Bull. Korean Math. Soc. **46** (2009), No. 4, pp. 657–671
DOI 10.4134/BKMS.2009.46.4.657

# A STRUCTURE THEOREM FOR COMPLETE INTERSECTIONS

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Abstract. Buchsbaum and Eisenbud proved a structure theorem for Gorenstein ideals of grade 3. In this paper we derive a class of the perfect ideals from a class of the complete matrices. From this we give a structure theorem for complete intersections of grade $g > 3$.

## 1. Introduction

Let $R$ be a Noetherian local ring and $I$ a perfect ideal of grade $g$ in $R$. Many people have been studying the algebra structure on the minimal free resolution of $R/I$, in particular, Gorenstein ideals, the ideals of type 1. In 1968, Burch [3] characterized perfect ideals of grade 2 by showing a structure theorem due to Hilbert in a special case: every perfect ideal of grade 2 generated by $n$ elements is the ideal of $(n-1)$st order minors of an $(n-1) \times n$ matrix. In 1977, Buchsbaum and Eisenbud [2] gave a structure theorem for Gorenstein ideals of grade 3 which says that every Gorenstein ideal of grade 3 in $R$ is generated by the maximal order Pfaffians of an alternating matrix. However a structure theorem for Gorenstein ideals of grade 4 is more complicated than that of grade 3 and not completely known. In 1987, Brown [1] described a structure theorem for a certain class of perfect ideals $I$ which have grade 3, type 2 and $\lambda(I) = \dim_k \Lambda_1^2 > 0$, where $\lambda(I)$ is a numerical invariant defined in [5]. In 1989, Sanchez [7] gave a structure theorem for type 3, grade 3 perfect ideals which have $\lambda(I) = \dim_k \Lambda_1^2 = 2$ or greater. In this paper we will describe a structure theorem for complete intersections of grade $g > 3$, which says that every complete intersection of grade $g > 3$ in $R$ is generated by the elements $x_i$'s, where $x_i^{g-1}$ is the determinant of the $(g-1) \times (g-1)$ diagonal matrix drawn from a complete matrix of grade $g$ for each $i$ $(1 \leq i \leq g)$.

In Section 2 we review some of the properties of alternating matrices, linkage theory, and a structure theorem for Gorenstein ideals of grade 3.

In Section 3 we give the concept of a complete matrix of grade 4 and provide a structure theorem for complete intersections of grade 4.

Received August 2, 2008.
2000 *Mathematics Subject Classification.* 13C05, 13C14, 13C40, 14M10.
*Key words and phrases.* complete intersection of grade $g$, structure theorem.

In Section 4 we introduce a complete matrix $f$ of grade $g > 3$, and define the ideal $\mathcal{K}_{g-1}(f)$ associated with $f$. Then we prove a structure theorem for complete intersections of grade $g > 3$. The structure theorem [4] for complete intersections of grade 4 is just a special case of our main Theorem 4.10. Throughout this paper, we assume that all rings are a Noetherian local ring with maximal ideal $\mathfrak{m}$ unless otherwise stated.

## 2. Gorenstein ideals of grade 3

The grade of a proper ideal $I$ in $R$ is the length of the maximal $R$-sequence contained in $I$. We say that an ideal $I$ of grade $g$ is *perfect* if grade $I = \operatorname{prodim}_R(R/I) = g$. If $I$ is a perfect ideal of grade $g$, then the *type* of $I$ is defined to be the dimension of the $R/\mathfrak{m}$-vector space $\operatorname{Ext}^g_R(R/\mathfrak{m}, R/I)$. A perfect ideal $I$ of grade $g$ is *Gorenstein* if $\operatorname{type} I = 1$, equivalently, if $\mathbb{F}$ is the minimal free resolution of $R/I$,

$$\mathbb{F} : 0 \longrightarrow F_g \xrightarrow{\varphi_g} F_{g-1} \xrightarrow{\varphi_{g-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0(= R) ,$$

then the rank of $F_g$ is 1. A perfect ideal $I$ of grade $g$ is a *complete intersection* if it is generated by $g$ elements, and is an *almost complete intersection* if it is minimally generated by $g + 1$ elements.

Let $R$ be a commutative ring, and $F$ a finite free $R$-module. An $R$-module homomorphism $\varphi : F \to F^*$ is said to be alternating if with respect to some (and therefore any) basis of $F$ and the corresponding dual basis of $F^*$, the matrix $\varphi$ is alternating, i.e., skew-symmetric and all its diagonal entries are 0. Now suppose that $\varphi$ is alternating, choose a basis of $F$ and the corresponding dual basis of this, and identify $\varphi$ with the corresponding matrix $(\varphi_{ij})$. If $\operatorname{rank} F$ is odd, then $\det \varphi = 0$, and if $\operatorname{rank} F$ is even, then there exists an element $\operatorname{Pf}(\varphi) \in R$, called the Pfaffian of $\varphi$, which is a polynomial function of the entries of $\varphi$, such that $\det \varphi = \operatorname{Pf}(\varphi)^2$. We set $\operatorname{Pf}(\varphi) = 0$ if $\operatorname{rank} F$ is odd. Pfaffians can be developed along a row just like the determinants. Denote by $\operatorname{Pf}_r(\varphi)$ the ideal generated by the $r$th order Pfaffians of $\varphi$. With these concepts Buchsbaum and Eisenbud gave a complete structure for Gorenstein ideals of grade 3:

**Theorem 2.1** ([2]). *Let $R$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$.*

(1) *Let $F$ be a free $R$-module with* $\operatorname{rank} F = n$, *where $n \geqslant 3$ is an odd integer. Let $\varphi : F^* \to F$ be an alternating map whose image is contained in $\mathfrak{m}F$. Suppose that $Pf_{n-1}(\varphi)$ has grade* 3. *Then $Pf_{n-1}(\varphi)$ is a Gorenstein ideal minimally generated by $n$ elements.*

(2) *Every Gorenstein ideal of grade* 3 *arises as in* (1).

Now we review some of the notions in the linkage theory formulated by Peskine and Szpiro in [6].

**Definition 2.2.** Let $I$ and $J$ be two ideals in a Gorenstein ring $R$ (not necessarily local).

(1) If there exists an $R$-regular sequence $\boldsymbol{\alpha} = \alpha_1, \alpha_2, \ldots, \alpha_g$ in $I \cap J$ such that $J = (\boldsymbol{\alpha}) : I$ and $I = (\boldsymbol{\alpha}) : J$, then $I$ and $J$ are said to be linked (with respect to $\boldsymbol{\alpha}$).

(2) If $I$ and $J$ are linked and if $\operatorname{Ass}(R/I) \cap \operatorname{Ass}(R/J) = \emptyset$, equivalently, if $I$ and $J$ are linked (with respect to $\boldsymbol{\alpha}$) and if $I \cap J = (\boldsymbol{\alpha})$, then $I$ and $J$ are said to be *geometrically linked.*

Let $R$ be a Gorenstein local ring of Krull dimension $g$ with maximal ideal $\mathfrak{m}$. If $I$ and $J$ are perfect ideals of grade $g$, then they are not geometrically linked because $(R/I)$ and $(R/J)$ are both zero-dimensional artinian local rings. Peskine and Szpiro gave a method of constructing a Gorenstein ideal of grade $g+1$ from two perfect ideals of grade $g$:

**Theorem 2.3** ([6])**.** *Let $R$ be a Gorenstein local ring with maximal ideal $\mathfrak{m}$. Let $I$ and $J$ be geometrically linked Cohen-Macaulay ideals of grade $g$ by a regular sequence $\mathbf{x} = x_1, x_2, \ldots, x_g$ and let $K = I + J$. Then $K$ is a Gorenstein ideal of grade $g+1$.*

Let $F$ be a free $R$-module with a basis $\{e_1, e_2, \ldots, e_n\}$ and let $I$ be an ideal generated by a regular sequence $\mathbf{x} = x_1, x_2, \ldots, x_n$. Let $\mathbb{K}(\mathbf{x})$ be the Koszul complex defined by $\mathbf{x} = x_1, x_2, \ldots, x_n$. Then

$$\mathbb{K}(\mathbf{x}) : 0 \longrightarrow \wedge^n F \xrightarrow{d_n} \wedge^{n-1} F \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \wedge^1 F \xrightarrow{d_1} \wedge^0 F$$

is the minimal free resolution of $R/I$, where $d_1(e_i) = x_i$ for each $i$ with $1 \leqslant i \leqslant n$, and for each $p$ with $1 \leqslant p \leqslant n$, $d_p : \wedge^p F \to \wedge^{p-1} F$ is given by

$$(2.1)\quad d_p(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} d_1(e_{i_j}) e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge e_{i_p}.$$

For example, if $\mathbf{x} = x_1, x_2, x_3, x_4, x_5$ is a regular sequence on $R$, then $d_2$ has the form

$$(2.2)\quad d_2 = \begin{bmatrix} -x_2 & -x_3 & -x_4 & -x_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & -x_3 & -x_4 & -x_5 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & x_2 & 0 & 0 & -x_4 & -x_5 & 0 \\ 0 & 0 & x_1 & 0 & 0 & x_2 & 0 & x_3 & 0 & -x_5 \\ 0 & 0 & 0 & x_1 & 0 & 0 & x_2 & 0 & x_3 & x_4 \end{bmatrix}.$$

The exterior algebra $\wedge F$ is a graded Hopf algebra such that $x \wedge y = (-1)^{pq} y \wedge x$ for $x \in \wedge^p F$ and $y \in \wedge^q F$ and $x \wedge x = 0$ for any homogeneous element $x$ of odd degree. It is well-known that the algebra structure on the Koszul complex which gives the minimal free resolution of a complete intersection is an exterior algebra.

## 3. Complete intersections of grade 4

In this section we start with a skew-symmetrizable matrix, and a complete matrix of grade 4 which play important roles in describing the complete intersections of grade 4.

**Definition 3.1.** Let $R$ be a commutative ring with identity. An $n \times n$ matrix $X = (x_{ij})$ over $R$ is said to be *generalized alternating or skew-symmetrizable* if there exist nonzero $n \times n$ diagonal matrices $D' = \operatorname{diag}(u_1, u_2, \ldots, u_n)$ and $D = \operatorname{diag}(w_1, w_2, \ldots, w_n)$ with entries in $R$ such that $D'XD$ is alternating. We denote by $\mathrm{GA}_n(R)$ the set of all skew-symmetrizable $n \times n$ matrices over $R$. If there is no ambiguity about the ring $R$, then $\mathrm{GA}_n(R)$ is denoted by $\mathrm{GA}_n$.

Notice that every alternating matrix is skew-symmetrizable. For an $n \times n$ skew-symmetrizable matrix $X$, we denote $\mathcal{A}(X)$ to be an alternating matrix $D'XD$ for some diagonal matrices $D'$ and $D$. To define a complete intersection of grade 4, we need to describe the submatrices of the given matrix in detail. A $p \times q$ submatrix of an $m \times n$ matrix $f$ is a matrix obtained from $f$ by taking the $pq$ entries at the intersections of the $i_1$th, $i_2$th, $\ldots$, $i_p$th rows and the $j_1$th, $j_2$th, $\ldots$, $j_q$th columns of $f$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_q \leq n$. The corresponding $p \times q$ submatrix of $f$ is denoted by

$$f(i_1, i_2, \ldots, i_p | j_1, j_2, \ldots, j_q).$$

Notice that the $p \times q$ matrix $f(i_1, i_2, \ldots, i_p | j_1, j_2, \ldots, j_q)$ consisting of the $pq$ entries at the intersection of these rows and columns of $f$ could not be a submatrix of $f$ unless $1 \leq i_1 < i_2 < \cdots < i_p \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_q \leq n$. Next we get into the skew-symmetrizable matrices and the special properties of the second differential map $d_2$ of the Koszul complex $\mathbb{K}(\mathbf{x})$

$$\mathbb{K}(\mathbf{x}) : 0 \longrightarrow \wedge^4 F \xrightarrow{d_4} \wedge^3 F \xrightarrow{d_3} \wedge^2 F \xrightarrow{d_2} \wedge^1 F \xrightarrow{d_1} \wedge^0 F$$

defined by a regular sequence $\mathbf{x} = x_1, x_2, x_3, x_4$ on $R$. With respect to the standard basis of $F$, $d_2$ has the following form

$$d_2 = \begin{bmatrix} -x_2 & -x_3 & -x_4 & 0 & 0 & 0 \\ x_1 & 0 & 0 & -x_3 & -x_4 & 0 \\ 0 & x_1 & 0 & x_2 & 0 & -x_4 \\ 0 & 0 & x_1 & 0 & x_2 & x_3 \end{bmatrix}.$$

**Proposition 3.2.** *With the notation as above, the second differential map $d_2$ of the Koszul complex satisfies the following properties*:

(1) *There are four disjoint pairs $(S, T)$ of two $4 \times 3$ submatrices of $d_2$*;

(2) *By removing a row and interchanging columns, each pair $(S, T)$ can be reduced to a pair $(\bar{S}, \bar{T})$ of $3 \times 3$ matrices such that $\bar{S}$ is a diagonal matrix whose determinant is the nonzero 3rd power element $x^3$ for some $x \in R$, and $\bar{T}$ is a skew-symmetrizable matrix with grade $\mathrm{Pf}_2(\mathcal{A}(\bar{T})) = 3$.*

*Proof.* Let $S_1 = d_2(1,2,3,4\,|\,1,2,3)$ and $T_1 = d_2(1,2,3,4\,|\,4,5,6)$ be the disjoint $4\times 3$ submatrices of $d_2$. Then the submatrix obtained by removing the first row of $S_1$ is a $3\times 3$ diagonal matrix $\bar{S}_1$ whose determinant is equal to $x_1^3$. Removing the first row and interchanging columns 1 and 3 of $T_1$, we have the $3\times 3$ matrix $\bar{T}_1$. Then $\bar{T}_1$ is skew-symmetrizable, since it becomes an alternating matrix by multiplying the second column of it by $-1$. Since $x_2, x_3, x_4$ is a regular sequence on $R$, $\mathrm{Pf}_2(\mathcal{A}(\bar{T}_1)) = (x_2, x_3, x_4)$ has grade 3. Similarly, we can take the disjoint submatrices of $d_2$;

$$S_2 = d_2(1,2,3,4\,|\,1,4,5) \text{ and } T_2 = d_2(1,2,3,4\,|\,2,3,6),$$
$$S_3 = d_2(1,2,3,4\,|\,2,4,6) \text{ and } T_3 = d_2(1,2,3,4\,|\,1,3,5),$$
$$S_4 = d_2(1,2,3,4\,|\,3,5,6) \text{ and } T_4 = d_2(1,2,3,4\,|\,1,2,4).$$

The similar argument gives us the $3\times 3$ diagonal matrix $\bar{S}_i$ whose determinant is equal to $x_i^3$ or $(-x_i)^3$, and the $3\times 3$ skew-symmetrizable matrix $\bar{T}_i$ with grade $\mathrm{Pf}_2(\mathcal{A}(\bar{T}_i)) = 3$ for $i = 2,3,4$. □

**Definition 3.3.** Let $R$ be a commutative ring with identity. A $4\times 6$ matrix $f$ over $R$ is said to be a *complete matrix of grade* 4 if

(1) $f$ has four distinct pairs $(S,T)$ of disjoint $4\times 3$ submatrices;

(2) By removing a row and interchanging columns, each pair $(S,T)$ is reduced to a pair $(\bar{S},\bar{T})$ of $3\times 3$ matrices such that $\bar{S}$ is a diagonal matrix whose determinant is a nonzero 3rd power element $x^3$ for some $x\in R$, and $\bar{T}$ is a skew-symmetrizable matrix with grade $\mathrm{Pf}_2(\mathcal{A}(\bar{T})) = 3$.

The following example illustrates Definition 3.3.

**Example 3.4.** Let $x, y, z$, and $w$ be a regular sequence on a commutative ring $R$. Let $f$ be a $4\times 6$ matrix given by

$$f = \begin{bmatrix} 0 & 0 & -y & -w & -z & 0 \\ 0 & -z & x & 0 & 0 & -w \\ -w & y & 0 & 0 & x & 0 \\ z & 0 & 0 & x & 0 & y \end{bmatrix}.$$

Then $f$ is a complete matrix of grade 4. To see this, we find four distinct pairs of disjoint $4\times 3$ submatrices $S_i$ and $T_i$ of $f$ satisfying the properties in Proposition 3.2. First we consider two submatrices of $f$;

$$S_1 = \begin{bmatrix} -y & -w & -z \\ x & 0 & 0 \\ 0 & 0 & x \\ 0 & x & 0 \end{bmatrix} \quad \text{and} \quad T_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -z & -w \\ -w & y & 0 \\ z & 0 & y \end{bmatrix},$$

that is, $S_1 = f(1,2,3,4\,|\,3,4,5)$ and $T_1 = f(1,2,3,4\,|\,1,2,6)$. So $S_1$ and $T_1$ are disjoint. By removing the first row and interchanging the second and the third columns of $S_1$ and $T_1$, we can get the $3\times 3$ matrices $\bar{S}_1 = S_1(2,3,4|1,3,2)$ and $\bar{T}_1 = T_1(2,3,4\,|\,1,3,2)$. Then $\bar{S}_1$ is a diagonal matrix whose determinant is a nonzero element $x^3$ and $\bar{T}_1$ is skew-symmetrizable since $\bar{T}_1\mathrm{diag}(1,-1,1)$

is alternating. It is easy to show that $\mathrm{Pf}_2(\mathcal{A}(\bar{T}_1))$ has grade 3. Similarly, we consider the submatrices of $f$;

$$S_2 = f(1,2,3,4\,|\,2,3,6) \text{ and } T_2 = f(1,2,3,4\,|\,1,4,5),$$
$$S_3 = f(1,2,3,4\,|\,1,2,5) \text{ and } T_3 = f(1,2,3,4\,|\,3,4,6),$$
$$S_4 = f(1,2,3,4\,|\,1,4,6) \text{ and } T_4 = f(1,2,3,4\,|\,2,3,5).$$

Clearly, $4\times 3$ submatrices $S_i$ and $T_i$ of $f$ are disjoint for $i = 2,3,4$. The similar argument gives us the following $3\times 3$ matrices;

$$\bar{S}_2 = S_2(1,3,4\,|\,2,1,3) \text{ and } \bar{T}_2 = T_2(1,3,4\,|\,1,2,3),$$
$$\bar{S}_3 = S_3(1,2,4\,|\,3,2,1) \text{ and } \bar{T}_3 = T_3(1,2,4\,|\,3,2,1),$$
$$\bar{S}_4 = S_4(1,2,3\,|\,2,3,1) \text{ and } \bar{T}_4 = T_4(1,2,3\,|\,1,3,2).$$

And $\det \bar{S}_2 = (-y)^3$, $\det \bar{S}_3 = z^3$ and $\det \bar{S}_4 = (-w)^3$ are nonzero 3rd power elements and

$$\mathrm{Pf}_2(\mathcal{A}(\bar{T}_1)) = (y,z,w),\ \ \mathrm{Pf}_2(\mathcal{A}(\bar{T}_2)) = (x,z,w),$$
$$\mathrm{Pf}_2(\mathcal{A}(\bar{T}_3)) = (x,y,w),\ \ \mathrm{Pf}_2(\mathcal{A}(\bar{T}_4)) = (x,y,z).$$

Since $x,y,z,w$ is a regular sequence on $R$, these four ideals have all grade 3. Hence the properties in Proposition 3.2 are satisfied.

We notice that if $f$ is a complete matrix of grade 4, then the matrix obtained from $f$ by interchanging rows of $f$ also becomes a complete matrix of grade 4.

**Theorem 3.5** ([4]). *Let $f = (f_{ij})$ be a $4\times 6$ complete matrix of grade 4.*

(1) *Every column of $f$ has exactly two nonzero entries.*
(2) *The number of nonzero rows in each $4\times 2$ submatrix of $f$ is greater than 2.*
(3) *Each pair $(\bar{S},\bar{T})$ of $3\times 3$ matrices given in Definition 3.3 is uniquely determined.*

Now we will define an ideal $\mathcal{K}_3(f)$ generated by the radical roots of the determinants of the $3\times 3$ diagonal matrices $\bar{S}$ derived from a given complete matrix $f$ of grade 4 in Theorem 3.5.

**Definition 3.6.** Let $f$ be a $4\times 6$ complete matrix of grade 4. Let $\bar{S}_i$ be a unique $3\times 3$ diagonal matrix reduced from the disjoint pair $(S_i,T_i)$ of $f$ such that $\det \bar{S}_i = x_i^3$ is nonzero for $i = 1,2,3,4$. We define $\mathcal{K}_3(f)$ to be the ideal generated by the $x_i$'s, that is,

$$\mathcal{K}_3(f) = (x_1,x_2,x_3,x_4).$$

Next let us show that the ideal $\mathcal{K}_3(f)$ defines a complete intersection of grade 4. Let $f$ be a complete matrix of grade 4. By Theorem 3.5 we may assume

$$f = \begin{bmatrix} f_{11} & f_{12} & f_{13} & 0 & 0 & 0 \\ f_{21} & 0 & 0 & f_{24} & f_{25} & 0 \\ 0 & f_{32} & 0 & f_{34} & 0 & f_{36} \\ 0 & 0 & f_{43} & 0 & f_{45} & f_{46} \end{bmatrix}.$$

Then we have

$$\begin{aligned} \bar{S}_1 &= f(2,3,4|1,2,3) \text{ and } \bar{T}_1 = f(2,3,4|6,5,4), \\ \bar{S}_2 &= f(1,3,4|1,4,5) \text{ and } \bar{T}_2 = f(1,3,4|6,3,2), \\ \bar{S}_3 &= f(1,2,4|2,4,6) \text{ and } \bar{T}_3 = f(1,2,4|5,3,1), \\ \bar{S}_4 &= f(1,2,3|3,5,6) \text{ and } \bar{T}_4 = f(1,2,3|4,2,1), \end{aligned}$$

i.e.,

$$\begin{aligned} \bar{S}_1 &= \begin{bmatrix} f_{21} & 0 & 0 \\ 0 & f_{32} & 0 \\ 0 & 0 & f_{43} \end{bmatrix} \quad \text{and} \quad \bar{T}_1 = \begin{bmatrix} 0 & f_{25} & f_{24} \\ f_{36} & 0 & f_{34} \\ f_{46} & f_{45} & 0 \end{bmatrix}, \\ \bar{S}_2 &= \begin{bmatrix} f_{11} & 0 & 0 \\ 0 & f_{34} & 0 \\ 0 & 0 & f_{45} \end{bmatrix} \quad \text{and} \quad \bar{T}_2 = \begin{bmatrix} 0 & f_{13} & f_{12} \\ f_{36} & 0 & f_{32} \\ f_{46} & f_{43} & 0 \end{bmatrix}, \\ \bar{S}_3 &= \begin{bmatrix} f_{12} & 0 & 0 \\ 0 & f_{24} & 0 \\ 0 & 0 & f_{46} \end{bmatrix} \quad \text{and} \quad \bar{T}_3 = \begin{bmatrix} 0 & f_{13} & f_{11} \\ f_{25} & 0 & f_{21} \\ f_{45} & f_{43} & 0 \end{bmatrix}, \\ \bar{S}_4 &= \begin{bmatrix} f_{13} & 0 & 0 \\ 0 & f_{25} & 0 \\ 0 & 0 & f_{36} \end{bmatrix} \quad \text{and} \quad \bar{T}_4 = \begin{bmatrix} 0 & f_{12} & f_{11} \\ f_{24} & 0 & f_{21} \\ f_{34} & f_{32} & 0 \end{bmatrix}. \end{aligned} \tag{3.2}$$

Since $\bar{T}_i \operatorname{diag}(u_{i_1}, u_{i_2}, u_{i_3})$ is alternating where $u_{i_k} \in \{\pm 1\}$, we have the following identities

$$\begin{aligned} &f_{24} = f_{46} \text{ or } -f_{46}, \; f_{25} = f_{36} \text{ or } -f_{36}, \; f_{34} = f_{45} \text{ or } -f_{45}, \\ &f_{12} = f_{46} \text{ or } -f_{46}, \; f_{13} = f_{36} \text{ or } -f_{36}, \; f_{32} = f_{43} \text{ or } -f_{43}, \\ &f_{11} = f_{45} \text{ or } -f_{45}, \; f_{13} = f_{25} \text{ or } -f_{25}, \; f_{21} = f_{43} \text{ or } -f_{43}, \\ &f_{11} = f_{34} \text{ or } -f_{34}, \; f_{12} = f_{24} \text{ or } -f_{24}, \; f_{21} = f_{32} \text{ or } -f_{32}. \end{aligned} \tag{3.3}$$

Thus (3.2) and (3.3) give us

(3.4)

$$\begin{aligned} &\det \bar{S}_1 = f_{21}f_{32}f_{43} = f_{21}^3 \text{ or } -f_{21}^3, \; \det \bar{S}_2 = f_{11}f_{34}f_{45} = f_{11}^3 \text{ or } -f_{11}^3, \\ &\det \bar{S}_3 = f_{12}f_{24}f_{46} = f_{12}^3 \text{ or } -f_{12}^3, \; \det \bar{S}_4 = f_{13}f_{25}f_{36} = f_{13}^3 \text{ or } -f_{13}^3, \end{aligned}$$

and

$$\begin{aligned} &\operatorname{Pf}_2(\mathcal{A}(\bar{T}_1)) = (f_{11}, f_{12}, f_{13}), \; \operatorname{Pf}_2(\mathcal{A}(\bar{T}_2)) = (f_{21}, f_{13}, f_{12}), \\ &\operatorname{Pf}_2(\mathcal{A}(\bar{T}_3)) = (f_{21}, f_{13}, f_{11}), \; \operatorname{Pf}_2(\mathcal{A}(\bar{T}_4)) = (f_{21}, f_{12}, f_{11}). \end{aligned} \tag{3.5}$$

Hence

$$\mathrm{Pf}_2(\mathcal{A}(\bar{T}_i)) \subseteq \mathcal{K}_3(f) = (f_{21}, f_{11}, f_{12}, f_{13}) \quad \text{for } i = 1, 2, 3, 4. \tag{3.6}$$

Thus we obtain the structure theorem for complete intersections of grade 4.

**Theorem 3.7** ([4]). *Let $R$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$.*

(1) *Let $F$ and $G$ be free $R$-modules with* rank $F = 6$ *and* rank $G = 4$. *Let $f = (f_{ij}) : F \to G$ be a complete matrix of grade* 4 *such that* $\mathrm{Im}\, f \subseteq \mathfrak{m}G$. *With the notation as in Theorem* 3.5, *we assume that* $\mathrm{Pf}_2(\mathcal{A}(\bar{T}_i)) + \mathrm{Pf}_2(\mathcal{A}(\bar{T}_j))$ *has grade* 4 *for some $i, j (i \neq j)$. Then the ideal $\mathcal{K}_3(f)$ is a complete intersection of grade* 4.

(2) *Let $I = (x_1, x_2, x_3, x_4)$ be a complete intersection of grade* 4 *and let*

$$\mathbb{F} : 0 \longrightarrow R \xrightarrow{\varphi_4} R^4 \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^4 \xrightarrow{\varphi_1} R$$

*be the minimal free resolution of $R/I$. Then $\varphi_2$ and the transpose of $\varphi_3$ satisfy the part* (1).

## 4. Complete intersections of grade $g > 4$

In this section we construct the ideal $\mathcal{K}_g(f)$ associated with a complete matrix $f$ of grade $g > 3$ and provide a structure theorem for complete intersections of grade $g > 3$. We begin this section with easy lemmas.

**Lemma 4.1.** *Let $R$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$. For any positive integer $g > 3$, let $\mathbf{x} = x_1, x_2, \ldots, x_g$ and $\mathbf{y}_i = x_1, x_2, \ldots, \widehat{x_i}, \ldots, x_g$ be regular sequences on $R$, where $\widehat{x_i}$ indicates that $x_i$ is to be omitted. Let $\mathbb{K}(\mathbf{x})$ and $\mathbb{K}(\mathbf{y}_i)$ be the Koszul complexes of $R/(\mathbf{x})$ and $R/(\mathbf{y}_i)$ for each $i = 1, 2, 3, \ldots, g$. Let*

$$\mathbb{K}(x_i) : 0 \longrightarrow R \xrightarrow{x_i} R$$

*be a complex of free $R$-modules and $R$-maps. Then*

(1) $\mathbb{K}(\mathbf{x}) \cong \mathbb{K}(x_i) \otimes \mathbb{K}(\mathbf{y}_i)$.

(2) *Let*

$$\mathbb{K}(\mathbf{y}_i) : 0 \longrightarrow F_{g-1} \xrightarrow{\varphi_{i\,g-1}} F_{g-2} \xrightarrow{\varphi_{i\,g-2}} \cdots \xrightarrow{\varphi_{i\,2}} F_1 \xrightarrow{\varphi_{i\,1}} R\ ,$$

*and*

$$\mathbb{K}(x_i) \otimes \mathbb{K}(\mathbf{y}_i) : 0 \longrightarrow R \otimes F_{g-1} \xrightarrow{\phi_{ig}} R \otimes F_{g-2} \oplus R \otimes F_{g-1} \xrightarrow{\phi_{ig-1}}$$

$$\cdots \xrightarrow{\phi_{i2}} R \otimes R \oplus R \otimes F_1 \xrightarrow{\phi_{i1}} R \otimes R.$$

*Then we have*

(4.1)

$$\phi_{i1} = \begin{bmatrix} x_i & \varphi_{i1} \end{bmatrix}, \quad \phi_{ik} = \left[\begin{array}{c|c} (-1)^{k-1}\varphi_{ik-1} & 0 \\ \hline x_i I & \varphi_{ik} \end{array}\right] \quad \textit{for} \quad k = 2, 3, \ldots, g-1,$$

$$\phi_{ig} = \begin{bmatrix} \varphi_{ig-1} \\ -x_i \end{bmatrix}.$$

*Proof.* Clear. □

**Lemma 4.2.** *With the notation as above, let* $t = \binom{g}{2}$. *Then, for each* $i$

(1) *Every column of* $\phi_{i2}$ *has exactly two nonzero entries.*
(2) *The number of nonzero rows in each* $g \times 2$ *submatrix of* $\phi_{i2}$ *is greater than* 2*, that is,* 3 *or* 4*.*

*Proof.* This follows from the matrix form of $\phi_{i2}$ (see (2.1) and (2.2)). □

Now we can describe the special properties of $\phi_{i2}$ in (4.1).

**Proposition 4.3.** *With the notation as above and hypotheses*:

(1) $\phi_{i2}$ *has* $g$ *disjoint pairs* $(S_k, T_k)$ *of a* $g \times (g-1)$ *submatrix* $S_k$ *and a* $g \times (t - g + 1)$ *submatrix* $T_k$;

(2) *By removing the* $i$*th row and interchanging columns of* $\phi_{i2}$*, each pair* $(S_k, T_k)$ *can be reduced to a pair* $(\bar{S}_k, \bar{T}_k)$*, where* $\bar{S}_k$ *is a* $(g-1) \times (g-1)$ *diagonal matrix whose determinant is* $x_k^{g-1}$*, up to sign, and* $\bar{T}_k$ *is the second differential map in the Koszul complex* $\mathbb{K}(\mathbf{y}_k)$*.*

*Proof.* (1) The first statement follows from the second statement.

(2) It is enough to prove the case $i = 1$. For the sake of simplicity, $\phi_{12}$ can be written as the form

$$\phi_{12} = \left[\begin{array}{c|c} -\varphi_{11} & 0 \\ \hline x_1 I & \varphi_{12} \end{array}\right]. \tag{4.2}$$

Let $S_1 = \phi_{12}(1, 2, \ldots, g \,|\, 1, 2, \ldots, g-1)$ and $T_1 = \phi_{12}(1, 2, \ldots, g \,|\, g, g+1, \ldots, t)$. Then clearly, $S_1$ and $T_1$ are disjoint. Taking $\bar{S}_1 = x_1 I$ and $\bar{T}_1 = \varphi_{12}$ as submatrices of $\phi_{12}$, it is clear that $\det \bar{S}_1 = (x_1)^{g-1}$ and $\bar{T}_1$ is the second differential map in the Koszul complex $\mathbb{K}(\mathbf{y}_1)$. Let $k > 1$ be an integer with $2 \le k \le g$. It follows from Lemma 4.2 that every row of $\phi_{12}$ consists of exactly $g-1$ nonzero entries and exactly $t - g + 1$ zero entries. Choose $S_k$ to be a $g \times (g-1)$ submatrix of $\phi_{12}$ such that all the entries of the $k$th row are nonzero, and $T_k$ to be a $g \times (t-g+1)$ submatrix of $\phi_{12}$ such that all the entries of the $k$th row are zero. Then clearly $S_k$ and $T_k$ are disjoint. Let $S'_k$ and $T'_k$ be the submatrices of $S_k$ and $T_k$ obtained by removing the $k$th row of $S_k$ and $T_k$, respectively. By the part (1) of Lemma 4.2, every column of $S'_k$ has exactly one nonzero entry. We observe from (4.2) that the nonzero entry in the $l$th column of $S'_k$ is either $x_k$ or $-x_k$ for $l = 1, 2, \ldots, g-1$. The part (2) of Lemma 4.2 implies that every row of $S'_k$ has exactly one nonzero entry. This implies that interchanging columns of $S'_k$ produces a $(g-1) \times (g-1)$ diagonal matrix $\bar{S}_k$ whose main diagonal entries are either $x_k$ or $-x_k$. Thus $\det \bar{S}_k = \pm x_k^{g-1}$. It follows from the construction of $T'_k$ and Lemma 4.2 that every column of $T'_k$ has exactly two nonzero entries and the number of nonzero rows in each $(g-1) \times 2$ submatrix of $T'_k$ is 3. Since $T'_k$ has $t - g + 1 = \binom{g-1}{2}$ columns and $g-1$ rows, interchanging columns of $T'_k$ (if necessary) gives us the second differential map $\bar{T}_k$ in the Koszul complex $\mathbb{K}(\mathbf{y}_k)$(see (4.2)). Actually, $\bar{T}_k$ has the form

$$\bar{T}_k = \left[\begin{array}{c|c} h_k & 0 \\ \hline d_1 & h'_k \end{array}\right],$$

where

$$\begin{aligned} h_k &= \begin{bmatrix} -x_2 & -x_3 & \cdots & -\widehat{x_k} & \cdots & -x_g \end{bmatrix}, \\ d_1 &= \operatorname{diag}(x_1, x_1, \ldots, x_1), \\ h'_k &= \text{the second differential map in the Koszul complex } \mathbb{K}(\mathbf{y}_{1k}) \text{ for} \\ &\qquad \mathbf{y}_{1k} = x_2, x_3, \ldots, \widehat{x_k}, \ldots, x_g. \end{aligned}$$

Thus we have the desired one $\bar{T}_k$. □

To define the ideal $\mathcal{K}_{g-1}(\phi_{i2})$ associated with the map $\phi_{i2}$ we need further properties of $\phi_{i2}$.

**Theorem 4.4.** (1) *With the notation as in Proposition* 4.3, *for each* $k$ $(1 \leq k \leq g)$, *a pair* $(\bar{S}_k, \bar{T}_k)$ *of matrices given in Proposition* 4.3 *is uniquely determined.*

(2) *If for each* $k$, $\mathcal{K}_{g-2}(\bar{T}_k)$ *is the ideal generated by the elements* $x_1, x_2, \ldots, \widehat{x_k}, \ldots, x_g$ *given in the proof of Proposition* 4.3, *then* $\mathcal{K}_{g-2}(\bar{T}_k)$ *has grade* $g-1$.

*Proof.* (1) This follows from Lemma 4.2.

(2) The second part is also clear since $x_1, x_2, \ldots, \widehat{x_k}, \ldots, x_g$ is a regular sequence on $R$. □

Thus Theorem 4.4 enables us to define a complete matrix of grade $g$. With an induction argument, we may call $\bar{T}_k$ given in Theorem 4.4 the complete matrix of grade $g-1$ in the following sense.

**Definition 4.5.** Let $R$ be a commutative ring with identity. Let $g > 3$ and $t = \binom{g}{2}$ be integers. A $g \times t$ matrix $f = (f_{ij})$ over $R$ is said to be *complete of grade* $g$ if

(1) $f$ has $g$ disjoint pairs $(S, T)$ of a $g \times (g-1)$ submatrix $S$ and a $g \times (t-g+1)$ submatrix $T$;

(2) By removing a row and interchanging columns, each pair $(S, T)$ can be reduced to a pair $(\bar{S}, \bar{T})$, where $\bar{S}$ is a $(g-1) \times (g-1)$ diagonal matrix with

$\det(\bar{S}) = x^{g-1}$ for some $x \in R$, and $\bar{T}$ is the complete matrix of grade $g-1$ with grade $\mathcal{K}_{g-2}(\bar{T}) = g-1$.

The following example illustrates Definition 4.5.

**Example 4.6.** Let $x, y, z, u, w$ be a regular sequence in a Noetherian local ring $R$. Let

$$f = \begin{bmatrix} y & z & u & w & 0 & 0 & 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 & z & u & w & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & -y & 0 & 0 & u & w & 0 \\ 0 & 0 & -x & 0 & 0 & -y & 0 & -z & 0 & w \\ 0 & 0 & 0 & -x & 0 & 0 & -y & 0 & -z & -u \end{bmatrix}.$$

The similar argument as in Example 3.4 shows that $f$ satisfies the properties in Proposition 4.3 and the part (2) of Theorem 4.4.

The following theorem is an easy generalization of Theorem 3.5.

**Theorem 4.7.** *Let $g > 3$ and $t = \binom{g}{2}$ be integers. A $g \times t$ matrix $f = (f_{ij})$ over $R$ is a complete matrix of grade $g$.*

1. *Every column of $f$ has exactly two nonzero entries.*
2. *The number of nonzero rows in each $g \times 2$ submatrix of $f$ is greater than 2.*
3. *Each pair $(\bar{S}, \bar{T})$ of matrices given in Definition 4.5 is uniquely determined.*

*Proof.* The proofs are essentially similar with those of Theorem 3.5. □

Now we define an ideal $\mathcal{K}_{g-1}(f)$ generated by the entries in the $(g-1)\times(g-1)$ matrices $\bar{S}$ derived from a given complete matrix $f$ of grade $g$ in Theorem 4.7.

**Definition 4.8.** Let $g > 3$ and $t = \binom{g}{2}$ be integers. Let $f$ be a $g \times t$ complete matrix of grade $g$. For $i = 1, 2, \ldots, g$, we let $\bar{S}_i$ be a unique $(g-1) \times (g-1)$ diagonal matrix extracted from $f$ in the part (3) of Theorem 4.7 such that $\det \bar{S}_i = x_i^{g-1}$ is nonzero for some $x_i \in R$. We define $\mathcal{K}_{g-1}(f)$ to be the ideal generated by the $x_i$'s, that is,

$$\mathcal{K}_{g-1}(f) = (x_1, x_2, \ldots, x_g).$$

Let $f = (f_{ij})$ be a $g \times t$ complete matrix of grade $g$. It follows from the properties (1) and (2) of Theorem 4.7 that interchanging columns of $f$ transforms $f$ to the following form.

$$f = \left[\begin{array}{c|c} h_1 & 0 \\ \hline d_1 & h_2 \end{array}\right], \tag{4.3}$$

where

$h_1 = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1g-1} \end{bmatrix}$,

$d_1 = \operatorname{diag}(f_{21}, f_{32}, \ldots, f_{g\,g-1})$, $h_2 =$ a complete matrix of grade $g-1$.

By applying the method of (3.5) and (3.6) in the case of a complete matrix of grade 4 to the given $f$, we have

$$\begin{aligned} &\mathcal{K}_{g-2}(\bar{T}_1) = (\widehat{f}_{21}, f_{11}, f_{12}, \ldots, f_{1\,g-1}), \text{ and} \\ (4.4)\quad &\mathcal{K}_{g-2}(\bar{T}_i) = (f_{21}, f_{11}, f_{12}, \ldots, \widehat{f}_{1\,i-1}, f_{1i}, \ldots, f_{1\,g-1}) \quad \text{for } i = 2, 3, \ldots g, \\ &\text{where } \widehat{f}_{1i} \text{ indicates that } f_{1i} \text{ is to be omitted.} \end{aligned}$$

Hence

$$(4.5)\qquad \mathcal{K}_{g-2}(\bar{T}_i) \subseteq \mathcal{K}_{g-1}(f) = (f_{21}, f_{11}, f_{12}, \ldots, f_{1\,g-1}) \quad \text{for each } i.$$

The following lemma will be used in proving the structure theorem for complete intersections of grade $g > 3$.

**Lemma 4.9.** *Let $\mathbf{x} = x_1, x_2, \ldots, x_g$ be a regular sequence on $R$ and $\mathbb{F}$ a minimal free resolution of $R/(\mathbf{x})$. If $\varphi_2$ is the second differential map of $\mathbb{F}$, then $\varphi_2$ is a complete matrix of grade $g$.*

*Proof.* Let $\mathbb{K}(\mathbf{x})$ be the Koszul complex defined by the regular sequence $\mathbf{x} = x_1, x_2, \ldots, x_g$ and $d_2$ the second differential map in $\mathbb{K}(\mathbf{x})$. We have shown in Proposition 4.3 and the part (2) of Theorem 4.4 that $d_2$ is a complete matrix of grade $g$. Let $F$ be the free $R$-module with the ordered basis $\{e_1 < e_2 < \cdots < e_g\}$. Then $\wedge^2 F$ is a free $R$-module with the ordered basis $\{e_1 \wedge e_2 < e_1 \wedge e_3 < \cdots < e_{g-1} \wedge e_g\}$. Let $t = \binom{g}{2}$ be an integer. Let

$$\mathbb{F} : 0 \longrightarrow F_g \xrightarrow{\varphi_g} F_{g-1} \xrightarrow{\varphi_{g-1}} \cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} R$$

be the minimal free resolution of $R/(\mathbf{x})$ such that $F_1$ and $F_2$ are free $R$-modules with the ordered bases $\{v_1 < v_2 < \cdots < v_g\}$ and $\{w_1 < w_2 < \cdots < w_t\}$, respectively. Then we have a commutative diagram

$$\begin{array}{ccc} \wedge^2 F & \xrightarrow{d_2} & \wedge^1 F \\ \big\downarrow \psi_2 & \circlearrowleft & \big\downarrow \psi_1 \\ F_2 & \xrightarrow{\varphi_2} & F_1 \end{array}$$

where $\psi_1$ and $\psi_2$ are order preserving isomorphisms as free $R$-modules. Since $\psi_1(e_k) = v_k$ for $k = 1, 2, \ldots, g$ and $\psi_2$ maps the $i$th basis element in $\wedge^2 F$ to the $i$th basis element $w_i$ in $F_2$ for $i = 1, 2, \ldots, t$, the commutativity implies that $d_2$ and $\varphi_2$ have the same matrix representation. Thus $\varphi_2$ is a complete matrix of grade $g$ since $d_2$ is a complete matrix of grade $g$. □

Now we can describe a structure theorem for complete intersections of grade $g > 3$.

**Theorem 4.10.** *Let $R$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$.*

(1) *Let $g > 3$ be an integer and $t = \binom{g}{2}$. Let $F$ and $G$ be free $R$-modules with rank $F = g$ and rank $G = t$. Let $f = (f_{ij}) : G \to F$ be a complete matrix of*

*grade $g$ whose image is contained in $\mathfrak{m}F$. With the notation as in Theorem 4.7, we assume that $\mathcal{K}_{g-2}(\bar{T}_i)+\mathcal{K}_{g-2}(\bar{T}_j)$ has grade $g$ for some $i, j (1 \le i \neq j \le g)$. Then the ideal $\mathcal{K}_{g-1}(f)$ is a complete intersection of grade $g$.*

(2) *Let $I=(x_1, x_2, \ldots, x_g)$ be a complete intersection of grade $g$ and let* (4.6)

$$\mathbb{F}: 0 \longrightarrow R \xrightarrow{\varphi_g} F_{g-1} \xrightarrow{\varphi_{g-1}} F_{g-2} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} R$$

*be the minimal free resolution of $R/I$. Then $\varphi_2$ and the transpose of $\varphi_{g-1}$ satisfy* (1).

*Proof.* (1) We showed in Theorem 3.7 that the first part of the theorem is true for the case of $g=4$. Let $f=(f_{ij})$ be a $g \times t$ complete matrix of grade $g$. As shown in Proposition 4.3, interchanging columns of $f$ transforms $f$ to the form of (4.3). So we may assume that $f$ has the form of (4.3). Then we have

$$\mathcal{K}_{g-1}(f)=(f_{21}, f_{11}, f_{12}, \ldots, f_{1\,g-1}). \tag{4.7}$$

Since $\mathcal{K}_{g-2}(\bar{T}_i)+\mathcal{K}_{g-2}(\bar{T}_j)$ has grade $g$ for some $i, j (i \neq j)$, it follows from (4.4) and (4.5) that $\mathcal{K}_{g-1}(f)$ is a complete intersection of grade $g$. Let $\mathbf{x}=f_{21}, f_{11}, f_{12}, \ldots, f_{1\,g-1}$. Then $\mathbf{y}_1=\widehat{f_{21}}, f_{11}, f_{12}, \ldots, f_{1\,g-1}$ and each $\mathbf{y}_i=f_{21}, f_{11}, f_{12}, \ldots, \widehat{f_{1\,i-1}}, \ldots, f_{1\,g-1}$ for $i>1$ are regular sequences. From (4.4), $f_{21}$ is regular on $R/\mathcal{K}_{g-2}(\bar{T}_1)$, and $f_{1i-1}$ is regular on $R/\mathcal{K}_{g-2}(\bar{T}_i)$ for $i>1$. Let $\mathbb{G}_i$ be a complex of free R-modules such that

$$\mathbb{G}_1: 0 \longrightarrow R \xrightarrow{f_{21}} R\,,$$

and for $i>1$,

$$\mathbb{G}_i: 0 \longrightarrow R \xrightarrow{f_{1i-1}} R\,.$$

Then by the part (1) of Lemma 4.1, $\mathbb{G}_i \otimes \mathbb{K}(\mathbf{y}_i)$ is a minimal free resolution of $R/\mathcal{K}_{g-1}(f)$.

(2) We showed in Theorem 3.7 that the part (2) holds for the case of $g=4$. Let $I=(x_1, x_2, \ldots, x_g)$ be a complete intersection of grade $g$ and $I'=(x_2, x_3, \ldots, x_g)$ be a complete intersection of grade $g-1$. The same argument as in the proof of the part (2) of Theorem 3.7 says that $\varphi_2$ in (4.6) is of the form

$$\varphi_2=\left[\begin{array}{c|c} \tilde{\varphi}_1 & 0 \\ \hline \tilde{d} & \tilde{\varphi}_2 \end{array}\right],$$

where

$$\tilde{\varphi}_1=\begin{bmatrix} -x_2 & -x_3 & \cdots & -x_g \end{bmatrix}, \quad \tilde{d}=\operatorname{diag}(x_1, x_1, \ldots, x_1),$$

and $\tilde{\varphi}_2$ is the second differential map of the minimal free resolution of $R/I'$. Lemma 4.9 says that $\tilde{\varphi}_2$ is a complete matrix of grade $g-1$. Since $x_1, x_2, \ldots, x_g$ is a regular sequence on $R$, Lemma 4.9 implies that $\varphi_2$ is a complete matrix of grade $g$. We observe that every row of $\varphi_2$ consists of $g-1$ nonzero entries and

$t-g+1$ zero entries. The similar argument as in the proof of Proposition 4.3 gives us the following : Let $(\bar{S}_i, \bar{T}_i)$ be a pair of a $(g-1)\times(g-1)$ diagonal matrix and a $(g-1)\times(t-g+1)$ complete matrix of grade $g-1$. Then for $i = 1, 2, \dots, g$,

$$\det \bar{S}_i = \pm x_i^{g-1}, \;\; \mathcal{K}_{g-2}(\bar{T}_i) = (x_1, x_2, \dots, \widehat{x_i}, x_{i+1}, \dots, x_g).$$

So we have

$$\mathcal{K}_{g-1}(\varphi_2) = (x_1, x_2, \dots, x_g), \text{ and } \mathcal{K}_{g-2}(\bar{T}_i) + \mathcal{K}_{g-2}(\bar{T}_j) = \mathcal{K}_{g-1}(\varphi_2)$$
$$\text{for some } i \neq j.$$

We know that each $\mathcal{K}_{g-2}(\bar{T}_i)$ has grade $g-1$, and $\mathcal{K}_{g-1}(\varphi_2)$ is a complete intersection of grade $g$. Hence $\varphi_2$ satisfies the part (1) of Theorem 4.10. Since every complete intersection is Gorenstein, $\mathbb{F} \cong \mathbb{F}^*$ as complexes. So $\mathbb{F}^*$ is the minimal free resolution of $R/I$. The same argument as in the proof of the part (2) of Theorem 3.7 for $\mathbb{K}(\mathbf{x})$ and $\mathbb{F}^*$ gives us the proof that the transpose of $\varphi_{g-1}$ is a complete matrix of grade $g$. □

It should be noticed that Theorem 3.7 is just the special case of $g = 4$ in Theorem 4.10. The following example illustrates how Theorem 4.10 works.

**Example 4.11.** Let $\mathbb{C}$ be the field of the complex numbers and $R$ the formal power series ring $\mathbb{C}[[x_{ij}, y, z, w, u | 1 \le i, j \le 3]]$ over $\mathbb{C}$ with indeterminates $x_{ij}, y, z, w, u$. Consider a $3\times 3$ matrix $X$ and a $3\times 3$ alternating matrix $Y$

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & w & z \\ -w & 0 & y \\ -z & -y & 0 \end{bmatrix}.$$

Define

$$Z_1 = \sum_{i=1}^{3} Y_i x_{i1}, \quad Z_2 = \sum_{i=1}^{3} Y_i x_{i2}, \quad Z_3 = \sum_{i=1}^{3} Y_i x_{i3}, \quad v = \det X.$$

Then $I = (Z_1, Z_2, Z_3, v)$ is an almost complete intersection of grade 3 of type 3 [1, 5]. Assume that $\mathbf{z} = Z_1, Z_2, Z_3$ is a regular sequence on $R$. Then

$$J = (\mathbf{z}) : I = (Y_1, Y_2, Y_3) = (y, z, w).$$

Since $v$ is not contained in the ideal $J$, $I \cap J = (\mathbf{z})$. Hence $I$ is geometrically linked to $J$ by a regular sequence $\mathbf{z}$. Thus by Theorem 2.3, $K = I + J = (y, z, w, v)$ is a complete intersection of grade 4. So $\mathbf{x} = y, z, w, v$ is a regular sequence on $R$. We may assume that $u$ is a regular element on $R/K$. Thus $H = (y, z, w, v, u)$ is a complete intersection of grade 5. Let

$$\mathbb{K}(u) : 0 \longrightarrow R \xrightarrow{\;u\;} R$$

be a complex of free $R$-modules and $R$-maps. Then $\mathbb{H} = \mathbb{K}(u) \otimes \mathbb{K}(\mathbf{x})$ described as in the part (2) of Lemma 4.1 is the minimal free resolution of $R/H$. Let $\phi_2$ be the second differential map in $\mathbb{H}$. Since $y, z, w, v, u$ is a regular sequence on $R$, by Lemma 4.9, $\phi_2$ is a complete matrix of grade 5. It is easy to show that $\mathcal{K}_4(\phi_2) = (u, y, z, w, v)$ is a complete intersection of grade 5. Moreover, we let $\bar{T}_i$ be a $4 \times 6$ complete matrix of grade 4 with the same notation, $\bar{T}_i$ in Definition 4.5. Then we have

$$\mathcal{K}_3(\bar{T}_1) = (y, z, w, v), \quad \mathcal{K}_3(\bar{T}_2) = (u, z, w, v), \quad \mathcal{K}_3(\bar{T}_3) = (u, y, w, v),$$
$$\mathcal{K}_3(\bar{T}_4) = (u, y, z, v), \quad \mathcal{K}_3(\bar{T}_5) = (u, y, z, w).$$

Hence $\mathcal{K}_3(\bar{T}_i) + \mathcal{K}_3(\bar{T}_j) = \mathcal{K}_4(\phi_2)$ for some $i \neq j$. This illustrates the Theorem 4.10.

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