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ON COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF I.I.D. RANDOM VARIABLES WITH APPLICATION TO MOVING AVERAGE PROCESSES

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Abstract. Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of i.i.d. random variables with $E|Y_1| < \infty$, $\{a_{ni}, -\infty < i < \infty, n \geq 1\}$ an array of real numbers. Under some conditions on $\{a_{ni}\}$, we obtain necessary and sufficient conditions for $\sum_{n=1}^{\infty} \frac{1}{n} P(|\sum_{i=-\infty}^{\infty} a_{ni}(Y_i - EY_i)| > n\epsilon) < \infty$. We examine whether the result of Spitzer [11] holds for the moving average process, and give a partial solution.

1. Introduction

Assume that $\{Y_i, -\infty < i < \infty\}$ is a doubly infinite sequence of identically distributed random variables. Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and

$$X_n = \sum_{i=-\infty}^{\infty} a_{i+n} Y_i, \quad n \geq 1$$

be the moving average process based on the sequence $\{Y_i\}$.

Under the independence assumption of the base sequence $\{Y_i\}$, many limiting results have been obtained. For example, Ibragimov [8] established the central limit theorem, Burton and Dehling [3] obtained a large deviation, and Li et al. [9] obtained the complete convergence. Under different dependence assumptions of the base sequence $\{Y_i\}$, Zhang [12], Baek et al. [1], and Li and Zhang [10] obtained the complete convergence results.

Note that even if $\{Y_i\}$ is a sequence of independent and identically distributed (i.i.d.) random variables, the moving average process $\{X_n\}$ are dependent random variables.

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For a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables, Baum and Katz [2] proved the following well known complete convergence theorem.

Theorem 1. *Suppose that* $\{X_n, n \geq 1\}$ *is a sequence of i.i.d. random variables. Then* $EX_1 = 0$ *and* $E|X_1|^{rp} < \infty (1 \leq p < 2, r \geq 1)$ *if and only if* $\sum_{n=1}^{\infty} n^{r-2} P(|\sum_{i=1}^{n} X_i| > n^{1/p}\epsilon) < \infty$ *for all* $\epsilon > 0$.

The case $r = 2$ and $p = 1$ of the above theorem was proved by Hsu and Robbins [6] and Erdös [4]. Spitzer [11] proved the above theorem for the case $r = 1$ and $p = 1$.

Li et al. [9] generalized Hsu-Robbins-Erdös result for the moving average process based on a sequence of i.i.d. random variables $\{Y_i, -\infty < i < \infty\}$. Zhang [12] and Baek et al. [1] generalized the result of Baum and Katz [2] for the moving average process based on a sequence of dependent random variables. If we omit the insignificant condition (slowly varying function), the result of Zhang [12] can be formulated as follows:

Theorem 2. *Let* $\{Y_i, -\infty < i < \infty\}$ *be a sequence of identically distributed and* ϕ*-mixing random variables with* $\sum_{n=1}^{\infty} \phi^{1/2}(n) < \infty$. *Suppose that* $\{X_n, n \geq 1\}$ *is the moving average process based on the sequence* $\{Y_i\}$. *If* $EY_1 = 0$ *and* $E|Y_1|^{rp} < \infty$ *for some* $1 \leq p < 2$ *and* $r \geq 1$, *then*

$$\sum_{n=1}^{\infty} n^{r-2} P(|\sum_{k=1}^{n} X_k| > n^{1/p}\epsilon) < \infty \quad \textit{for all } \epsilon > 0.$$

Baek et al. [1] proved Theorem 2 for the negatively associated random variables. However, the proofs of Zhang [12] and Baek et al. [1] are mistakenly based on the fact that

$$\sum_{i=1}^{n} i^{r-1-1/p} = O(n^{r-1/p}). \tag{1}$$

Note that (1) holds only for $r - 1/p > 0$. From the conditions $1 \leq p < 2$ and $r \geq 1$, the proofs of Zhang [12] and Baek et al. [1] are valid except for the case $r = 1$ and $p = 1$. Thus it is natural to ask whether the result of Spitzer [11] holds for the moving average process.

Question. Can we generalize the result of Spitzer [11] for the moving average process? Namely, if $\{X_n, n \geq 1\}$ is the moving average process based on a sequence of i.i.d. random variables $\{Y_i, -\infty < i < \infty\}$ with $EY_1 = 0$, then $\sum_{n=1}^{\infty} \frac{1}{n} P(|\sum_{k=1}^{n} X_k| > n\epsilon) < \infty$ for all $\epsilon > 0$?

In this paper, we obtain new complete convergence results for weighted sums of i.i.d. random variables. As corollaries, we derive a partial solution to the question.

Throughout this paper, the symbol C denotes a positive constant which is not necessarily the same one in each appearance.

2. Preliminaries

The following two lemmas will be used to prove our main results. Lemma 1 is due to Etemadi [5].

Lemma 1. *If $X_1, \ldots, X_n$ are independent random variables, then for any $t > 0$*

$$\max_{1\le l\le n} P(|\sum_{i=1}^{l} X_i| > t) \ge \frac{1}{4}\sum_{i=1}^{n} P(|X_i| > 8t)\Big\{1 - P(\max_{1\le l\le n}|\sum_{i=1}^{l} X_i| > 4t)\Big\}.$$

Hu et al. [7] proved the following lemma which is a version of the famous Hoffmann-Jørgensen inequality for independent, but not necessarily symmetric, random variables.

Lemma 2. *If $X_1, \ldots, X_n$ are independent random variables, then for every integer $j \ge 1$ and $t > 0$*

$$P(|\sum_{i=1}^{n} X_i| > 6^j t) \le C_j P(\max_{1\le i\le n}|X_i| > \frac{t}{4^{j-1}}) + D_j \max_{1\le l\le n}\Big[P(|\sum_{i=1}^{l} X_i| > \frac{t}{4^j})\Big]^{2^j},$$

where C_j and D_j are positive constants depending only on j.

3. Complete convergence for weighted sums

Throughout this section, let $\{Y_i, -\infty < i < \infty\}$ be a sequence of i.i.d. random variables with $E|Y_1| < \infty$, $\{a_{ni}, -\infty < i < \infty, n \ge 1\}$ an array of real numbers. Under some conditions on $\{a_{ni}\}$, we will find necessary and sufficient conditions for (2).

$$\sum_{n=1}^{\infty} \frac{1}{n} P(|\sum_{i=-\infty}^{\infty} a_{ni}(Y_i - EY_i)| > n\epsilon) < \infty \quad \text{for all } \epsilon > 0. \tag{2}$$

Lemma 3. *Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of i.i.d. random variables with $E|Y_1| < \infty$. Let $\{a_{ni}, -\infty < i < \infty, n \ge 1\}$ be a bounded array of real numbers satisfying*

$$\sum_{i=-\infty}^{\infty} |a_{ni}| = O(n). \tag{3}$$

Then for $\delta > 0$ and $\gamma = 1/\sup_{i,n}|a_{ni}|$

$$\begin{aligned} &P\big(\sum_{i=-\infty}^{\infty} |a_{ni}Y_i| I(|Y_i| > n\gamma\delta) > n\delta/8\big) \\ &\ge \frac{1}{8}\sum_{i=-\infty}^{\infty} P(|a_{ni}Y_i| > n\delta) \text{ for all large } n. \end{aligned}$$

Proof. We first note by (3) that

$$E\big(\sum_{i=-\infty}^{\infty} |a_{ni}Y_i|I(|Y_i| > n\gamma\delta)\big) \le CnE|Y_1| < \infty.$$

Thus $\sum_{i=-\infty}^{\infty} |a_{ni}Y_i|I(|Y_i| > n\gamma\delta)$ converges a.s.

Lemma 1 implies that

$$(4) \quad \max_{k\le l\le m} P(|\sum_{i=k}^{l} a_{ni}Y_iI(|Y_i| > n\gamma\delta)| > n\delta/8)$$
$$\ge \frac{1}{4}\sum_{i=k}^{m} P(|a_{ni}Y_i|I(|Y_i| > n\gamma\delta) > n\delta)\Big\{1 - P(\max_{k\le l\le m}|\sum_{i=k}^{l} a_{ni}Y_iI(|Y_i| > n\gamma\delta)| > n\delta/2)\Big\}.$$

By Markov's inequality and (3), we get that

$$\begin{aligned} &P(\max_{k\le l\le m}|\sum_{i=k}^{l} a_{ni}Y_iI(|Y_i| > n\gamma\delta)| > n\delta/2) \\ &\le P(\sum_{i=k}^{m}|a_{ni}Y_i|I(|Y_i| > n\gamma\delta) > n\delta/2) \\ &\le \frac{2}{n\delta}\sum_{i=k}^{m}|a_{ni}|E|Y_1|I(|Y_1| > n\gamma\delta) \\ &\le CE|Y_1|I(|Y_1| > n\gamma\delta) \to 0 \end{aligned}$$

as $n \to \infty$. Hence there exists a positive integer N such that

$$P(\max_{k\le l\le m}|\sum_{i=k}^{l} a_{ni}Y_iI(|Y_i| > n\gamma\delta)| > n\delta/2) \le 1/2$$

if $n > N$. It follows by (4) that for $n > N$

$$\begin{aligned} &P(\sum_{i=k}^{m}|a_{ni}Y_i|I(|Y_i| > n\gamma\delta) > n\delta/8) \\ &\ge \max_{k\le l\le m} P(|\sum_{i=k}^{l} a_{ni}Y_iI(|Y_i| > n\gamma\delta)| > n\delta/8) \\ &\ge \frac{1}{8}\sum_{i=k}^{m} P(|a_{ni}Y_i|I(|Y_i| > n\gamma\delta) > n\delta). \end{aligned}$$

Letting $k \to -\infty$ and $m \to \infty$, we have that for $n > N$

$$\begin{aligned} P(\sum_{i=-\infty}^{\infty}|a_{ni}Y_i|I(|Y_i| > n\gamma\delta) > n\delta/8) &\ge \frac{1}{8}\sum_{i=-\infty}^{\infty} P(|a_{ni}Y_i|I(|Y_i| > n\gamma\delta) > n\delta) \\ &= \frac{1}{8}\sum_{i=-\infty}^{\infty} P(|a_{ni}Y_i| > n\delta). \end{aligned}$$

Thus the result is proved. □

Lemma 4. *Let* $\{Y_i, -\infty < i < \infty\}$ *be a sequence of i.i.d. random variables with* $E|Y_1| < \infty$. *Let* $\{a_{ni}, -\infty < i < \infty, n \geq 1\}$ *be a bounded array of real numbers satisfying* (3). *Then, for* $\delta > 0$, *the following statements hold*:

(i) $\frac{1}{n}\sum_{i=-\infty}^{\infty} |a_{ni}|E|Y_i|I(|Y_i| > n\delta) \to 0$ *as* $n \to \infty$.

(ii) $\sum_{n=1}^{\infty} \frac{1}{n}P(|\sum_{i=-\infty}^{\infty} a_{ni}(Y_iI(|Y_i| \leq n\delta) - EY_iI(|Y_i| \leq n\delta))| > n\epsilon) < \infty$ *for all* $\epsilon > 0$.

Proof. For (i), we have by (3) that

$$\frac{1}{n}\sum_{i=-\infty}^{\infty} |a_{ni}|E|Y_i|I(|Y_i| > n\delta) \leq CE|Y_1|I(|Y_1| > n\delta) \to 0$$

as $n \to \infty$. Hence (i) holds.

For (ii), we get by Markov's inequality, $|a_{ni}| = O(1)$, and (3) that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n}P(|\sum_{i=-\infty}^{\infty} a_{ni}(Y_iI(|Y_i| \leq n\delta) - EY_iI(|Y_i| \leq n\delta))| > n\epsilon)\\
&\leq \frac{1}{\epsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^3}E\Big|\sum_{i=-\infty}^{\infty} a_{ni}(Y_iI(|Y_i| \leq n\delta) - EY_iI(|Y_i| \leq n\delta))\Big|^2\\
&\leq \frac{1}{\epsilon^2}\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{i=-\infty}^{\infty} |a_{ni}|^2E|Y_1|^2I(|Y_1| \leq n\delta)\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{n^2}E|Y_1|^2I(|Y_1| \leq n\delta)\\
&= C\sum_{i=1}^{\infty}E|Y_1|^2I((i-1)\delta < |Y_1| \leq i\delta)\sum_{n=i}^{\infty}\frac{1}{n^2}\\
&\leq CE|Y_1| < \infty.
\end{aligned}$$

Hence (ii) holds. □

The following theorem gives a necessary and sufficient condition for (2).

Theorem 3. *Let* $\{Y_i, -\infty < i < \infty\}$ *be a sequence of i.i.d. random variables with* $E|Y_1| < \infty$. *Let* $\{a_{ni}, -\infty < i < \infty, n \geq 1\}$ *be a bounded array of real numbers satisfying* (3). *Then* (2) *is equivalent to*

$$\sum_{n=1}^{\infty}\frac{1}{n}P(|\sum_{i=-\infty}^{\infty} a_{ni}Y_iI(|Y_i| > n)| > n\epsilon) < \infty \quad \textit{for all } \epsilon > 0.$$

Proof. It suffices to show that

$$\frac{1}{n}\sum_{i=-\infty}^{\infty} a_{ni}EY_iI(|Y_i| > n) \to 0 \tag{5}$$

and

(6)
$$\sum_{n=1}^{\infty}\frac{1}{n}P(|\sum_{i=-\infty}^{\infty}a_{ni}(Y_iI(|Y_i|\le n)-EY_iI(|Y_i|\le n))|>n\epsilon)<\infty \quad \text{for all } \epsilon>0.$$

By Lemma 4, (5) and (6) are satisfied. □

The following theorem gives a necessary condition for (2).

Theorem 4. *Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of i.i.d. non-negative random variables with $EY_1 < \infty$. Let $\{a_{ni}, -\infty < i < \infty, n \ge 1\}$ be a bounded array of non-negative real numbers satisfying* (3). *If* (2) *holds, then*

(7)
$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=-\infty}^{\infty}P(|a_{ni}Y_i|>n\epsilon)<\infty \quad \textit{for all } \epsilon>0.$$

Proof. Let $\gamma = 1/\sup_{i,n} a_{ni}$. For $-\infty < i < \infty$ and $n \ge 1$, define

$$Y'_{ni} = Y_iI(|Y_i|\le n\gamma\epsilon), \quad Y''_{ni} = Y_i - Y'_{ni}.$$

Observe that

$$\sum_{i=-\infty}^{\infty}a_{ni}Y''_{ni} = \sum_{i=-\infty}^{\infty}a_{ni}(Y_i-EY_i) - \sum_{i=-\infty}^{\infty}a_{ni}(Y'_{ni}-EY'_{ni}) + \sum_{i=-\infty}^{\infty}a_{ni}EY''_{ni}.$$

It follows by Lemma 3, Lemma 4, and (2) that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=-\infty}^{\infty}P(|a_{ni}Y_i|>n\epsilon)\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n}P(\sum_{i=-\infty}^{\infty}a_{ni}Y''_{ni}>n\epsilon/8)\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n}P(|\sum_{i=-\infty}^{\infty}a_{ni}(Y_i-EY_i)|>n\epsilon/16)\\
&\quad + C\sum_{n=1}^{\infty}\frac{1}{n}P(|\sum_{i=-\infty}^{\infty}a_{ni}(Y'_{ni}-EY'_{ni})+\sum_{i=-\infty}^{\infty}a_{ni}EY''_{ni}|>n\epsilon/16)\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n}P(|\sum_{i=-\infty}^{\infty}a_{ni}(Y_i-EY_i)|>n\epsilon/16)\\
&\quad + C\sum_{n=1}^{\infty}\frac{1}{n}P(|\sum_{i=-\infty}^{\infty}a_{ni}(Y'_{ni}-EY'_{ni})|>n\epsilon/32)<\infty.
\end{aligned}$$

Thus the result is proved. □

The following theorem gives a sufficient condition for (2).

Theorem 5. *Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of i.i.d. random variables with $E|Y_1| < \infty$. Let $\{a_{ni}, -\infty < i < \infty, n \geq 1\}$ be a bounded array of real numbers satisfying*

$$\sum_{i=-\infty}^{\infty} |a_{ni}| = O(\frac{n}{\log^\alpha n}) \quad \text{for some } \alpha > 1. \tag{8}$$

Then (2) *holds.*

Proof. We observe by (8) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} P(|\sum_{i=-\infty}^{\infty} a_{ni} Y_i I(|Y_i| > n)| > n\epsilon) \\
&\leq \frac{1}{\epsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} E\Big|\sum_{i=-\infty}^{\infty} a_{ni} Y_i I(|Y_i| > n)\Big| \\
&\leq \frac{1}{\epsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{i=-\infty}^{\infty} |a_{ni}| E|Y_1| I(|Y_1| > n) \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{n \log^\alpha n} E|Y_1| I(|Y_1| > n) < \infty,
\end{aligned}$$

since $\alpha > 1$. Thus the result follows from Theorem 3. □

The following theorem is a partial converse of Theorem 4.

Theorem 6. *Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of i.i.d. random variables with $E|Y_1| < \infty$. Let $\{a_{ni}, -\infty < i < \infty, n \geq 1\}$ be a bounded array of real numbers satisfying*

$$\sum_{i=-\infty}^{\infty} |a_{ni}| = O(\frac{n}{\log^\alpha n}) \quad \text{for some } \alpha > 0. \tag{9}$$

If (7) *holds, then* (2) *holds.*

Proof. Take a positive integer j such that $\alpha 2^j > 1$. By Lemma 2, we have that

$$\begin{aligned}
&P(|\sum_{i=k}^{m} a_{ni} Y_i I(|Y_i| > n)| > n\epsilon) \\
&\leq P(\sum_{i=k}^{m} |a_{ni} Y_i| I(|Y_i| > n) > n\epsilon) \\
&\leq C_j P(\max_{k \leq i \leq m} |a_{ni} Y_i| I(|Y_i| > n) > n\epsilon/(4^{j-1} 6^j)) \\
&\quad + D_j \max_{k \leq l \leq m} \left[P(\sum_{i=k}^{l} |a_{ni} Y_i| I(|Y_i| > n) > n\epsilon/(4^j 6^j))\right]^{2^j}
\end{aligned}$$

$$\leq C_j \sum_{i=k}^{m} P(|a_{ni}Y_i|I(|Y_i|>n) > n\epsilon/(4^{j-1}6^j))$$
$$+ D_j \Big[P(\sum_{i=k}^{m} |a_{ni}Y_i|I(|Y_i|>n) > n\epsilon/(4^j6^j))\Big]^{2^j}$$
$$\leq C_j \sum_{i=k}^{m} P(|a_{ni}Y_i| > n\epsilon/(4^{j-1}6^j)) + D_j \Big[\frac{4^j6^j}{n\epsilon} \sum_{i=k}^{m} |a_{ni}|E|Y_1|I(|Y_1|>n)\Big]^{2^j}.$$

Letting $k \to -\infty$ and $m \to \infty$, it follows that

$$P(|\sum_{i=-\infty}^{\infty} a_{ni}Y_iI(|Y_i|>n)| > n\epsilon)$$
$$\leq C_j \sum_{i=-\infty}^{\infty} P(|a_{ni}Y_i| > n\epsilon/(4^{j-1}6^j)) + D_j \Big[\frac{4^j6^j}{n\epsilon} \sum_{i=-\infty}^{\infty} |a_{ni}|E|Y_1|I(|Y_1|>n)\Big]^{2^j}.$$

Hence we have by (7) and (9) that

$$\sum_{n=1}^{\infty} \frac{1}{n} P(|\sum_{i=-\infty}^{\infty} a_{ni}Y_iI(|Y_i|>n)| > n\epsilon)$$
$$\leq C_j \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} P(|a_{ni}Y_i| > n\epsilon/(4^{j-1}6^j))$$
$$+ D_j \sum_{n=1}^{\infty} \frac{1}{n} \Big[\frac{4^j6^j}{n\epsilon} O(\frac{n}{\log^\alpha n}) E|Y_1|I(|Y_1|>n)\Big]^{2^j} < \infty,$$

since $\alpha 2^j > 1$. Thus the result follows from Theorem 3. □

Remark 1. Condition (9) in Theorem 6 is stronger than condition (3) in Theorem 4, and so Theorem 6 is a partial converse of Theorem 4.

4. Complete convergence of moving average processes

In this section, we give a partial solution to the question proposed in the introduction.

Corollary 1. *Let $\{Z_i, -\infty < i < \infty\}$ be a sequence of i.i.d. non-negative random variables with $EZ_1 < \infty$. Let $\{a_i, -\infty < i < \infty\}$ be a summable sequence of non-negative real numbers. Set $Y_i = Z_i - EZ_i$. Suppose that $\{X_n, n \geq 1\}$ is the moving average process based on the sequence $\{Y_i\}$, i.e., $X_n = \sum_{i=-\infty}^{\infty} a_{i+n}Y_i$. If $\sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} P(|\sum_{k=1}^{n} a_{i+k}Z_i| > n\delta) = \infty$ for some $\delta > 0$, then $\sum_{n=1}^{\infty} \frac{1}{n} P(|\sum_{k=1}^{n} X_k| > n\delta) = \infty$ for some $\delta > 0$.*

Proof. Set $a_{ni} = \sum_{k=1}^{n} a_{i+k}$. Then $\sum_{k=1}^{n} X_k = \sum_{i=-\infty}^{\infty} a_{ni}(Z_i - EZ_i)$. Since $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, it can be easily obtained that

$$|a_{ni}| = O(1) \quad \text{and} \quad \sum_{i=-\infty}^{\infty} |a_{ni}| = O(n).$$

Thus the result follows from Theorem 4. □

Remark 2. If we can find $\{a_i\}$ and $\{Z_i\}$ such that

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} P(|\sum_{k=1}^{n} a_{i+k} Z_i| > n\delta) = \infty$$

for some $\delta > 0$, then the answer can be false. Unfortunately, we fail to find such sequences, and so Corollary 1 gives a partial solution to the question.

Corollary 2. *Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers satisfying*

$$\sum_{i=-\infty}^{\infty} |\sum_{k=1}^{n} a_{i+k}| = O(\frac{n}{\log^{\alpha} n}) \quad \textit{for some } \alpha > 0.$$

Suppose that $\{X_n, n \geq 1\}$ is the moving average process based on a sequence of i.i.d. random variables $\{Y_i, -\infty < i < \infty\}$ with $EY_1 = 0$. If

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} P(|\sum_{k=1}^{n} a_{i+k} Y_i| > n\epsilon) < \infty$$

for all $\epsilon > 0$, then $\sum_{n=1}^{\infty} \frac{1}{n} P(|\sum_{k=1}^{n} X_k| > n\epsilon) < \infty$ for all $\epsilon > 0$.

Proof. The result follows from Theorem 6. □

Remark 3. If $\{a_i, -\infty < i < \infty\}$ is an absolutely summable sequence of real numbers, it follows that $\sum_{i=-\infty}^{\infty} |\sum_{k=1}^{n} a_{i+k}| = O(n)$. The case $\alpha = 0$ is of importance for Corollary 2. If it is possible, then the answer can be true.

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