

ON THE COMPLEX OSCILLATION OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we investigate the growth of solutions and the existence of subnormal solutions for a class of higher order linear differential equations. We obtain some results which improve and extend the results of Chen-Shon [2] and Gundersen-Steinbart [5].

1. Introduction

In the study of the oscillation theory of complex differential equations, the growth of solutions is a very important property. For linear differential equations of the form

$$(1) \quad f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z)f = 0,$$

where $a_0(z), \dots, a_{n-1}(z)$ are polynomials, it is known that every entire solution of equation (1) must be of finite order, and if some of the coefficients $a_j(z)$ ($0 \leq j \leq n-1$) are replaced by transcendental entire functions, then equation (1) has at least one solution of infinite order. This can be proved by mainly using the Wiman-Valiron theory [6, 8].

In this paper, we assume that the reader is familiar with the fundamental results and the standard notation of the value distribution theory of meromorphic functions [8, 10]. We denote the order of growth of f by $\sigma(f)$. In addition, let us define inductively, for $r \in [0, +\infty)$, $\exp^{[1]} r = e^r$ and $\exp^{[n+1]} r = \exp(\exp^{[n]} r)$, $n \in \mathbb{N}$. For all r sufficiently large, define $\log^{[1]} r = \log r$ and $\log^{[n+1]} r = \log(\log^{[n]} r)$, $n \in \mathbb{N}$. To express the rate of growth of meromorphic functions of infinite order, we recall the following definitions [7].

Definition 1.1. The iterated p -order $\sigma_p(f)$ of a meromorphic function f is defined by

$$\sigma_p(f) = \limsup_{r \rightarrow \infty} \frac{\log^{[p]} T(r, f)}{\log r}, \quad p \in \mathbb{N}.$$

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Definition 1.2. Suppose $f \neq 0$ is a solution of equation (1) and if f satisfies the condition

$$\limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{r} = 0,$$

then we say that f is a nontrivial subnormal solution of equation (1).

Chiang and Gao [3] gave the definition of the e-type order of meromorphic function as follows:

Definition 1.3. Let f be a meromorphic function. Define

$$\sigma_e(f) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{r}$$

to be the e-type order of f .

From Definition 1.3, we know that if $\sigma_e(f) = 0$, then f must be subnormal. The following results are obviously.

- (i) If $0 < \sigma_e(f) < \infty$, then $\sigma_2(f) = 1$,
- (ii) if $\sigma_2(f) < 1$, then $\sigma_e(f) = 0$,
- (iii) if $\sigma_2(f) = \infty$, then $\sigma_e(f) = \infty$.

Thus the iterated order and the e-type order can be applied to get a more precise estimate of the growth of meromorphic function with infinite order. Wittich [9] investigated the subnormal solution of the equation

$$(2) \quad f'' + P(e^z)f' + Q(e^z)f = 0,$$

where $P(z)$ and $Q(z)$ are nonconstant polynomials in z .

Theorem A. *If $f \neq 0$ is a subnormal solution of (2), then f must have the form*

$$f(z) = e^{cz}(h_0 + h_1e^z + \cdots + h_me^{mz}),$$

where $m \geq 0$ is an integer, c, h_0, \dots, h_m are constants and $h_0, h_m \neq 0$.

Based on the comparison of degrees of P and Q , Gundersen and Steinbart [5] improved Theorem A and obtained the following theorem.

Theorem B. *The following statements hold with regard to the subnormal solutions of equation (2),*

- (i) *if $\deg P > \deg Q$ and $Q \neq 0$, then any subnormal solution $f \neq 0$ of equation (2) must have the form*

$$f(z) = \sum_{k=0}^m h_k e^{-kz},$$

where $m \geq 1$ is an integer and h_0, \dots, h_m are constants, $h_0, h_m \neq 0$,

- (ii) *if $\deg P \geq 1$ and $Q = 0$, then any subnormal solution of equation (2) must be a constant,*
- (iii) *if $\deg P < \deg Q$, then the subnormal solution of equation (2) is $f = 0$.*

Gundersen and Steinbart [5] also considered non-homogeneous differential equations

$$(3) \quad f'' + P(e^z)f' + Q(e^z)f = R_1(e^z) + R_2(e^{-z}),$$

where $P(z)$, $Q(z)$, $R_1(z)$ and $R_2(z)$ are polynomials in z and obtained the following theorem.

Theorem C. *Suppose $f \neq 0$ is a subnormal solution of (3) and $\deg P > \deg Q$.*

(i) *If $\deg P > \deg R_1$, then f must have the form*

$$f(z) = S_1(e^z) + S_2(e^{-z}),$$

where $S_1(z)$ and $S_2(z)$ are polynomials in z ;

(ii) *if $\deg P \leq \deg R_1$, then f must have the form*

$$f(z) = cz \sum_{k=0}^m a_k e^{-kz} + S_1(e^z) + S_2(e^{-z}),$$

where $m \geq 0$ is an integer, c, a_0, \dots, a_m are constants with $a_0, a_m \neq 0$, and $S_1(z), S_2(z)$ are polynomials in z .

They also raised the following question in [5]:

Question. Whether equation (3) in Theorem C can be generalized to equation

$$(4) \quad f'' + [P_1(e^z) + P_2(e^{-z})]f' + [Q_1(e^z) + Q_2(e^{-z})]f = R_1(e^z) + R_2(e^{-z})?$$

With regard to above question, Chen and Shon [2] investigated the existence of subnormal solutions of equation (4) and its corresponding homogeneous equation

$$(5) \quad f'' + [P_1(e^z) + P_2(e^{-z})]f' + [Q_1(e^z) + Q_2(e^{-z})]f = 0.$$

They obtained some results about the subnormal solutions and the growth of solutions of (4) and (5). Their results can be stated as follows.

Theorem D. *Let $P_j(z), Q_j(z)$ ($j = 1, 2$) be polynomials in z . If*

$$(6) \quad \deg Q_1 > \deg P_1 \quad \text{or} \quad \deg Q_2 > \deg P_2,$$

then the differential equation (5) has no nontrivial subnormal solutions, and every nontrivial solution of (5) satisfies $\sigma_2(f) = 1$.

Theorem E. *Let $P_j(z), Q_j(z)$ ($j = 1, 2$) be polynomials in z , and $R_1 + R_2 \neq 0$. If $P_j(z), Q_j(z)$ satisfy the conditions (6), then*

(i) *equation (4) has at most a subnormal solution f_0 , and $f_0 = S_1(e^z) + S_2(e^{-z})$, where $S_1(z)$ and $S_2(z)$ are polynomials in z ;*

(ii) *all solutions f of equation (4) satisfy $\sigma_2(f) = 1$ except the subnormal solution in (i).*

The main purpose of this paper is to improve Theorems D, E to higher order periodic differential equations. Furthermore, we replace the exponent z or $-z$

with nonconstant entire functions to the former theorems. We obtained the following results.

Theorem 1.1. *Let $P_j(z), Q_j(z)$ ($j = 0, 1, \dots, n-1$) be polynomials in z . If*

$$\deg P_0 > \deg P_j \quad \text{or} \quad \deg Q_0 > \deg Q_j \quad j = 1, \dots, n-1,$$

then the n -th order differential equation

$$(7) \quad f^{(n)} + [P_{n-1}(e^z) + Q_{n-1}(e^{-z})]f^{(n-1)} + \dots + [P_0(e^z) + Q_0(e^{-z})]f = 0$$

has no nontrivial subnormal solution, and every nontrivial solution satisfies $\sigma_2(f) = 1$.

Remark 1.1. Obviously, equation (7) is more general than equation (5) in Theorem D. The following example shows that, in Theorem 1.1, if there exists $\deg P_i = \deg P_j$ and $\deg Q_i = \deg Q_j$, $i \neq j$, then equation (7) may have a nontrivial subnormal solution.

Example. A subnormal solution $f = e^{-z}$ satisfies the following equation

$$f^{(n)} + f^{(n-1)} + \dots + f'' + (e^{2z} + e^{-2z})f' + (e^{2z} + e^{-2z})f = 0,$$

where n is an odd number.

Remark 1.2. If $\deg P_j > \deg P_0$, then equation (7) may have a nontrivial subnormal solution, which can be seen by the following example.

Example. A subnormal solution $f = e^{-z} - 1$ satisfies the equation

$$f''' + e^z f'' + (e^{2z} - e^{-z})f' - (e^z + e^{-z})f = 0.$$

Corollary 1.1. *Under the assumptions of Theorem 1.1, the following equation*

$$(8) \quad f^{(n)} + [P_{n-1}(e^z) + Q_{n-1}(e^{-z})]f^{(n-1)} + \dots + [P_1(e^z) + Q_1(e^{-z})]f' + [P_0(e^z) + Q_0(e^{-z})]f = R_1(e^z) + R_2(e^{-z}),$$

has at most a subnormal solution f_0 , and other solutions f satisfy $\sigma_2(f) = 1$.

Theorem 1.2. *Let $A(z), B(z)$ be nonconstant entire functions, and let $P(z), Q(z)$ be polynomials in z . Then*

$$(9) \quad f'' + P(e^A)f' + Q(e^B)f = 0$$

has no nontrivial subnormal solutions except two cases:

- (1) $A(z), B(z)$ are polynomials in z , $\deg A = \deg B$ and $\deg P = \deg Q$.
- (2) $A(z), B(z)$ are transcendental entire functions, $\deg P = \deg Q$ and $\sigma_p(A(z)) = \sigma_p(B(z))$.

Remark 1.3. From the conclusion of Theorem A, we easily find that $\deg P = \deg Q$, which can be seen a special case of (1) in Theorem 1.2.

2. Some lemmas

The following two results easily get from the classical Wiman-Valiron theory.

Lemma 2.1. *Let A_i ($i = 0, 1, \dots, n - 1$) be finite order entire functions. If f is a solution of differential equation*

$$f^{(n)} + A_{n-1}f^{(n-1)} + \dots + A_0f = 0,$$

then $\sigma_2(f) \leq \max\{\sigma(A_i)\}$.

Lemma 2.2. *Let $f(z)$ be an entire function of order σ , and let $\nu_f(r)$ be the central index of f . Then*

$$\sigma = \limsup_{r \rightarrow \infty} \frac{\log^+ \nu_f(r)}{\log r}.$$

Lemma 2.3 ([4]). *Let f be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exists a set $E \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and m, n ($0 \leq m < n$) such that for all $|z| = r \notin (0, 1) \cup E$, we have*

$$\left| \frac{f^{(n)}(z)}{f^{(m)}(z)} \right| \leq \left(\frac{T(\alpha r, f)}{r} \log^\alpha r \log T(\alpha r, f) \right)^{n-m}.$$

Lemma 2.4. *Let f be a transcendental entire function, and let s be a positive integer. Then there exists one sequence $\{r_k\} (r_k \rightarrow \infty)$ such that $\left| \frac{f(z)}{f^{(s)}(z)} \right| \leq r_k^{2s}$ holds for enough large r_k , as $|z| = r_k$.*

Proof. Using the similarly method as the proof of Lemma 14 in [1], we get that there exists one sequence $\{r_k\} (r_k \rightarrow \infty)$ such that

$$\frac{f^{(s)}(z)}{f(z)} = \left(\frac{\nu_f(r_k)}{z} \right)^s (1 + o(1)), \quad n \in \mathbb{N},$$

holds for enough large $\{r_k\}$, as $|z| = r_k$ and $|f(z)| = M(r_k, f)$.

On the other hand, since f is an entire function, from Lemma 2.2, which implies that for any $\varepsilon > 0$, we have $\nu_f(r_k) \geq r_k^{\sigma - \varepsilon}$, where $\sigma - \varepsilon$ can be replaced with a large enough real number M if $\sigma = \infty$. By calculation, we get $\left| \frac{f(z)}{f^{(s)}(z)} \right| \leq r_k^{(1 - \sigma + \varepsilon)s} \leq r_k^{2s}$. □

Lemma 2.5 ([8]). *Let $g : (0, +\infty) \rightarrow R, h : (0, +\infty) \rightarrow R$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

3. Proofs of main results

Proof of Theorem 1.1. Suppose that

$$\begin{aligned} P_j(z) &= a_{jm_j}z^{m_j} + a_{jm_{j-1}}z^{m_{j-1}} + \cdots + a_{j1}z + a_{j0}, \\ Q_j(z) &= b_{jn_j}z^{n_j} + b_{jn_{j-1}}z^{n_{j-1}} + \cdots + b_{j1}z + b_{j0}, \end{aligned}$$

where $j = 0, 1, \dots, n-1$ and $m_j, n_j \in \mathbb{Z}$.

Let $f \neq 0$ be a solution of equation (7). Then f must be an entire function. From Lemma 2.1, we get

$$\sigma_2(f) \leq \max\{\sigma(P_j(e^z)), \sigma(Q_j(e^z))\} = 1.$$

The following, we will prove $\sigma_2(f) \geq 1$. By Lemma 2.3, we see that there exists a subset $E \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$ such that for all z satisfying $|z| \notin (0, 1) \cup E$, and we have

$$(10) \quad \left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B[T(2r, f)]^{j+1}, \quad j = 1, 2, \dots, n-1.$$

If $\deg P_0 > \deg P_j$, then we take $z = r$. If $\deg Q_0 > \deg Q_j$, we take $z = -r$. Combining (7) with (10), for a sufficiently large r and $r \notin (0, 1) \cup E$, we obtain

$$\begin{aligned} |P_0(e^z) + Q_0(e^{-z})| &= |a_{0m_0}|e^{m_0r}(1 + o(1)) \\ &\leq \left| \frac{f^{(n)}}{f} \right| + |P_{n-1}(e^z) + Q_{n-1}(e^{-z})| \left| \frac{f^{(n-1)}}{f} \right| + \cdots \\ &\quad + |P_1(e^z) + Q_1(e^{-z})| \left| \frac{f'}{f} \right| \\ &\leq B[T(2r, f)]^{n+1} + |a_{n-1m_{n-1}}|e^{m_{n-1}r}B[T(2r, f)]^n(1 + o(1)) \\ &\quad + \cdots + |a_{1m_1}|e^{m_1r}B[T(2r, f)]^2(1 + o(1)) \\ (11) \quad &\leq B[T(2r, f)]^{n+2}Me^{\max\{m_1, \dots, m_{n-1}\}r}(1 + o(1)), \end{aligned}$$

where $M = \max\{|a_{n-1m_{n-1}}|, \dots, |a_{1m_1}|\}$. Since $m_0 > \max\{m_1, \dots, m_{n-1}\}$, utilizing (11) and Lemma 2.5, we get $\sigma_2(f) \geq 1$. Thus, $\sigma_2(f) = 1$.

Now we will prove any solution $f (\neq 0)$ is not subnormal. Otherwise, if f is subnormal, from Definition 1.2, for any given ε , $0 < \varepsilon < \frac{1}{4n+8}$, f satisfies

$$(12) \quad T(r, f) \leq e^{r\varepsilon}.$$

As we take $z = r \notin (0, 1) \cup E$, from (11) and (12), we obtain that

$$\begin{aligned} |a_{0m_0}|e^{m_0r}(1 + o(1)) &\leq Be^{(2n+4)r\varepsilon}Me^{\max\{m_1, \dots, m_{n-1}\}r}(1 + o(1)) \\ &\leq Be^{\frac{1}{2}r}Me^{\max\{m_1, \dots, m_{n-1}\}r}(1 + o(1)). \end{aligned}$$

Since $m_0 > \max\{m_1, \dots, m_{n-1}\}$, we get a contradiction as $r \rightarrow \infty$. Hence, equation (7) has no nontrivial subnormal solution, thus we have completed the proof of Theorem 1.1. \square

Proof of Corollary 1.1. If f_1 and f_2 are two distinct subnormal solutions of equation (8), so $f_2 - f_1$ is a subnormal solution of the corresponding homogeneous equation (7), which contradicts the results of Theorem 1.1.

By Theorem 1.1, we see that all solutions of the corresponding homogeneous equation (7) of (8) are of $\sigma_2(f) = 1$. By the variation of parameters, we see that all solutions of equation (8) satisfy $\sigma_2(f) \leq 1$. If $\sigma_2(f) < 1$, then f clearly satisfies $\sigma_e(f) = 0$, so f is subnormal. Hence we see that all other solutions f of equation (8) satisfy $\sigma_2(f) = 1$ at most one subnormal solution. □

Proof of Theorem 1.2. We divide into two cases:

Case 1. $A(z), B(z)$ are polynomials. Suppose that p, q, m, n are constants,
 $A(z) = A_p z^p + A_{p-1} z^{p-1} + \dots + A_0$, $B(z) = B_q z^q + B_{q-1} z^{q-1} + \dots + B_0$,
 $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_0$, $Q(z) = b_n z^n + b_{n-1} z^{n-1} + \dots + b_0$.
 Subcase 1.1. $m = n$.

(1) $p = q$. In this case, we give an example to illustrate that there exists nontrivial subnormal solution. The subnormal solution $f = e^{-z}$ satisfies the equation

$$f'' + (e^{z^k} + 1)f' + e^{z^k} f = 0, \quad k \in \mathbb{N}.$$

(2) $p > q$. If f is a subnormal solution, from (9), then we get

$$(13) \quad |P(e^A)| \leq \left| \frac{f''}{f'} \right| + |Q(e^B)| \left| \frac{f}{f'} \right|.$$

From Lemma 2.3, then there exists a set $E_1 \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$ such that for all $|z| = r \notin (0, 1) \cup E_1$, we have

$$\left| \frac{f''(z)}{f'(z)} \right| \leq B(T(2r, f))^2.$$

From Lemma 2.4, there exists $r_k \notin (0, 1) \cup E_1$ ($r_k \rightarrow \infty$), we get

$$\left| \frac{f(z)}{f'(z)} \right| \leq r_k^{2s}.$$

Take suitable $z = r_k$ or $z = -r_k$ to ensure that $A_p z^p > 0$. From (10) and (13), we get

$$(14) \quad \begin{aligned} |a_m| e^{m|A_p r_k^p|} (1 + o(1)) &\leq B(T(2r_k, f))^2 + |b_n| e^{n|B_q r_k^q|} r_k^{2s} (1 + o(1)) \\ &\leq 2B(T(2r_k, f))^2 |b_n| e^{n|B_q r_k^q|} r_k^{2s} (1 + o(1)). \end{aligned}$$

Since $m = n$, $p > q$, combining (12) with $0 < \varepsilon < \frac{1}{12}$, then (14) changes into the following form

$$|a_m| e^{m(|A_p r_k^p| - |B_q r_k^q|)} (1 + o(1)) \leq 2B e^{4r_k \varepsilon} r_k^{2s} < 2B e^{\frac{1}{3} r_k} r_k^{2s}.$$

Obviously, it is a contradiction. So, there is no subnormal solution in this case.

(3) $p < q$. If f is a subnormal solution, from Lemma (2.3), there exists a set $E_2 \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$ such that for all $|z| = r \notin (0, 1) \cup E_2$, take either $z = r$ or $z = -r$ to assure that $B_q z^q > 0$, we get

$$\begin{aligned}
 |Q(e^B)| &= |b_n|e^{nB_q r^q} (1 + o(1)) \\
 &\leq \left| \frac{f''}{f} \right| + |P(e^A)| \left| \frac{f'}{f} \right| \\
 &\leq B(T(2r, f))^3 + |a_m|e^{m|A_p r^p|} B(T(2r, f))^2 (1 + o(1)) \\
 (15) \quad &\leq 4B(T(2r, f))^3 |a_m|e^{m|A_p r^p|} (1 + o(1)).
 \end{aligned}$$

Using the similar argument as the proof of Theorem 1.1, we get a contradiction.

Subcase 1.2. $m < n$.

(1) $p = q$. If f is a solution of equation (9), we assert that $\sigma_2(f) = p$, and f is not a subnormal solution. Since Lemma 2.1, then we get

$$\sigma_2(f) \leq \max\{\sigma(P(z)), \sigma(Q(z))\} = p.$$

Using the similar proof of Theorem 1.1, we get the following inequality

$$(16) \quad |b_n|e^{(n-m)r^p} (1 + o(1)) \leq 4B(T(2r, f))^3 |a_m| (1 + o(1)).$$

Since $m < n$, then (16) gives $\sigma_2(f) \geq p$. So $\sigma_2(f) = p$. If f is a subnormal solution, combining (12) and (16), we get a contradiction.

(2) $p < q$. We can get a similar inequality to (16) as follows,

$$(17) \quad |b_n|e^{nr^q - mr^p} \leq 4B(T(2r, f))^3 |a_m| (1 + o(1)).$$

If f is a subnormal solution, combining (12) and (17), we also get a contradiction as $r \rightarrow \infty$.

(3) $p > q$. In this case, using the similar method as (2) of Subcase 1.1, we obtain that there is no subnormal solution of equation (9).

Subcase 1.3. $m > n$. Similar argument as Subcase 1.2, we get that there is no subnormal solution of equation (9).

Case 2. If either $A(z)$ or $B(z)$ is a transcendental entire function, without loss of generality, we assume that $A(z)$ is a transcendental entire function and f is a solution of (9). From (9), Lemma 2.3 and Lemma 2.4, we get that there exists $E_3 \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$ such that for all $|z| = r_k \notin (0, 1) \cup E_3$, we have

$$(18) \quad |a_m|e^{mM(r_k, A(z))} \leq 4B(T(2r_k, f))^3 |b_n|e^{nM(r_k, B(z))} r_k^2 (1 + o(1)),$$

where $M(r_k, f) = \max_{|z|=r_k} |f(z)|$.

Subcase 2.1. Suppose that $\sigma_p(A(z)) > \sigma_p(B(z))$, thus from (18) and $A(z)$ is transcendental entire, we get $\sigma_e(f) = \infty$. Hence f is not a subnormal solution.

Subcase 2.2. Suppose that $\sigma_p(A(z)) < \sigma_p(B(z))$. Similarly as above, we get $\sigma_e(f) = \infty$, hence f is not a subnormal solution.

Subcase 2.3. Suppose that $\sigma_p(A(z)) = \sigma_p(B(z))$ and $m = n$. We give the following example to illustrate there may exist a subnormal solution in this case. A subnormal solution $f = e^{-z}$ satisfies the equation

$$f'' + (\exp^{[n+1]} z + 1)f' + (\exp^{[n+1]} z)f = 0.$$

Thus, we have completed the proof of Theorem 1.2. \square

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