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ON THE HYERS-ULAM-RASSIAS STABILITY OF JENSEN'S EQUATION

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Abstract. J. Wang [21] proposed a problem: whether the Hyers-Ulam-Rassias stability of Jensen's equation for the case $p, q, r, s \in (\beta, \frac{1}{\beta}) \setminus \{1\}$ holds or not under the assumption that G and E are β-homogeneous F-space $(0 < \beta \leq 1)$. The main purpose of this paper is to give an answer to Wang's problem. Furthermore, we proved that the stability property of Jensen's equation is not true as long as p or q is equal to β, $\frac{1}{\beta}$, or $\frac{\beta_2}{\beta_1}$ $(0 < \beta_1, \beta_2 \leq 1)$.

1. Introduction

Let G denote a linear space and E denote a real or complex Hausdorff topological vector space. $f\colon \mathbf{G} \to \mathbf{E}$ is a mapping. We call the following equation

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \theta, \tag{1}$$

as Jensen's equation.

More than half a century ago, S. M. Ulam [20] posed the following problem:

> Give a group G, and a metric group E with the metric $d(\cdot,\cdot)$ and a positive number $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a function $f\colon G \to E$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $h\colon G \to E$ with $d(h(x), f(x)) < \varepsilon$ for all $x \in G$?

In 1941, the case of approximately additive mapping was solved by D. H. Hyers [6] for G and E being Banach spaces. Next, Th. M. Rassias [13] generalized the conclusion of Hyers' by introducing the unbounded Cauchy difference as follows:

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Theorem. *Let $f\colon G \to E$ be a mapping between Banach spaces subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p) \quad (\forall x, y \in G),$$

where ε, p are constants with $\varepsilon > 0$ and $0 \le p < 1$. Then there exists a unique additive mapping $T\colon G \to E$ such that

$$\|f(x) - T(x)\| \le \frac{2\varepsilon}{2 - 2^p}\|x\|^p \quad (\forall x \in G).$$

If, in addition $f(tx)$ is continuous in t for each fixed $x \in G$, then T is linear.

The proof given in [13] also works when $p < 0$. In 1991, Z. Gajda [3] following the spirit of the proof of Th. M. Rassias's Theorem for the unbounded Cauchy difference by replacing n by $-n$ solved Th. M. Rassias's question by proving the stability theorem for all real values of p that are strictly greater than one. And in this paper, Z. Gajda found firstly that the stability problem does not hold when $p = 1$.

The remarkable generalization of Th. M. Rassias for D. H. Hyer's Theorem promoted greatly the development of the stability problems of functional equations. It stimulated a number of mathematicians to study the stability problems of various functional equations. For more detailed information of such a field one can refer to [4], [14], [15], and [16].

In this paper, we deal with the stability of the Jensen's functional Eq.(1).

The first result on the stability of Jensen's equation was carried out by Z. Kominek [9]. New generalizations of Jensen's functional equation were given by Th. M. Rassias [12]. In 1998, S.-M. Jung [8] gave an important generalization of the Z. Kominek's result. In fact, he proved the following theorem:

Theorem. *Let E_1 be a real normed space and let E_2 be a real Banach space. Assume that $\delta, \theta \ge 0$ are fixed, and let $p > 0$ be given with $p \ne 1$. Suppose a mapping $f\colon E_1 \to E_2$ satisfied the functional inequality*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \le \delta + \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$. Furthermore, assume $f(0) = 0$ and $\delta = 0$ in above inequality for the case of $p > 1$. Then there exists a unique additive function $T\colon E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \le \delta + \|f(0)\| + \frac{\theta}{2^{1-p} - 1}\|x\|^p \quad (\textit{for} \ \ p < 1)$$

or

$$\|f(x) - T(x)\| \le \frac{2^{p-1}\theta}{2^{p-1} - 1}\|x\|^p \quad (\textit{for} \ \ p > 1)$$

for all $x \in E_1$.

Later, many results concerning the stability of Jensen's equation were obtained by numerous authors, such as [11], [19], and [10]. J. Wang [22], [24] attempted to weaken the condition of the space. She proved a generalized conclusion of S.-M. Jung. In the following, we introduce Wang's result [24]:

Corollary I. *Let G be an F^*-space and E be an F-space with the property that there exists $0 < \beta \le 1$ such that $\|\frac{u}{3}\| \le \frac{\|u\|}{3^\beta}$ for all $u \in E$. If $\phi(x,y) = \delta + \varepsilon_1\|x\|^p + \varepsilon_2\|y\|^q (\delta, \varepsilon_1, \varepsilon_2 \ge 0, p, q < \beta)$, then there exists a unique additive mapping $T\colon G \to E$ such that*

$$\begin{aligned} &\|T(x) - f(x) + f(\theta)\| \\ \le\ & \frac{2\delta}{3^\beta - 1} + \frac{2\varepsilon_1}{3^\beta - 3^p}\|x\|^p + \frac{(1+3^q)\varepsilon_2}{3^\beta - 3^q}\|x\|^q \end{aligned}$$

for any $x \in G$. If there exists at least one of p, q such that it is strictly less than 0, then the domain of T is $G \setminus \{\theta\}$ instead of G.

Corollary II. *Let G be an F^*-space with the property that there exists $0 < \beta \le 1$ such that $\|\frac{x}{3}\| \le \frac{\|x\|}{3^\beta}$ for all $x \in G$, and E be an F-space. Assume that $f(\theta) = \theta$. If $\phi(x,y) = \delta + \varepsilon_1\|x\|^p + \varepsilon_2\|y\|^q (\varepsilon_1, \varepsilon_2 \ge 0, p, q > \frac{1}{\beta})$, then there exists a unique additive mapping $T\colon G \to E$ such that*

$$\begin{aligned} &\|T(x) - f(x)\| \\ \le\ & \frac{2\varepsilon_1}{3^{p\beta} - 3}\|x\|^p + \frac{(1+3^{q\beta})\varepsilon_2}{3^{q\beta} - 3}\|x\|^q \end{aligned}$$

for any $x \in G$.

In above corollaries, $\phi(x,y) = \frac{1}{2}f(\frac{x+y}{2}) - f(x) - f(y)$.

J. Wang noticed that these results hold for p, $q < \beta$ or p, $q > \frac{1}{\beta}$. She raised the following question: What does it hold if p, q satisfy $\beta < p$, $q < \frac{1}{\beta}$ under the assumption that G and E are β-homogeneous F-spaces $(0<\beta\le 1)$? In Section 2 of the present paper, by still using the ideas from the papers of Hyers [6], Rassias [13], Rassias and Šemrl [16], we provide the stability of Eq.(1) for $\beta_2 < p$, $q < \frac{1}{\beta_1}$ $(p, q \ne \frac{\beta_2}{\beta_1})$ in β-homogeneous F-space. In Section 3, we show that the stability of Jensen's equation is not satisfied as long as p or q equals β_2, $\frac{1}{\beta_1}$ or $\frac{\beta_2}{\beta_1}(0 < \beta_1, \beta_2 \le 1)$.

2. Stability of Eq.(1) for $\beta_2 < p$, $q < \frac{1}{\beta_1}$ $(p, q \ne \frac{\beta_2}{\beta_1})$

From now on, we let $\mathbb{N}$ denote the set of positive integers set and $\mathbb{R}$ denotes the set of real numbers set, respectively. Meanwhile, We assume p, q to be different real numbers.

Firstly, we introduce the definition of F-space and β-homogeneous (see [18]).

Let X be a linear space. A non-negative valued function $\|\cdot\|$ defined on X is called an F-norm if it obeys the following rules:

(n1) $\|x\| = 0$ if and only if $x = 0$;

(n2) $\|ax\| = \|x\|$ for all a, $|a| = 1$;

(n3) $\|x + y\| \le \|x\| + \|y\|$;

(n4) $\|a_n x\| \longrightarrow 0$ provided $a_n \longrightarrow 0$;
(n5) $\|a x_n\| \longrightarrow 0$ provided $x_n \longrightarrow 0$.

A space X with an F-norm is called an F^*-space. An F-pseudonorm ($\|x\| = 0$ does not necessarily imply that $x = 0$ in (n1)) is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{R}$. A complete F^*-space is said to be an F-space.

Theorem 2.1. *Let G and E be a β_1-homogeneous F^*-space and a β_2-homogeneous F-space, respectively. Suppose that $f\colon G \to E$ is a mapping with the property that*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \le \varepsilon_1 \|x\|^p + \varepsilon_2 \|y\|^q, \tag{2}$$

where $\beta_1,\ \beta_2 \in (0,\ 1],\ \varepsilon_1, \varepsilon_2 \in (0, \infty)$ and $p,\ q \in (\beta_2, \frac{1}{\beta_1}) \setminus \{\frac{\beta_2}{\beta_1}\}$. Then there exists a unique additive mapping $T\colon G \to E$ such that

$$\|T(x) - f(x)\| \le \frac{2\varepsilon_1}{3^{\beta_2} - 3^{\beta_1 p}} \|x\|^p + \frac{(1 + 3^{\beta_1 q})\varepsilon_2}{3^{\beta_2} - 3^{\beta_1 q}} \|x\|^q, \tag{3}$$

in the case $\beta_2 < p,\ q < \frac{\beta_2}{\beta_1}$, while in the case $\frac{\beta_2}{\beta_1} < p,\ q < \frac{1}{\beta_1}$

$$\|T(x) - f(x)\| \le \frac{2\varepsilon_1}{3^{\beta_1 p} - 3^{\beta_2}} \|x\|^p + \frac{(1 + 3^{\beta_1 q})\varepsilon_2}{3^{\beta_1 q} - 3^{\beta_2}} \|x\|^q. \tag{4}$$

Moreover, if for each fixed $x \in G$, there exists a real numbers $\delta_x > 0$, such that $f(tx)$ is continuous on $[0, \delta_x]$, then $T(x)$ is linear.

Proof. Let $g(x) = f(x) - f(\theta)$. Then g also satisfies (2). From this, we can assume that $f(\theta) = \theta$ without loss of generality.

When $\beta_2 < p,\ q < \frac{\beta_2}{\beta_1}$, we claim that

$$\begin{aligned} &\|3^{-n} f(3^n) - f(x)\| \\ &\le \sum_{k=1}^{n} 3^{k(\beta_1 p - \beta_2)} \cdot 2 \cdot 3^{-\beta_1 p} \varepsilon_1 \|x\|^p \\ &\quad + \sum_{k=1}^{n} (3^{k(\beta_1 q - \beta_2)} \cdot 3^{-\beta_1 q} + 3^{k(\beta_1 q - \beta_2)}) \varepsilon_2 \|x\|^q \end{aligned} \tag{5}$$

holds for any integer n. The verification of (5) follows by induction on n. Indeed, for $n = 1$, we set $y = -x$, then

$$\| - f(x) - f(-x)\| \le \varepsilon_1 \|x\|^p + \varepsilon_2 \|x\|^q.$$

Replacing x by $-x$ and y by $3x$, (5) implies

$$\|2f(x) - f(-x) - f(3x)\| \le \varepsilon_1 \|x\|^p + \varepsilon_2 \cdot 3^{\beta_1 q} \|x\|^q.$$

Taking the two inequality into account, then

$$\begin{aligned}\|3^{-1}f(3x)-f(x)\| &= \|3^{-1}[f(3x)+f(-x)-2f(x)-f(-x)-f(x)]\|\\ &\le 3^{-\beta_2}[\|f(3x)+f(-x)-2f(x)\|+\|-f(-x)-f(x)\|]\\ &\le 2\cdot 3^{-\beta_2}\cdot\varepsilon_1\|x\|^p+(1+3^{\beta_1 q})\cdot 3^{-\beta_2}\varepsilon_2\|x\|^q.\end{aligned}$$

Assume that the formula is true for $n=m$, we want to examine the case when $n=m+1$. We have

$$\begin{aligned}&\|3^{-(m+1)}f(3^{(m+1)}x)-f(x)\|\\ &=\|3^{-1}[3^{-m}f(3^m(3x))-f(3x)]+3^{-1}f(3x)-f(x)\|\\ &\le 3^{-\beta_2}\left[\sum_{k=1}^{m}3^{k(\beta_1 p-\beta_2)}\cdot 2\cdot 3^{-\beta_1 p}\varepsilon_1\|3x\|^p+\sum_{k=1}^{m}(3^{k(\beta_1 q-\beta_2)}\cdot 3^{-\beta_1 q}+3^{k(\beta_1 q-\beta_2)})\varepsilon_2\|3x\|^q\right]\\ &\quad+2\cdot 3^{-\beta_2}\cdot\varepsilon_1\|x\|^p+(1+3^{\beta_1 q})\cdot 3^{-\beta_2}\varepsilon_2\|x\|^q\\ &=\sum_{k=1}^{m+1}3^{k(\beta_1 p-\beta_2)}\cdot 2\cdot 3^{-\beta_1 p}\varepsilon_1\|x\|^p+\sum_{k=1}^{m+1}(3^{k(\beta_1 q-\beta_2)}\cdot 3^{-\beta_1 q}+3^{k(\beta_1 q-\beta_2)})\varepsilon_2\|x\|^q.\end{aligned}$$

Therefore (5) is proved.

Let

$$T(x)=\lim_{n\to\infty}\frac{f(3^n x)}{3^n}. \tag{6}$$

It is easy to see that T exists. In fact,

$$\begin{aligned}&\left\|\frac{f(3^m x)}{3^m}-\frac{f(3^n x)}{3^n}\right\|\\ &=\left\|\frac{1}{3^n}\left[\frac{f(3^{m-n}(3^n x))}{3^{m-n}}-f(3^n x)\right]\right\|\\ &\le\frac{1}{3^{n\beta_2}}\left[\sum_{k=1}^{m-n}3^{k(\beta_1 p-\beta_2)}2\cdot 3^{-\beta_1 p}\varepsilon_1\|3^n x\|^p+\sum_{k=1}^{m-n}(3^{k(\beta_1 q-\beta_2)}\cdot 3^{-\beta_1 q}+3^{k(\beta_1 q-\beta_2)})\varepsilon_2\|3^n x\|^q\right]\\ &\le\frac{1}{3^{n(\beta_2-\beta_1 p)}}\left[\frac{2\varepsilon_1}{3^{\beta_2}-3^{\beta_1 p}}\|x\|^p+\frac{(1+3^{\beta_1 q})\varepsilon_2}{3^{\beta_2}-3^{\beta_1 q}}\|x\|^q\right]\end{aligned}$$

for any $m>n,\ m,n\in\mathbb{N}$. By virtue of $\beta_2-\beta_1 p>0$, it follows that

$$\lim_{n\to\infty}\left\|\frac{f(3^m x)}{3^m}-\frac{f(3^n x)}{3^n}\right\|=0.$$

Thus $\{\frac{f(3^n x)}{3^n}\}$ is a Cauchy sequence. However the F-space is complete, thus $\{\frac{f(3^n x)}{3^n}\}$ converges. It follows that $T(x)=\lim_{n\to\infty}\frac{f(3^n x)}{3^n}$ exists. Hence by letting $n\to\infty$ in (5), one obtains

$$\|T(x)-f(x)\|\le\frac{2\varepsilon_1}{3^{\beta_2}-3^{\beta_1 p}}\|x\|^p+\frac{(1+3^{\beta_1 q})\varepsilon_2}{3^{\beta_2}-3^{\beta_1 q}}\|x\|^q.$$

Now we shall deal with the additivity of T. On account of (6), one has

$$T(3^m x) = \lim_{n\to\infty} \frac{f(3^n(3^m x))}{3^{n+m}} \cdot 3^m = 3^m T(x). \tag{7}$$

And employing the condition (2), we set

$$\begin{aligned} &\left\| 2T(\frac{x+y}{2}) - T(x) - T(y) \right\| \\ =\ & \lim_{n\to\infty} \left\| 2\cdot\frac{1}{3^n} f\left(\frac{3^n x + 3^n y}{2}\right) - \frac{1}{3^n} f(3^n x) - \frac{1}{3^n} f(3^n y) \right\| \\ \le\ & \lim_{n\to\infty} \left(\frac{\varepsilon_1}{3^{n(\beta_2-\beta_1 p)}} \|x\|^p + \frac{\varepsilon_2}{3^{n(\beta_2-\beta_1 q)}} \|y\|^q \right) \\ =\ & 0. \end{aligned} \tag{8}$$

By (3), (6), and (7), it follows

$$\begin{aligned} &\|2T(2x) - 4T(x)\| = \|2T(2x) - T(3x) - T(x)\| \\ =\ & \left\| 3^{-n} \left[2T(3^n\cdot 2x) - T(3^n \cdot 3x) - T(3^n x) \right] \right\| \\ \le\ & 3^{-n\beta_2} \left(\|2T(3^n\cdot 2x) - 2f(3^n\cdot 2x)\| + \|T(3^n\cdot 3x) - f(3^n\cdot 3x)\| \right) \\ & + 3^{-n\beta_2} \left(\|T(3^n x) - f(3^n x)\| + \left\| 2f\left(\frac{3^n(3x+x)}{2}\right) - f(3^n\cdot 3x) - f(3^n x) \right\| \right) \\ \le\ & \frac{2\varepsilon_1 \|x\|^p}{3^{n(\beta_2-\beta_1 p)}\cdot(3^{\beta_2} - 3^{\beta_1 p})} (2^{\beta_1 p+\beta_2} + 3^{\beta_1 p} + 1) \\ & + \frac{(1+3^{\beta_1 q})\varepsilon_2 \|x\|^q}{3^{n(\beta_2-\beta_1 q)}\cdot(3^{\beta_2} - 3^{\beta_1 p})} (2^{\beta_1 q+\beta_2} + 3^{\beta_1 q} + 1) \\ & + \frac{3^{\beta_1 p}\cdot\varepsilon_1}{3^{n(\beta_2-\beta_1 p)}} \|x\|^p + \frac{\varepsilon_2}{3^{n(\beta_2-\beta_1 q)}} \|x\|^q. \end{aligned}$$

Clearly, $\|2T(2x) - 4T(x)\| \to 0$ as $n \to \infty$. Thus, $2T(2x) = 4T(x)$. From (8), we get

$$T(x+y) = \frac{1}{2}(T(2x) + T(2y)) = T(x) + T(y).$$

We will prove the uniqueness of T. Suppose that $H\colon G \to E$ is another additive mapping satisfying (3) for all $x \in G$. It follows that

$$\begin{aligned} &\|T(x) - H(x)\| = \frac{1}{n^{\beta_2}} \|T(nx) - H(nx)\| \\ =\ & \frac{1}{n^{\beta_2}} \|T(nx) - f(nx) - H(nx) + f(nx)\| \\ \le\ & \frac{1}{n^{\beta_2}} (\|T(nx) - f(nx)\| + \|H(nx) - f(nx)\|) \\ \le\ & \frac{4\varepsilon_1 \|x\|^p}{n^{(\beta_2-\beta_1 p)}\cdot(3^{\beta_2} - 3^{\beta_1 p})} + \frac{2(1+3^{\beta_1 q})\varepsilon_2 \|x\|^q}{n^{(\beta_2-\beta_1 q)}\cdot(3^{\beta_2} - 3^{\beta_1 q})}, \end{aligned}$$

and so $\|T(x) - H(x)\| \to 0$ as $n \to \infty$. Hence $T(x) = H(x)$ for all $x \in G$.

This finishes the first step of the proof.

When $\frac{\beta_2}{\beta_1} < p$, $q < \frac{1}{\beta_1}$, we claim that

$$\begin{aligned} &\|3^n f(3^{-n}) - f(x)\| \\ &\leq \sum_{k=1}^{n} 3^{k(\beta_2-\beta_1 p)} \cdot 2 \cdot 3^{-\beta_2}\varepsilon_1 \|x\|^p \\ &\quad + \sum_{k=1}^{n} (3^{k(\beta_2-\beta_1 q)} \cdot 3^{-\beta_2} + 3^{(k-1)(\beta_2-\beta_1 q)})\varepsilon_2 \|x\|^q. \end{aligned} \tag{9}$$

Note that substituting $3^{-n}x$ by x in (5) and later multiplying both sides by $3^{n\beta_2}$, we can yield the above formula (9).

Define $T(x) = \lim_{n\to\infty} 3^n f(3^{-n}x)$. The rest of the proofs follows as that in the case of $\beta_2 < p$, $q < \frac{\beta_2}{\beta_1}$, and therefore we omit it.

Consequently, we obtain

$$\|T(x) - f(x)\| \leq \frac{2\varepsilon_1}{3^{\beta_1 p} - 3^{\beta_2}} \|x\|^p + \frac{(1 + 3^{\beta_1 q})\varepsilon_2}{3^{\beta_1 q} - 3^{\beta_2}} \|x\|^q.$$

Moreover, if for each fixed $x \in G$, there exists a real number $\delta_x > 0$, such that $f(tx)$ is continuous on $[0, \delta_x]$, we claim that $f(tx)$ is bounded on $[0, \delta_x]$. Otherwise, if this were not the case then for any $n \in \mathbb{N}$, there exists $t_n \in [0, \delta_x]$ such that $\|f(t_n x)\| \geq n$. For the bounded sequence $\{t_n\}$, we could apply the Bolzano-Weierstass theorem to find a convergent subsequence $\{t_{n_k}\}$ and $t_0 \in [0, \delta_x]$ such that $\lim_{k\to\infty} t_{n_k} = t_0$. It follows that $\lim_{k\to\infty} t_{n_k} x = t_0 x$ for each fixed $x \in G$. Since $f(tx)$ is continuous in t_0, we can conclude that $\lim_{k\to\infty} f(t_{n_k} x) = f(t_0 x)$. Thus, we get a contradiction to $\lim_{k\to\infty} \|f(t_{n_k})\| = \infty$. The remaining proof follows a similar argument as in the proof of [16], hence we obtain that $T(x)$ is linear. Thus, claim is given. □

Remark 1. Let G and E be a β_1-homogeneous F^*-space and a β_2-homogeneous F-space, respectively. Suppose that $f\colon G \to E$ satisfies

$$\left\| 2f(\frac{x+y}{2}) - f(x) - f(y) \right\| \leq \delta.$$

Then there exists a unique additive mapping $T\colon G \to E$ such that

$$\|T(x) - f(x)\| \leq \frac{2\delta}{3^{\beta_2} - 1}$$

for all $x \in G$.

Now we construct an F-norm satisfying the condition that there exists $0 < \beta < 1$ such that $\|\frac{x}{3}\| \leq \frac{\|x\|}{3^\beta}$ but not β-homogeneity. So, the condition of spaces G and E in theorem can be weakened.

Example 1. We define the non-negative function $\|\cdot\|$ in $\mathbb{R}$ by

$$\|x\| = \begin{cases} |x|^\beta & |x| \le 1 \\ |x| & |x| > 1 \end{cases} \qquad (\forall x \in \mathbb{R}).$$

Then $\|\cdot\|$ is an F-norm with the property that $\|\frac{x}{n}\| \le \frac{\|x\|}{n^\beta}$ $(n \in \mathbb{N})$, but not the β-homogeneity.

Proof. We have only to show that $\|\cdot\|$ satisfies the triangle inequality. To establish one, we shall consider three cases. In the case where $|x| > 1, |y| > 1$, one has

$$\|x+y\| = |x+y| \le |x| + |y| = \|x\| + \|y\|.$$

In the case where $|x| < 1, |y| < 1$, and likewise $|x+y| \le 1$,

$$\|x+y\| = |x+y|^\beta \le (|x|+|y|)^\beta \le |x|^\beta + |y|^\beta = \|x\| + \|y\|,$$

or $|x| < 1, |y| < 1$ and likewise $|x+y| > 1$, and therefore

$$\|x+y\| = |x+y| \le |x| + |y| \le |x|^\beta + |y|^\beta = \|x\| + \|y\|.$$

While in the case where $|x| > 1, |y| < 1$ or $|x| < 1, |y| > 1$, we might as well suppose that $|x| > 1, |y| < 1$. Then if $|x+y| \le 1$ holds, we obtain

$$\|x+y\| = |x+y|^\beta \le |x|^\beta + |y|^\beta \le |x| + |y|^\beta = \|x\| + \|y\|.$$

However, if $|x+y| > 1$ then,

$$\|x+y\| = |x+y| \le |x| + |y| \le |x| + |y|^\beta = \|x\| + \|y\|.$$

Therefore $\|\cdot\|$ is an F-norm.

Now we will prove that $\|\frac{x}{n}\| \le \frac{\|x\|}{n^\beta}$ for any $n \in \mathbb{N}$. Indeed, when $|x| \le n$, then

$$\left\|\frac{x}{n}\right\| = \left|\frac{x}{n}\right|^\beta = \frac{1}{n^\beta}|x|^\beta = \frac{1}{n^\beta}\|x\|^\beta$$

and also when $|x| > n$, one has

$$\left\|\frac{x}{n}\right\| = \frac{|x|}{n} \le \frac{|x|}{n^\beta} = \frac{\|x\|}{n^\beta}.$$

It follows that $\|\frac{x}{n}\| \le \frac{\|x\|}{n^\beta}$ for any $x \in \mathbb{R}$.

It is easy to see that the $\|\cdot\|$ is not β-homogeneous.

Therefore the proof is completed. □

3. Instability of Eq.(1)

We will first cite the counterexample constructed by Z. Gajda [3].

Example 2. For a fixed $\varepsilon > 0$ and $\mu = \frac{\varepsilon}{6}$, define a function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n} \quad x \in \mathbb{R},$$

where the function $\phi\colon \mathbb{R} \to \mathbb{R}$ is given by

$$\phi(x) = \begin{cases} \mu & x \leq 1, \\ \mu x & -1 < x < 1, \\ -\mu & x \leq -1. \end{cases}$$

Theorem 3.1. *The function f defined above satisfies*

$$|f(x+y) - f(x) - f(y)| \leq \varepsilon(|x| + |y|^{\frac{1}{2}}) \tag{10}$$

for all $x, y \in \mathbb{R}$. However

$$\sup\left\{\frac{|f(x) - T(x)|}{|x|} : \ x \in \mathbb{R}\backslash\{0\}\right\} = \infty$$

for each additive mapping $T\colon \mathbb{R} \to \mathbb{R}$.

Proof. The inequality (10) is trivially fulfilled if $x = y = 0$.

Now, we assume that $|x| + |y|^{\frac{1}{2}} < 1$. Then $|x| < 1, |y|^{\frac{1}{2}} < 1$. There exists an $N \in \mathbb{N}$ such that

$$2^{N-1}(|x| + |y|^{\frac{1}{2}}) < 1, \quad 2^{N}(|x| + |y|^{\frac{1}{2}}) \geq 1.$$

Since $|x|+|y| \leq |x|+|y|^{\frac{1}{2}}$, we get $2^{N-1}(|x|+|y|) \leq 2^{N-1}(|x|+|y|^{\frac{1}{2}}) < 1$. Hence,

$$|2^{N-1}(x+y)| \leq 2^{N-1}(|x| + |y|) < 1 \quad \text{and} \quad |2^{N-1}x| < 1, \quad |2^{N-1}y| < 1,$$

which means that for each $n \in \{0, 1, 2, \ldots, N-1\}$, $2^{n-1}x, 2^{n-1}y, 2^{n-1}(x+y) \in (-1, 1)$. Since ϕ is a linear mapping on the interval, we infer that

$$\phi(2^n(x+y)) = \phi(2^n x) + \phi(2^n y)$$

for $n = 0, 1, \ldots, N-1$. As a result, we obtain

$$\begin{aligned} \frac{|f(x+y) - f(x) - f(y)|}{|x| + |y|^{\frac{1}{2}}} &\leq \sum_{n=0}^{\infty} \frac{|\phi(2^n(x+y)) - \phi(2^n x) - \phi(2^n y)|}{2^n(|x| + |y|^{\frac{1}{2}})} \\ &= \sum_{n=N}^{\infty} \frac{|\phi(2^n(x+y)) - \phi(2^n x) - \phi(2^n y)|}{2^n(|x| + |y|^{\frac{1}{2}})} \\ &\leq \sum_{k=0}^{\infty} \frac{3\mu}{2^k \cdot 2^N(|x| + |y|^{\frac{1}{2}})} \leq \sum_{k=0}^{\infty} \frac{3\mu}{2^k} = 6\mu. \end{aligned}$$

Finally, assume that $|x| + |y|^{\frac{1}{2}} \geq 1$. Then because of the boundedness of f, we have

$$\frac{|f(x+y) - f(x) - f(y)|}{|x| + |y|^{\frac{1}{2}}} \leq 6\mu = \varepsilon,$$

since

$$|f(x)| \leq \sum_{n=0}^{\infty} = 2\mu, \quad x \in \mathbb{R}.$$

Thus, we conclude that f satisfies (10) for all $x, y \in \mathbb{R}$. The proof of the last assertion in the theorem follows the same argument as in [3]. □

Remark 2. Let the function f be as before.

(i) If $G = (\mathbb{R}, \|\cdot\|_1)$ with the Euclidean metric $\|\cdot\|_1 = |\cdot|$ and $E = (\mathbb{R}, \|\cdot\|_2)$ with the β-homogeneous norm $\|\cdot\|_2 = |\cdot|^\beta$, then

$$\|f(x+y) - f(x) - f(y)\|_2 \leq \varepsilon^\beta(\|x\|_1^\beta + \|y\|_1^{\frac{\beta}{2}})$$

for any x, $y \in \mathbb{R}$, however

$$\sup\left\{\frac{\|f(x) - T(x)\|_2}{\|x\|_1^\beta} :\ x \in \mathbb{R}\backslash\{0\}\right\} = \infty$$

for each additive mapping $T\colon G \to E$.

(ii) If $G = (\mathbb{R}, \|\cdot\|_1)$ with the β-homogeneous norm $\|\cdot\|_1 = |\cdot|^\beta$ and $E = (\mathbb{R}, \|\cdot\|_2)$ with the Euclidean metric $\|\cdot\|_2 = |\cdot|$, then

$$\|f(x+y) - f(x) - f(y)\|_2 \leq \varepsilon(\|x\|_1^{\frac{1}{\beta}} + \|y\|_1^{\frac{1}{2\beta}})$$

for any x, $y \in \mathbb{R}$, however

$$\sup\left\{\frac{\|f(x) - T(x)\|_2}{\|x\|_1^{\frac{1}{\beta}}} :\ x \in \mathbb{R}\backslash\{0\}\right\} = \infty$$

for each additive mapping $T\colon G \to E$.

(iii) If $G = (\mathbb{R}, \|\cdot\|_1)$ with the β_1-homogeneous norm $\|\cdot\|_1 = |\cdot|_1^\beta$ and $E = (\mathbb{R}, \|\cdot\|_2)$ with the β_2-homogeneous norm $\|\cdot\|_2 = |\cdot|_2^\beta$, then

$$\|f(x+y) - f(x) - f(y)\|_2 \leq \varepsilon^{\beta_2}(\|x\|_1^{\frac{\beta_2}{\beta_1}} + \|y\|_1^{\frac{\beta_2}{2\beta_1}})$$

for any x, $y \in \mathbb{R}$, however

$$\sup\left\{\frac{\|f(x) - T(x)\|_2}{\|x\|_1^{\frac{\beta_2}{\beta_1}}} :\ x \in \mathbb{R}\backslash\{0\}\right\} = \infty$$

for each additive mapping $T\colon G \to E$.

Remark 3. Set $\mu = \frac{\varepsilon}{8}$. By using a similar proof as in the Theorem 3.1 for Jensen's equation, we can also get

$$\left|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right| \leq \varepsilon(|x| + |y|^{\frac{1}{2}}),$$

however

$$\sup\left\{\frac{|f(x) - T(x)|}{|x|} :\ x \in \mathbb{R}\backslash\{0\}\right\} = \infty$$

for each additive mapping $T\colon \mathbb{R} \to \mathbb{R}$.

Thus, we can obtain a conclusion similar to remark 2 relating to Jensen's equation. This leads to the fact that the stability of Jensen's equation does not hold as long as one of the numbers p, q equals β, $\frac{1}{\beta}$ or $\frac{\beta_2}{\beta_1}$ $(0 < \beta_1, \beta_2 \leq 1)$.

In summary, under the condition that G and E are F-spaces with certain property, one is interested to prove that the Hyers-Ulam-Rassias stability is fulfilled in three cases: $(\triangle_1)$ p, $q < \beta_2$ (see [24]), $(\triangle_2)$ p, $q > \frac{1}{\beta_1}$ (see [24]) and $(\triangle_3)$ $\beta_2 < p$, $q < \frac{1}{\beta_1}$ $(p, q \neq \frac{\beta_2}{\beta_1})$, but this fails as long as p or q is equal to $\beta_2, \frac{1}{\beta_1}$ or $\frac{\beta_2}{\beta_1}$ $(0 < \beta_1, \beta_2 \leq 1)$.

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