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LEFT-INVARIANT MINIMAL UNIT VECTOR FIELDS ON A LIE GROUP OF CONSTANT NEGATIVE SECTIONAL CURVATURE

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Abstract. We find all left-invariant minimal unit vector fields and strongly normal unit vector fields on a Lie group which is isometric to the hyperbolic space.

1. Introduction

A smooth unit vector field on a Riemannian manifold (M, g) is a cross section of its unit sphere bundle $T^1(M)$ and hence can be viewed as a submanifold of $T^1(M)$. If the manifold M is compact and $T^1(M)$ is equipped with a natural Riemannian metric g_s called the Sasaki metric, then the volume of the unit vector field is defined as the volume of this submanifold.

For the problem of determining unit vector fields which have minimal volume, Gluck and Ziller showed that the unit vector fields of minimal volume on S^3 are precisely the Hopf vector fields and no others ([7]). But in the higher dimensional spheres, $S^{2n+1}, k \geq 2$, this is not the case ([4], [8], [10]).

The problem of finding unit vector fields of the minimum volume seems to be very difficult, so it is natural to consider the problem of finding the critical values or critical points of the volume functional.

Gil-Medriano and Llinares-Fuster proved that a unit vector field is a critical point of the volume functional if and only if the corresponding immersion in (T^1M, g_s) is minimal ([3]). So we call such unit vector fields minimal even though the manifold is not compact.

Many examples of Riemannian manifolds and Lie groups equipped with left-invariant minimal unit vector fields are provided ([1], [2], [3], [5], [6], [12], [13]). But there are very few manifolds on which we know all the minimal unit vector fields. Even for almost examples of the Lie groups, not all the left-invariant minimal unit vector fields are provided but only some of them are found to be minimal.

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The aim of this paper is to provide a Lie group with all the left-invariant minimal unit vector fields as follows. For an integer $n > 1$, a Lie group G_n is defined as follows.

$$G_n := \left\{ \begin{pmatrix} 1 & 0 \\ v & sI_{n-1} \end{pmatrix} \in GL(n, \mathbb{R}) \,|\, v \in \mathbb{R}^{n-1}, s > 0 \right\},$$

where $v \in \mathbb{R}^{n-1}$ and I_{n-1} is the $(n-1) \times (n-1)$ identity matrix. Then the Lie algebra $\mathfrak{g}_n$ of G_n consists of the $n \times n$ matrices of the form

$$\begin{pmatrix} 0 & 0 \\ v & sI_{n-1} \end{pmatrix}, v \in \mathbb{R}^{n-1}, s \in \mathbb{R}.$$

Let $\{e_1, e_2, \ldots, e_{n-1}\}$ be the usual orthonormal basis for $\mathbb{R}^{n-1}$. Put

$$E_i = \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, i < n, \qquad E_n = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix}.$$

Then the set $\{E_1, E_2, \ldots, E_{n-1}, E_n\}$ is an orthonormal bases for $\mathfrak{g}_n$.

The set of all left-invariant minimal unit vector fields on the Lie group G_n is as follows which is the main result of this paper.

Theorem 1.1. *For the Lie group G_2 every left-invariant unit vector field is minimal. For the Lie group $G_n, n > 2$, the set of left-invariant minimal unit vector fields is $\{\pm E_n\} \cup (\mathcal{S} \cap E_n^{\perp})$, where $\mathcal{S}$ is the unit sphere of $\mathfrak{g}_n$.*

In Section 2 we give some basic notions and facts. In Section 3 we prove the main theorem and in Section 4 we find the set of all strongly normal unit vector fields on the Lie group G_n.

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2. Minimal unit vector fields

Let (M, g) be a smooth Riemannian manifold, ∇ be the Levi-Civita connection on (M, g) and R be the associated Riemannian curvature tensor with the sign convention $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$.

We assume that the set $\chi^1(M)$ of unit vector fields on M is non-empty. For $V \in \chi^1(M)$, let L_V be a tensor field defined by

$$L_V := I + (\nabla V)^* \nabla V,$$

where I is the identity map and $(\nabla V)^*$ is the adjoint. Then L_V is positive definite and symmetric. Put $f(V) = (\det L_V)^{\frac{1}{2}}$. For a closed manifold M, we define the volume functional $F : \chi^1(M) \longrightarrow \mathbb{R}$ by

$$F(V) := \int_M f(V) dv,$$

where dv is the volume form on (M, g).

Now let K_V be a $(1,1)$-tensor field define by

$$K_V = f(V) \cdot L_V^{-1} \circ (\nabla V)^*$$

and a 1-form ω_V associated to V by

$$\omega_V(X) = \mathrm{tr}(Z \longmapsto \nabla_Z K_V)(X).$$

For an orthonormal basis $\{E_1, E_2, \ldots, E_n\}$ of the tangent space, $\omega_V(X)$ is given as follows.

$$\omega_V(X) = \sum_{i=1}^{n} g((\nabla_{E_i} K_V)(X), E_i).$$

In [3] it is shown that a unit vector field V is a minimal immersion if and only if the 1-form ω_V annihilates the distribution $\mathcal{H}^V$ consisting of tangent vectors orthogonal to V. Moreover it is shown that V is a critical point of the volume functional if and only if the map $V : M \longrightarrow (T^1M, g_s)$ is a minimal immersion, where (T^1M, g_s) is the unit tangent bundle T^1M equipped with the Sasaki metric g_s. For this reason we define minimal unit vector fields on a manifold which is not necessarily compact as follows.

Definition 2.1. A unit vector field V on a Riemannian manifold (M, g) is called minimal if $\omega_V(X) = 0$ for all $X \in \mathcal{H}^V$.

From now on we consider left-invariant unit vector fields on Lie groups. Let G be an n-dimensional connected Lie group equipped with a left-invariant metric and $\mathfrak{g}$ be its Lie algebra. Let $\mathcal{S}$ be the unit sphere of $\mathfrak{g}$ with respect to the inner product $\langle , \rangle$ which is determined by the left-invariant metric on G. Since $V \in \mathcal{S}, \nabla V, L_V, K_V$ and ω_V are invariant by left translation, the function f can be considered as a function on $\mathcal{S}$.

The distribution $\mathcal{H}^V$ is invariant by left translation and can be identified with the orthogonal complement $V^\perp$ of V in $\mathfrak{g}$ and thus $V^\perp$ may be naturally identified with the tangent space $T_V\mathcal{S}$ of the unit sphere $\mathcal{S}$ at V. Thus a left-invariant unit vector field V is minimal if and only if the 1-form ω_V on $\mathfrak{g}$ vanishes on $V^\perp \cong T_V\mathcal{S}$([13]).

Proposition 2.1 ([13], Proposition 2.1)**.** *For $X \in T_V\mathcal{S}$ we have*

$$\omega_V(X) = -df_V(X) - \mathrm{tr}\, ad_{K_V X}$$

and V is minimal if and only if

$$df_V(X) = -\mathrm{tr}\, ad_{K_V X}$$

for all $X \in T_V\mathcal{S}$.

Thus on a unimodular Lie group G, i.e., $\mathrm{tr}\, ad_X = 0$, for all $X \in \mathfrak{g}$, a left-invariant unit vector field V is minimal if and only if V is a critical point of the function f on $\mathcal{S}$.

For a non-unimodular Lie group G with a left-invariant metric, we denote by $\mathcal{U}$ its *unimodular kernel*, i.e.,

$$\mathcal{U} = \{X \in \mathfrak{g} \,|\, \mathrm{tr}\, ad_X = 0\}.$$

Then $\mathcal{U}$ is an ideal of codimension 1 since $\mathrm{tr}\, ad_X$ is a linear functional. For a unit vector H which is orthogonal to $\mathcal{U}$, the linear transformation ad_H restricted to $\mathcal{U}$ is a derivation of $\mathcal{U}$. So we have the following.

Proposition 2.2 ([13], Proposition 2.5). *Let $\mathcal{U}$ be the unimodular kernel of a non-unimodular Lie group such that $ad_H|_{\mathcal{U}}$ is a symmetric endomorphism of $\mathcal{U}$ with respect to $\langle , \rangle$. Then a left-invariant unit vector field V is minimal if and only if it is a critical point of the function f on $\mathcal{S}$.*

We shall use this proposition 2.2 to find all the left-invariant minimal unit vector fields on the Lie group G_n which is isometric to the hyperbolic space $\mathbb{H}^n$.

3. Proof of the main theorem

For an integer $n > 1$, define a Lie group G_n as follows.

$$G_n := \left\{ \begin{pmatrix} 1 & 0 \\ v & sI_{n-1} \end{pmatrix} \in GL(n, \mathbb{R}) \,|\, v \in \mathbb{R}^{n-1}, s > 0 \right\},$$

where $v \in \mathbb{R}^{n-1}$ and I_{n-1} is the $(n-1) \times (n-1)$ identity matrix. Then the Lie algebra $\mathfrak{g}_n$ of G_n consists of the $n \times n$ matrices of the form

$$\begin{pmatrix} 0 & 0 \\ v & sI_{n-1} \end{pmatrix}, v \in \mathbb{R}^{n-1}, s \in \mathbb{R}.$$

Let $\{e_1, e_2, \ldots, e_{n-1}\}$ be the usual orthonormal basis for $\mathbb{R}^{n-1}$. Put

$$E_i = \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, \, i < n, \qquad E_n = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix}.$$

Then we have $[E_i, E_j] = 0$ for $1 \le i, j < n$, and $[E_n, E_k] = E_k$ for $1 \le k \le n$.

Let G_n be equipped with a left-invariant metric such that $\{E_1, E_2, \ldots, E_n\}$ is an orthonormal basis for $\mathfrak{g}_n$.

Let ∇ be the Levi-Civita connection of G_n. Then for $X, Y, Z \in \mathfrak{g}$ it satisfies the following identity([9]).

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} \{ \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle \}. \tag{1}$$

For $1 \le i, j < n, \ 1 \le k \le n$, it is easy to show that

$$\nabla_{E_i} E_j = \delta_{ij} E_n, \quad \nabla_{E_i} E_n = -E_i, \quad \nabla_{E_n} E_k = 0. \tag{2}$$

Thus $K_{E_i \wedge E_j} = \langle R(E_i, E_j)E_i, E_j \rangle = -1$ and the Lie group $(G_n, \langle , \rangle)$ has constant negative sectional curvature -1. In fact it is simply connected and complete. Thus the Lie group $(G_n, \langle , \rangle)$ is isometric to the hyperbolic space H^n([11]).

Proof of the Theorem 1.1. For $X = \sum_{i=1}^n a_i E_i$, we have

$$\mathrm{tr} ad_X = \sum_{j=1}^n \langle ad_X E_j, E_j \rangle = n a_n.$$

So the unimodular kernel $\mathcal{U}$ is the set $\{X \in \mathfrak{g} \,|\, X = \sum_{i=1}^{n-1} a_i E_i\}$ and a unit vector orthogonal to $\mathcal{U}$ is E_n.

And for an element $X \in \mathcal{U}$ we have $ad_H(X) = X$ and thus $ad_H|_{\mathcal{U}} = Id|_{\mathcal{U}}$. Therefore $ad_H|_{\mathcal{U}}$ is a symmetric endomorphism of $\mathcal{U}$ with respect to $\langle , \rangle$. So by the Proposition 2.2, a left-invariant unit vector field V is minimal if and only if V is a critical point of f on $\mathcal{S}$.

Let $W = \sum_{i=1}^{n-1} z_i E_i$ with $\sum_{i=1}^{n-1} z_i^2 = 1$ and $V = xE_n + yW$ with $x^2 + y^2 = 1$. Then for $j < n$, $\nabla_{E_j} V = yz_j E_n - xE_j$ and $\nabla_{E_n} V = 0$. So we have

$$\nabla V = \sum_{i=1}^{n-1} (yz_j \otimes \alpha_j - xE_j \otimes \alpha_j),$$

where $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the dual coframe field of $\{E_1, E_2, \ldots, E_n\}$. The matrix form of ∇V is as follows.

$$\nabla V = \begin{pmatrix} -x & 0 & \cdots & 0 & 0 \\ 0 & -x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -x & 0 \\ yz_1 & yz_2 & \cdots & yz_{n-1} & 0 \end{pmatrix}.$$

So we have

$$f(V) = \sqrt{\det(I + (\nabla V)^* \nabla V)} = \sqrt{2}(1 + x^2)^{\frac{n-2}{2}}.$$

If $n = 2$, the function f becomes a constant function and thus every left-invariant unit vector field is minimal.

Assume that $n > 2$ and let

$$\begin{aligned} f(x, y, z_1, \ldots, z_{n-1}) &= (1 + x^2)^r, \\ g_1(x, y, z_1, \ldots, z_{n-1}) &= x^2 + y^2 - 1 = 0, \\ g_2(x, y, z_1, \ldots, z_{n-1}) &= z_1^2 + \cdots + z_{n-1}^2 - 1 = 0, \end{aligned}$$

where $r = \frac{n-2}{2}$.

By the Lagrange multiplier method, we have to solve the following simultaneous equation.

$$\begin{cases} \nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \\ g_1 = 0 \\ g_2 = 0. \end{cases}$$

Since

$$\begin{aligned} \nabla f &= (2rx(1 + x^2)^{r-1}, 0, \ldots, 0), \\ \nabla g_1 &= (2x, 2y, 0, \ldots, 0), \\ \nabla g_2 &= (0, 0, 2z_1, \ldots, 2z_{n-1}), \end{aligned}$$

the above equation becomes as follows:

$$(2rx(1 + x^2)^{r-1}, 0, \ldots, 0) = (2\lambda x, 2\lambda y, 2\mu z_1, \ldots, 2\mu z_{n-1}).$$

So we have the following:

$$2\lambda x = 2rx(1+x^2)^{r-1}, \tag{3}$$
$$2\lambda y = 0, \tag{4}$$
$$2\mu z_i = 0, \quad i = 1, \dots, n-1, \tag{5}$$
$$x^2 + y^2 = 1, \tag{6}$$
$$z_1^2 + \cdots + z_{n-1}^2 = 1. \tag{7}$$

In the equation (5), if $\mu \neq 0$, then we have $z_i = 0$, $i = 1, \dots, n-1$, and it contradicts to the equation (7). So we have $\mu = 0$,

In (4), we have $\lambda = 0$ or $y = 0$.

(i) If $\lambda = 0$, then $x = 0$ and thus $y = \pm 1$.

(ii) If $\lambda \neq 0$, then $y = 0$ and thus $x = \pm 1, \lambda = 2r$.

Therefore the set of critical points of f are as follows.

$$\{(\pm 1, 0, z_1, \dots, z_{n-1}), (0, \pm 1, z_1, \dots, z_{n-1})\},$$

where $z_1, \dots, z_{n-1}$ are arbitrary real numbers which satisfies the equation (5). This completes the proof. □

4. Strongly normal unit vectors

A unit vector field V on a Riemannian manifold (M, g) is called *normal* if R_{XY} preserves $\mathcal{H}^V$ for all $X, Y \in \mathcal{H}^V$, i.e., $g(R_{XY}Z, V) = 0$ for all $X, Y, Z \in \mathcal{H}^V$. And $V \in \chi^1(M, g)$ is called *strongly normal* if $g((\nabla_X A_V)Y, Z) = 0$ for all $X, Y, Z \in \mathcal{H}^V$. Here, $A_V = -\nabla V$. A vector field V is called *killing* if A_V is skew-symmetric. It is easy to see that strongly normal unit vector field is normal and a unit Killing vector field is strongly normal if and only if normal.

The set of all strongly normal unit vector fields on G_n is given as follows.

Theorem 4.1. *For $n \geq 2$, the set of all strongly normal unit vector fields on the Lie group G_n is $\{\pm E_n\} \cup (\mathcal{S} \cap E_n^{\perp})$, where $\mathcal{S}$ is the unit sphere of $\mathfrak{g}_n$.*

Proof. Let $X = \sum_{i=1}^n a_i E_i$, $Y = \sum_{j=1}^n b_j E_j$, $Z = \sum_{k=1}^n a_k E_k$, and $V = \sum_{l=1}^n p_l E_l$. And assume that a_i, b_j, c_k, p_l satisfies the conditions

$$||X|| = ||Y|| = ||Z|| = ||V|| = 1$$

and

$$\langle X, V \rangle = \langle Y, V \rangle = \langle Z, V \rangle = 0.$$

Then

$$(\nabla_X A_V)(Y) = -\nabla_X(\nabla_Y V) + \nabla_{\nabla_X Y} V$$
$$= \sum_{i=1}^{n-1}(-a_i b_n p_i + a_i b_i p_n)E_n$$
$$= (\sum_{i=1}^{n} a_i b_i) p_n E_n.$$

So we have

$$\langle(\nabla_X A_V)(Y), Z\rangle = (\sum_{i=1}^{n} a_i b_i) c_n p_n.$$

If $p_n = 0$, then V is strongly normal. Thus every vector in the set $\mathcal{S} \cap E_n^{\perp}$ is strongly normal. If $p_1 = p_2 = \cdots = p_{n-1} = 0$, then $a_n = b_n = c_n = 0$ and thus E_n is strongly normal.

Now assume that $p_n \cdot p_i \neq 0$, for some i, $1 \leq i < n$. Put

$$X = Y = Z = \frac{1}{\sqrt{p_i^2 + p_n^2}} \left(-p_n E_i + p_i E_n\right).$$

Then $\langle(\nabla_X A_V)(Y), Z\rangle = p_n \cdot p_i \neq 0$. So the vector $V = \sum_{i=1}^n p_i E_l$ with $p_n \cdot p_i \neq 0$, $1 \leq i < n$, is not strongly normal. This completes the proof. □

By the Theorem 1.1 and the Theorem 4.1, for $n > 2$, the set of minimal unit vector fields and the set of strongly normal unit vector fields on the Lie group G_n are the same.

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