f-derivation of X. This completes the proof. □

In the similar fashion, we can establish the following.

Proposition 3.3. *Let X be a p-semisimple BCI-algebra and $d_f, d_f^{'}$ are (r,l)-f-derivations of X. Then $d_f\wedge\, d_f^{'}$ is also a (r,l)-f-derivation of X.*

By using the Propositions 3.2, 3.3, we conclude the following.

Proposition 3.4. *If $d_f, d_f^{'} \in \mathrm{Der}(X)$, then $d_f\wedge\, d_f^{'} \in \mathrm{Der}(X)$. Also, $(d_f\wedge (d_f^{'}\wedge\, d_f^{''}))(x*y) = ((d_f\wedge\, d_f^{'})\wedge\, d_f^{''})(x*y)$.*

Let $d_f, d_f^{'}, d_f^{''} \in \mathrm{Der}(X)$. Then by definition,

$$\begin{aligned}
&((d_f \wedge d_f^{'}) \wedge d_f^{''})(x*y)\\
&= (d_f \wedge d_f^{'})(x*y) \wedge d_f^{''}(x*y)
\end{aligned}$$

$= (d''_f(x*y)) * (d''_f(x*y) * (d_f \wedge d'_f)(x*y))$
$= (d_f \wedge d'_f)(x*y)$
$= d_f(x*y) \wedge d'_f(x*y)$
$= (d_f(x) * f(y) \wedge f(x) * d_f(y)) \wedge (d'_f(x) * f(y) \wedge f(x) * d'_f(y))$
$= d_f(x) * f(y) \wedge d'_f(x) * f(y)$
$= d_f(x) * f(y).$

Also consider the following,

$(d_f \wedge (d'_f \wedge d''_f))(x*y)$
$= d_f(x*y) \wedge (d'_f \wedge d''_f)(x*y)$
$= d_f(x*y) \wedge ((d'_f(x*y) \wedge d''_f(x*y))$
$= d_f(x*y) \wedge d'_f(x*y)$
$= (d_f(x) * f(y) \wedge f(x) * d_f(y)) \wedge (d'_f(x) * f(y) \wedge f(x) * d'_f(y))$
$= d_f(x) * f(y) \wedge d'_f(x) * f(y)$
$= d_f(x) * f(y).$

This shows that $(d_f \wedge d'_f) \wedge d''_f = d_f \wedge (d'_f \wedge d''_f)$. Thus $\mathrm{Der}(X)$ forms a semigroup.

Corollary 3.5. *If X is a p-semisimple BCI-algebra, then* $(\mathrm{Der}(X), \wedge)$ *is a semigroup.*

Let we take f as an identity map, i.e., $d_f = d_I = d$ in the following examples.

Example 3.6. If $X = \{0, a, b\}$ and binary operation $*$ is defined as

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

then it forms a p-semisimple BCI-algebra (see [3]). Moreover if I is an identity mapping, $I \in \mathrm{Epi}(X)$, then $\mathrm{Der}(X)$ associated to I is only $\{I\}$, which is indeed a semigroup in view of Definition 3.1.

However, we can also observe from the following example that if X is not a p-semisimple BCI-algebra, then $\mathrm{Der}(X)$ may also form a semigroup.

Example 3.7. Let $X = \{0, a, b\}$ be a commutative BCK-algebra with binary operation $*$ as defined in the following table:

$*$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Let $\mathrm{Der}(X)$ set of all derivations of X. If $d \in \mathrm{Der}(X)$, then $d(0) = 0$ since X is a BCK-algebra.

Now 0 can be associated under d in one way. Now $d(0) = 0$ and X is a commutative BCK-algebras therefore $d(x * y) = d(x) * y$ and $d(x) \leq x$. This shows that a and b can be associated under d in 2 ways each and hence $\mid \mathrm{Der}(X) \mid \leq 1 \times 2 \times 2 = 4$, which are the following:

$d_1(x) = 0$ for all $x \in X$, $d_1 \in \mathrm{Der}(X)$.
$d_2(0) = 0, d_2(a) = 0, d_2(b) = b,$
$d_3(0) = 0, d_3(a) = a, d_3(b) = 0,$
$d_4(0) = 0, d_4(a) = a, d_4(b) = b,$
$\mathrm{Der}(X) = \{d_1, d_2, d_3, d_4\}$.

Since X is a commutative BCK-algebra therefore $\mathrm{Der}(X) = \{d_1, d_2, d_3, d_4\}$ form a commutative semigroup as shown in the following table:

$\wedge$	d_1	d_2	d_3	d_4
d_1	d_1	d_1	d_1	d_1
d_2	d_1	d_2	d_1	d_2
d_3	d_1	d_1	d_3	d_3
d_4	d_1	d_2	d_3	d_4

Remark 3.8. If X is a commutative BCI-algebra, then from the Definition 1.2 if f is considered as identity map, then it follows that

$$d(x) \leq x * d(0).$$

This shows that $d(x), x * d(0)$ always lies in the same branch. If $d(0) = 0$, then $d(x) \leq x$. This shows that $d(x)$ and x lies in the same branch. We will use these observation in the following example.

The following example reflect that if X is a commutative BCI-algebra which is not a p-semisimple BCI-algebra, then under the binary operation defined in Definition 3.1, $\mathrm{Der}(X)$ does not form a semigroup. However, it is a groupoid.

Example 3.9. If $X = \{0, a, b\}$ and binary operation $*$ is defined as

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

$d(0) = 0$ and $d(0) = b$, then in the light of above said remarks

Case 1: $d(0) = 0$, then $d(x) \leq x * d(0)$ implies that $d(x) \leq x$, so

(i) $d(a) \leq a \Rightarrow$ Either $d(a) = 0$ or $d(0) = a$,
(ii) $d(b) \leq b * 0 = b$.

Then possible derivations are

$$\begin{aligned} d_1(0) &= 0,\ d_1(a) = 0,\ d_1(b) = b, \\ d_2(0) &= 0,\ d_2(a) = a,\ d_2(b) = b, \\ D_1(X) &= \{d_1, d_2\}. \end{aligned}$$

Case 2: $d(0) = b$, so,

(i) $d(a) \leq a * d(0) = a * b = b$
$\Rightarrow d(a) \leq b$
$\Rightarrow d(a) = b,$

(ii) $d(b) \leq b * b = 0.$

Then possible derivation is

$$\begin{aligned} & f_1(0) = b, f_1(a) = b,_1 (b) = 0, \\ & D_2(X) = \{f_1\}, \\ & D(X) = D_1(X) \cup D_2(X) = \{d_1, d_2 . f_1\}, \end{aligned}$$

$\wedge$	d_1	d_2	f_1
d_1	d_1	d_1	d_1
d_2	d_1	d_2	d_1
f_1	f_1	f_1	f_1

Proposition 3.10. *Let X be a p-semisimple BCI-algebra and $d_f, d_f^{'}$ are f-derivations of X. Then $d_f(x) * f(x) = d_f^{'}(x) * f(x)$ if $d_f(x), d_f^{'}(x)$ lie in the same branch.*

Proof. Let $x, y \in X$, then by (14), Definition 3.1 and Theorem 1.1, we have

$$\begin{aligned} (d_f * d_f^{'})(x * y) &= d_f(x * y) \wedge d_f^{'}(x * y) \\ &= d_f(x) * f(y) \wedge f(x) * d_f(y) \qquad (3.1) \\ &= d_f(x) * f(y). \end{aligned}$$

Now consider

$$\begin{aligned} (d_f * d_f^{'})(x * y) &= d_f(x * y) \wedge d_f^{'}(x * y) \\ &= d_f^{'}(x * y) \wedge d_f(x * y) \\ &= d_f^{'}(x * y) \qquad (3.2) \\ &= d_f^{'}(x) * f(y) \wedge f(x) * d_f^{'}(y) \\ &= d_f^{'}(x) * f(y). \end{aligned}$$

From (3.1) and (3.2), $d_f^{'}(x) * f(y) = d_f(x) * f(y)$, hence for $y = x$, $d_f^{'}(x) * f(x) = d_f(x) * f(x)$.

This completes the proof. □

References

[1] H. A. S. Abujabal and N. O. Al-shehri, *Some results on derivations of BCI-algebras*, Jr. of Natural sciences and Mathematics **46** (2006), no. 2, 13–19.

[2] M. Aslam and A. B. Thaheem, *A note on p-semisimple BCI-algebras*, Math. Japon. **36** (1991), no. 1, 39–45.

[3] M. Aslam, *Ideal theory of BCK-algebras*, Quaid-I-Azam University Islamabad, Pakistan. Ph. D. Thesis, 1992.

[4] M. Daoji, *BCI-algebras and Abelian groups*, Math. Japon. **32** (1987), no. 5, 693–696.

[5] W. A . Dudek, *On BCI- algebras with condition* (s), Math. Japon. **31** (1986), no. 1, 26–29.

[6] C. S. Hoo, *BCI- algebras with condition* (s), Math. Japon. **32** (1987), no. 5, 749–756.

[7] K. Iseki, *On BCI-algebras*, Math. Sem. Notes **8** (1980), 125–130.

[8] Y. B. Jun and X. L. Xin, *On derivations of BCI-algebras*, Inform. Sci. **159** (2004), 167–176.

[9] P. H. Lee and T. K. Lee, *On derivations of prime rings*, Chinese J. Math. **9** (1981), 107–110.

[10] J. Meng, Y. B. Jun, and E. H. Roh, *BCI-algebras of order 6*, Math. Japon. **47** (1998), no. 1, 33–43.

[11] E. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.

[12] L. Tiande and X. Changchang, *p-Radical in BCI-algebras*, Math. Japon. **30** (1985), no. 4, 511–517.

[13] X . L. Xin, E. H. Roh, and J. C. Li , *Some results on the BCI G part of algebras*, Far East J. Math. Sci. SpecialVolume (Part III), (1997), 363–370.

[14] J. E. Whitesitt, *Boolean algebra and its Applications*, New York, Dover, 1995.

[15] J. M. Zhan and Y. L. Liu, *On f derivation of BCI-algebras*, Int. Jour. Math. Mathematical Sciences. **11** (2005), 1675–1684.

[16] Q. Zhang, *Some other characterizations of p semisimple BCI-algebras*, Math . Japon. **36** (1991), no. 5, 815–817.

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