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Analysis of Revenue-Sharing Contracts for Service Facilities

Ruey Huei Yeh †
Department of Industrial Management
National Taiwan University of Science and Technology, Taipei 106, TAIWAN
E-mail: rhyeh@ mail.ntust.edu.tw

Yi-Fang Lin
Department of Industrial Management
National Taiwan University of Science and Technology, Taipei 106, TAIWAN
E-mail: d9301205@ mail.ntust.edu.tw

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Abstract. There are customer services jointly provided by two facilities so that each customer will complete the course made up of both facilities' sub-services. The two facilities are assumed invested respectively by an infrastructure owner and one subordinate facility owner, whose partnership is built on their capital investments. This paper presents a mathematical model of Stackelberg competition between the two facility owners to derive their optimal Nash equilibrium. In this study, each facility owner's profit is consisted of fixed revenue fractions of sold services, operating costs (including depreciation cost) and maintenance costs of her facility. The maintenance costs of one facility are incurred both by failures and deterioration due to usage. Moreover, for both facilities, failures are rectified immediately by minimal repairs and preventive maintenance is carried out at a fixed time epoch. Additional assumptions are also employed to develop the model such as customer arrivals are manipulated to follow a Poisson process, and each facility's lifetime is independently Weibull-distributed. The Stackelberg game proceeds as follows. At the first stage of decision making process, the infrastructure owner (acting as a leader) decides the allocation of revenue shares based on her self-interest. After observing the allocation of revenue shares, the subordinate facility owner determines her own optimal price of services. This paper investigates actions and reactions of the two partners in the system. Then analytical conditions are proposed to achieve a unique optimal Nash equilibrium. Finally, some suggestions for further research are discussed.

Keywords: Service Facilities, Revenue-sharing Contracts, Maintenance, Poisson Process, Weibull Lifetime

1. INTRODUCTION

With emphasis on the mutual benefits, inter-firm partnership has been increasingly attracted attention. One interesting partnership is a service channel providing a course of customer services by a number of facilities. The partnership is built on the capital investments of those facilities. This kind of facility-partnership is common and important in capital-intensive industries such as healthcare and telecommunications industries. In fact, this study is inspired by a scenario in the healthcare industry: Due to a financial shortage, the radiology department of some hospital only has basic medical devices (infrastructure). Having no advanced imaging facilities such as computed tomography (CT) and magnetic resonance imaging (MRI) equipment, the radiology department cannot earn technical revenues from CT/MRI examinations. Hence the department head offered a revenue-sharing contract to an imaging equipment firm to start up an imaging center. This partnership is a typical leader-follower relationship because the imaging equipment owner cannot independently practice her medical services without the medical orders given by a radiologist. Therefore, the imaging equipment owner has to accept the share of revenues allocated by the head of the radiology department unless she leaves the joint alliance.

The cost structure in such service chain is mainly consisted of maintenance and depreciation costs of service facilities. However, there are limited literatures that

† : Corresponding Author

deal with service channels along with maintenance costs of service facilities.

There are two streams of literature that are relevant to this paper. One is supply chain contracting problems, the other is maintenance models. Extensive discussions about the virtues and characterizations of revenue-sharing contracts can be found in Cachon and Lariviere (2005), and Lariviere and Porteus (2001). To model the competitive behaviors of the players as leaders and followers in a decentralized channel, Stackelberg game formulation is adopted in many recent studies such as Wang and Gerchak (2003), Bernstein and DeCroix (2004), and Gurnani and Gerchak (2007).

Relevant literature on maintenance models is introduced as follows. To ensure smooth operation of the service system, the facility owners should carry out appropriate maintenance actions. Maintenance models have been extensively explored. Barlow and Proschan (1965), Scarf (1997), and Wang (2002) provide comprehensive literature reviews of various maintenance models and policies.

There are two main types of maintenance actions -- corrective maintenance (CM) and preventive maintenance (PM). CM actions rectify failed facility to restore its operation; PM actions are arranged and performed to reduce failures of operational facility. Minimal repair is the most widely adopted action among all types of CM (Nakagawa and Kowada, 1983). After each minimal repair, the facility is brought back to operational status while its failure rate remains unchanged.

Although a great number of maintenance models have been developed, few models reflect the situation that facility is intermittently operated. Hsu (1992) and Dohi *et al.* (2001) employ renewal theory to derive (approximate) optimal PM policies over infinite time horizon under an intermittently used environment.

It seems that there are abundant literatures relevant to this study. However, no methodologies of the above mentioned papers can be applied directly to the underlying environment of our concern.

The rest of this paper is organized as follows. The mathematical models of action/reaction of the facility owners are developed in Section 2, and analyzed in Section 3. An optimal Nash equilibrium is also achieved in Section 3. Finally, some concluding remarks are drawn in the last section.

2. MATHEMATICAL FORMULATION

Consider a service channel in which a complete course of customer services is provided by two facilities. The two facilities are invested individually by two partners, an infrastructure owner and one subordinate facility owner. In this system, each facility owner's profit is consisted of fixed revenue fractions of sold services, operating costs (including depreciation cost) and maintenance costs of her facility. The maintenance costs of one facility are incurred both by failures and deterioration. Facility failures are rectified immediately by minimal repairs and preventive maintenance is carried out at a fixed time epoch. Additionally, customer arrivals are assumed to follow a Poisson process, and each facility's lifetime is independently Weibull-distributed. The decision process proceeds as follows. First, the infrastructure owner decides the allocation of revenue shares. After observing the allocation of revenue shares, the subordinate facility owner determines the price of the services. The question is if there exists an optimal equilibrium for the two partners' action and reaction. Before proceeding further, some notations and assumptions are introduced as follows.

2.1 Modeling Notations and Assumptions

The decision variables are service price (unit revenue per customer served) and revenue shares received by partner i which are denoted by R and r_i $(r_1+r_2=1)$, respectively. For i = 1 or 2, other notations are listed below.

λ	demand rate of services; customer arrival rate
K	Poisson process with rate λ
μ_i	mean processing rate of facility i
$F_i(t)$	lifetime distribution of facility i
$h_i(t)$	hazard function of the facility i
$H_i(t)$	cumulative hazard function of facility i
T	period of PM for both facilities
$N_i(\lambda;T)$	number of failures of facility i in the time interval [0, T]
c_i	unit operating cost of facility i
C_{p_i}	PM cost and other setup cost of facility i
C_{m_i}	CM (minimal repair) cost of facility i
π_i	profit of facility i

The demand rate of services is assume to be the following log-linear (Cobb-Douglas) demand

$$\lambda = \lambda_0 R^{-a}, \tag{1}$$

where λ_0 is the potential customer arrival rate, *i.e.,* a potential demand for the service. By definition, the parameter a is exactly the price elasticity of the service. Additionally, the demand rate of services is also assumed to be price elastic so that $a > 1$ throughout this paper. For the economic interpretation of the price elasticity a, the reader is referred to So and Song (1998). Other fundamental assumptions are listed as follows.

A1. Unmet demand for services is lost.
A2. Each customer will complete the course of all services provided by the two facilities.
A3. The hazard functions of both facilities are increasing.
A4. Time to carry a CM action is negligible relative to the PM period T for each facility.
A5. Time to carry a PM action is negligible relative to the PM period T for each facility.

A6. No CM actions would be intentionally delayed. A CM action is performed immediately upon facility failure.
A7. Operating conditions of the two facilities are independent. Therefore, one facility will not be affected upon another facility's failure.
A8. Demand and all profit parameters such as cost, revenue, operating conditions of facilities are common knowledge to both partners.

2.2 Revenue-sharing Contract and Stackelberg Game Formulation

At first, acting as a leader in this service channel, the infrastructure owner decides the allocation of revenue shares based on her self-interest. Then, after observing the revenue shares, the subordinate facility owner determines the price of services based on her own self-interest. This constitutes a Stackelberg game. Each party has to decide her own optimal action/reaction to maximize her own expected profit function. For simplicity, let the subordinate facility owner be denoted by facility owner 1 and the infrastructure owner be denoted by facility owner 2.

2.3 Expected Profits of the Facility Owners

According to the context, both profit functions are consisted of four components: service revenue, operating cost, CM cost, and PM cost. Therefore, given revenue shares r_i, the profit functions of the two partners are

$$\pi_i(\lambda) = (r_i R - c_i)K(T) - C_{m_i} N_i(\lambda;T) - C_{p_i}, \tag{2}$$

where $i = 1, 2$. Substituting the inverse demand function $R = R(\lambda)$ given by (1) into the profit function (2) and taking the expectation, the expected profit function $E\left[\pi_i(\lambda)\right]$ reduces to

$$\begin{aligned} E\left[\pi_i(\lambda)\right] &= (r_i R - c_i)T\lambda - C_{m_i} E\left[N_i(\lambda;T)\right] - C_{p_i} \\ &= T\left[r_i \lambda_0^{1/a} \lambda^{1-1/a} - c_i \lambda\right] - C_{m_i} E\left[N_i(\lambda;T)\right] - C_{p_i}. \end{aligned} \tag{3}$$

Similarly, after allocating revenue share r_1 to the subordinate facility owner, the profit function of the infrastructure owner (the 2nd facility owner) becomes

$$\pi_2(r_1) = \left[(1-r_1)R - c_2\right]K(T) - C_{m_2} N_2(\lambda;T) - C_{p_2}. \tag{4}$$

The corresponding expectation is

$$\begin{aligned} E\left[\pi_2(r_1)\right] = T&\left[(1-r_1)\lambda_0^{1/a}\lambda^{1-1/a} - c_2\lambda\right] \\ &- C_{m_2} E\left[N_2(\lambda;T)\right] - C_{p_2}. \end{aligned} \tag{5}$$

3. MODEL ANALYSIS

It is obvious that the expected number of facility failures $E\left[N_i(\lambda;T)\right]$ plays an important role in these two expected profit functions. Before the analysis of the two partners' decisions, some analytical properties of $E\left[N_i(\lambda;T)\right]$ are explored.

3.1 Analytical Properties of the Expected Number of Facility Failures

The expected number of facility failures is determined by the lifetime distribution and the maintenance actions for the facility. The lifetime of facility *i,* for $i = 1, 2$, is assumed to be Weibull-distributed and thus its probability density function can be written as $f_i(t) = \alpha_i \beta_i(\alpha_i t)^{\beta_i - 1} e^{-(\alpha_i t)^{\beta_i}}$ with scale parameter $\alpha_i > 0$ and shape parameter $\beta_i > 0$, where $t > 0$. Then it follows from definitions that the failure rate function is $h_i(t) = \alpha_i \beta_i (\alpha_i t)^{\beta_i - 1}$ and the cumulative failure rate function is $H_i(t) = (\alpha_i t)^{\beta_i}$. Note that h is increasing in t for $\beta_i > 1$, decreasing for $\beta_i < 1$, and constant for $\beta_i = 1$. Since each hazard rate $h_i(t)$ is assumed to be increasing in t, we consider the case where $\beta_i > 1$ throughout this paper. It should be mentioned that the Weibull distribution is the most widely used lifetime distribution model. Because of its shape and scale parameters, the Weibull distribution can describe or approximate diverse types of lifetimes (Lawless, 2003). It is also the reason why the Weibull distribution is adopted in this paper.

Because facility failures are rectified by minimal repairs, in a continuously used environment without PM actions, the failure process of facility i is a nonhomogeneous Poisson process (NHPP) with intensity function h_i (Nakagawa and Kowada, 1983). Hence, the expected number of facility failures during the time interval [0, T] becomes $\int_0^T h_i(s)ds = H_i(T)$. However, in this intermittently used environment, the expected number of facility failures during [0, T] should be derived in another different approach.

Let T_{S_i} denote the total time serving customers during the period T. Then the expected number of facility failures becomes $E\left[N_i(\lambda;T)\right] = E\left[H_i(T_{S_i})\right]$. It turns our focus to the distribution of T_{S_i}. Under assumptions A3 and A4, since the processing time of facility i is exponentially distributed with parameter μ_i and the total number of customers entering the system is Poisson-distributed with parameter $T\lambda$, the conditional expectation of $E\left[N_i(\lambda;T)\right]$ is approximately

$$\begin{aligned} E[H_i(T_{S_i}) | X = k] &= E[(\alpha_i T_{S_i})^{\beta_i} | X = k] \\ &= \alpha_i^{\beta_i} \int_0^\infty \frac{\mu_i^k}{\Gamma(k)} t^{\beta_i + k - 1} e^{-\mu_i t} dt. \end{aligned}$$

where X follows the Poisson distribution with mean $T\lambda$. Therefore, the expected number of facility failures can be rewritten as the series

$$E\left[N_i(\lambda;T)\right] = (\alpha_i/\mu_i)^{\beta_i} \sum_{k=1}^{\infty} \frac{\Gamma(\beta_i + k)}{\Gamma(k)} e^{-T\lambda} \frac{(T\lambda)^k}{k!}. \tag{6}$$

Proposition 1. *Suppose that the lifetime of facility i is Weibull-distributed with a shape parameter* $\beta_i > 1$. *Then, the expected number of facility failures* $E[N_i(\lambda; T)]$ *is strictly increasing and strictly convex in demand rate* λ.

Proof. The analytical properties are obtained by applying the assumption $\beta_i > 1$ to the following derivatives of $E[N_i(\lambda; T)]$. That is,

$$\frac{d}{d\lambda}E[N_i(\lambda; T)]$$
$$= (\alpha_i/\mu_i)^{\beta_i} \sum_{k=1}^{\infty} \frac{\Gamma(\beta_i + k)}{\Gamma(k)} T e^{-T\lambda} \left[\frac{(T\lambda)^{k-1}}{(k-1)!} - \frac{(T\lambda)^k}{k!} \right] \quad (7)$$
$$= T\beta_i (\alpha_i/\mu_i)^{\beta_i} \sum_{k=0}^{\infty} \frac{\Gamma(\beta_i + k)}{\Gamma(k+1)} e^{-T\lambda} \frac{(T\lambda)^k}{k!} > 0, \text{ and}$$
$$\frac{d^2}{d\lambda^2}E[N_i(\lambda; T)] = T^2 \beta_i (\alpha_i/\mu_i)^{\beta_i} e^{-T\lambda}$$
$$\times \left\{ -\Gamma(\beta_i) + \sum_{k=1}^{\infty} \frac{\Gamma(\beta_i + k)}{\Gamma(k+1)} \left[\frac{(T\lambda)^{k-1}}{(k-1)!} - \frac{(T\lambda)^k}{k!} \right] \right\} \quad (8)$$
$$= T^2 \beta_i (\beta_i - 1)(\alpha_i/\mu_i)^{\beta_i} \sum_{k=0}^{\infty} \frac{\Gamma(\beta_i + k)}{\Gamma(k+2)} e^{-T\lambda} \frac{(T\lambda)^k}{k!} > 0.$$

This completes the proof.

It is worth noted that when β_i is integer-valued, $\Gamma(\beta_i + k)/\Gamma(k)$ is simply a polynomial in k, *i.e.,* $\Gamma(\beta_i + k)/\Gamma(k) = k(k+1)\cdots(k+\beta_i - 1)$, and therefore $E[N_i(\lambda; T)]$ has an explicit form which is a polynomial of $T\lambda$ of order β_i. To see this more precisely, the reader is referred to Johnson *et al*. (2005) for the details to find the moments of a Poisson distribution and apply the technique to Equation (6).

3.2 Subordinate Facility Owner's Decision: Price of Services

For a given revenue share r_1, the subordinate facility owner has to determine a service price R, which can be expressed as a function of the customer demand rate λ, to solve the following maximization problem

$$\max_{0<\lambda<\mu_1} E[\pi_1(\lambda)]$$
$$= T\left[r_1 \lambda_0^{1/a} \lambda^{1-1/a} - c_1\lambda \right] - C_{m_1} E[N_1(\lambda; T)] - C_{p_1}. \quad (9)$$

To obtain the first-order condition and the second-order condition of the maximization problem (9), the first two derivatives of $E[\pi_1(\lambda)]$ are calculated as follows:

$$\frac{d}{d\lambda}E[\pi_1(\lambda)] = T\left[(1-1/a) r_1 \lambda_0^{1/a} \lambda^{-1/a} - c_1 \right]$$
$$- C_{m_1} \frac{d}{d\lambda} E[N_1(\lambda;T)]$$
$$= T\left[(1-1/a) r_1 \lambda_0^{1/a} \lambda^{-1/a} - c_1 \right]$$
$$- C_{m_1} T \beta_1 (\alpha_1/\mu_1)^{\beta_1} \sum_{k=0}^{\infty} \frac{\Gamma(\beta_1 + k)}{\Gamma(k+1)} e^{-T\lambda} \frac{(T\lambda)^k}{k!},$$
$$\frac{d^2}{d\lambda^2}E[\pi_1(\lambda)] = -T\left(1/a - 1/a^2\right) r_1 \lambda_0^{1/a} \lambda^{-1/a-1}$$
$$- C_{m_1} \frac{d^2}{d\lambda^2} E[N_1(\lambda; T)]$$
$$= -T\left(1/a - 1/a^2\right) r_1 \lambda_0^{1/a} \lambda^{-1/a-1}$$
$$- C_{m_1} T^2 \beta_1 (\beta_1 - 1)(\alpha_1/\mu_1)^{\beta_1} \sum_{k=0}^{\infty} \frac{\Gamma(\beta_1 + k)}{\Gamma(k+2)} e^{-T\lambda} \frac{(T\lambda)^k}{k!}.$$

The second-order condition

$$\frac{d^2}{d\lambda^2}E[\pi_1(\lambda)] < 0$$

always holds since the service demand is price elastic (*i.e.,* $a > 1$) and the hazard rate h_1 is a strictly IFR (*i.e.,* $\beta_1 > 1$). To investigate the first-order condition $\frac{d}{d\lambda}E[\pi_1(\lambda)] = 0$, let

$$\xi_i(\lambda) = (1-1/a) r_i, \quad (10)$$

where

$$\xi_i(\lambda) = \left[C_{m_i} \beta_i (\alpha_i/\mu_i)^{\beta_i} \sum_{k=0}^{\infty} \frac{\Gamma(\beta_i + k)}{\Gamma(k+1)} e^{-T\lambda} \frac{(T\lambda)^k}{k!} + c_i \right] \lambda_0^{-1/a} \lambda^{1/a}$$
$$= \left\{ \frac{C_{m_i}}{T} \frac{d}{d\lambda} E[N_i(\lambda; T)] + c_i \right\} R(\lambda)^{-1}.$$

Analytical properties of ξ_i are summarized in Lemma 1 and Lemma 2.

Lemma 1. (a) ξ_i *is positive, strictly increasing, and differentiable in* λ. (b) ξ_i *admits all values between* $\xi_i(0^+)$ *and* $\xi_i(\mu_i)$.

Proof. (a) follows from direct computations of $\frac{d}{d\lambda}\xi_i$ and Proposition 1. (b) follows from the intermediate value theorem.

Lemma 2. (a) ξ_i^{-1} *is positive, strictly increasing, and differentiable.* (b) ξ_i^{-1} *admits all values between 0 and* μ_i.

It is noteworthy that the first-order condition (10) reveals the trade-off between maintenance cost and service revenue minus operating cost: The more customers served, the more service revenue. But, it also incurs higher facility deteriorating degree and thus more CM cost. On the other hand, the less customers served, the less service revenue. But, we have lower facility deteriorating degree and thus less CM cost.

The following proposition summarizes conditions for existence of the optimal solution to the maximization problem (9) in this section.

Proposition 2. *Under the conditions that the shape parameter* $\beta_1 > 1$ *and* $\xi_1(0^+) < (1-1/a)r_1 < \xi_1(\mu_1)$. *We have that* $E[\pi_1(\lambda)]$ *admits a unique maximum at*

$$\lambda_1^* = \xi_1^{-1}\left[(1-1/a)r_1\right].$$

More precisely, $E[\pi_1(\lambda)]$ *is strictly increasing on* $(0, \lambda_1^*]$, *strictly decreasing on* $[\lambda_1^*, \mu_1)$, *and strictly concave on* $(0, \mu_1)$.

The optimal solution λ_1^* has the following properties which can be verified through (10).

Corollary 1. *Suppose that the shape parameter* $\beta_1 > 1$ *and* $\xi_1(0^+) < (1-1/a)r_1 < \xi_1(\mu_1)$. *Then,*
(a) λ_1^* *is decreasing in CM cost* C_{m_1};
(b) λ_1^* *is increasing in unit revenue* r_1;
(c) λ_1^* *is decreasing in unit operating cost* c_1;

3.3 Infrastructure Owner's Decision: Allocation of Revenue Shares

As the dominated decision maker, the infrastructure owner has to determine a revenue share r_1 to the subordinate facility owner to solve the following maximization problem

$$\begin{aligned}&\max_{0<r_1<1} E[\pi_2(r_1)] \\ &= T\left[(1-r_1)\lambda_0^{1/a}\lambda^{1-1/a} - c_2\lambda\right] - C_{m_2}E[N_2(\lambda;T)] - C_{p_2}\end{aligned} \tag{11}$$

Proposition 3. *In the optimal Nash equilibrium to the Stackelberg game, the infrastructure owner's dominate strategy* r_1^* *satisfies*

$$\xi_1^{-1}\left[(1-1/a)r_1\right] \geq \xi_2^{-1}\left[(1-1/a)r_2\right].$$

Proof. According to Proposition 2, the corresponding subordinate strategy is determined by $\xi_1^{-1}\left[(1-1/a)r_1^*\right]$. Assume the contrary, $\xi_1^{-1}\left[(1-1/a)r_1\right] < \xi_2^{-1}\left[(1-1/a)r_2\right]$. Applying Lemma 2, r_1 can be increased to a larger value such that $\xi_1^{-1}\left[(1-1/a)r_1\right] = \xi_2^{-1}\left[(1-1/a)r_2\right]$. That is, this increase of revenue share r_1 reduces the infrastructure owner's expected profit $E[\pi_2(r_1)]$ without affecting the subordinate strategy, which is a contradiction. This completes the proof.

Proposition 3 indicates that the infrastructure owner's optimal individual service price (or the inverse demand of services) has no effect on the formulation of subordinate strategy. It also implies that the response function of $\lambda = \lambda(r_1)$ to revenue share r_1 is given by $\xi_1(\lambda) = (1-1/a)r_1$. Conversely, the inverse response function $r_1 = r_1(\lambda)$ is given by $r_1(\lambda) = (1-1/a)^{-1}\xi_1(\lambda)$. Therefore, substituting $r_1(\lambda) = (1-1/a)^{-1}\xi_1(\lambda)$ into the maximization problem (11), the optimization target of the infrastructure owner becomes $\max_\lambda E\{\pi_2[r_1(\lambda)]\}$, or $\max_\lambda E[\pi_2(\lambda)]$. That is,

$$\begin{aligned}E[\pi_2(\lambda)] &= E\{\pi_2[r_1(\lambda)]\} \\ &= T\left[R(\lambda) - (1-1/a)^{-1}\left\{\frac{C_{m_1}}{T}\frac{d}{d\lambda}E[N_1(\lambda;T)] + c_1\right\} - c_2\right]\lambda \\ &\quad - C_{m_2}E[N_2(\lambda;T)] - C_{p_2} \\ &= T\left[(\lambda_0/\lambda)^{1/a} - (1-1/a)^{-1}\left\{\frac{C_{m_1}}{T}\frac{d}{d\lambda}E[N_1(\lambda;T)] + c_1\right\} - c_2\right]\lambda \\ &\quad - C_{m_2}E[N_2(\lambda;T)] - C_{p_2}.\end{aligned} \tag{12}$$

The transformed problem (12) leads to the following proposition.

Proposition 4. *Assume that each lifetime of the facilities in the system is Weibull-distributed with shape parameter greater than 2. Suppose that the demand given by (1) is price elastic. Then, the infrastructure owner has a unique optimal allocation of revenue shares to the subordinate facility owners.*

Proof. To skip the tedious details, only a sketch of the proof is stated here. First, take the first order derivative and the second order derivative of $E[\pi_2(\lambda)]$ in (12) by direct computations. Then, apply the assumptions β_1, $\beta_2 > 2$ and $a > 1$ to the first-order condition and second-order condition of problem (12), the result follows as stated in this proposition.

Finally, we state a concluding remark that a unique optimal Nash equilibrium for the Stackelberg game in this service channel can be achieved under the analytical conditions in Propositions 2 and 4.

4. Numerical Example

To illustrate a more concrete application of the main results in this paper, the special case with shape parameters $\beta_1 = \beta_2 = 3$, scale parameters $\alpha_1 = \alpha_2 = \alpha$, and mean processing rates $\mu_1 = \mu_2 = \mu$ is considered in this section. The above conditions imply that both facilities obey a same simple failure pattern so that both profit functions (maximization targets) also admit explicit forms.

To present a more practical demonstration, a series of direct computations corresponding to the solution procedures in Section 3 is carried out as follows. Since $\Gamma(\beta_i + k)/\Gamma(k) = k(k+1)(k+2)$ for $\beta_i = 3$, $E[X] = T\lambda$, $E[X^2] = T\lambda(1+T\lambda)$, and $E[X^3] = T\lambda\left[1 + 3T\lambda + (T\lambda)^2\right]$ for a random variable $X \sim Poi(T\lambda)$, the expected number of facility failures (6) reduces to the explicit form

$$E[N_i(\lambda;T)] = (\alpha/\mu)^3 \sum_{k=1}^{\infty} k(k+1)(k+2)e^{-T\lambda}\frac{(T\lambda)^k}{k!} \tag{13}$$

$$=(\alpha/\mu)^3\left[(T\lambda)^3+6(T\lambda)^2+6(T\lambda)\right].$$

Hence, the expected profit function (3) also reduces to

$$E\left[\pi_i(\lambda)\right]=T\left[r_i\lambda_0^{1/a}\lambda^{1-1/a}-c_i\lambda\right] \\ -C_{m_i}(\alpha/\mu)^3\left[(T\lambda)^3+6(T\lambda)^2+6(T\lambda)\right]-C_{p_i}. \quad (14)$$

Now the first-order condition $\frac{d}{d\lambda}E\left[\pi_1(\lambda)\right]=0$ of the subordinate facility owner's decision problem (optimal service price) becomes

$$(1-1/a)r_1 \\ =\left\{c_1+C_{m_1}(\alpha/\mu)^3\left[3(T\lambda)^2+12(T\lambda)+6\right]\right\}(\lambda/\lambda_0)^{\frac{1}{a}}. \quad (15)$$

Substituting the inverse response function $r_1=r_1(\lambda)$ obtained from (15) into the expected profit function (5) of the infrastructure owner, and then combining it with (14), the decision problem (optimal revenue share) is transformed into

$$E\left[\pi_2(\lambda)\right]=E\left\{\pi_2\left[r_1(\lambda)\right]\right\} \\ =T\left\{\left[1-r_1(\lambda)\right]R(\lambda)-c_2\right\}\lambda-C_{m_2}E\left[N_2(\lambda;T)\right]-C_{p_2} \\ =-C_{p_2}+T\lambda_0^{1/a}\lambda^{1-1/a} \\ -(\alpha/\mu)^3\left[3(1-1/a)^{-1}C_{m_1}+C_{m_2}\right](T\lambda)^3 \\ -(\alpha/\mu)^3\left[12(1-1/a)^{-1}C_{m_1}+6C_{m_2}\right](T\lambda)^2 \\ -\left\{\left[(1-1/a)^{-1}c_1-c_2\right]+(\alpha/\mu)^3\left[6(1-1/a)^{-1}C_{m_1}+6C_{m_2}\right]\right\}(T\lambda).$$

Hence the first-order condition $\frac{d}{d\lambda}E\left\{\pi_2\left[r_1(\lambda)\right]\right\}=0$ is

$$(1-1/a)\lambda_0^{1/a}\lambda^{-1/a} \\ =3(\alpha/\mu)^3\left[3(1-1/a)^{-1}C_{m_1}+C_{m_2}\right](T\lambda)^2 \\ +12(\alpha/\mu)^3\left[2(1-1/a)^{-1}C_{m_1}+C_{m_2}\right](T\lambda) \\ +\left\{\left[(1-1/a)^{-1}c_1-c_2\right]+6(\alpha/\mu)^3\left[(1-1/a)^{-1}C_{m_1}+C_{m_2}\right]\right\}. \quad (16)$$

Now it is proceeded to a numerical solution realized at the following parameters: a = 4, λ_0 = 75, α = 4, μ = 100, T = 0.5 (month), c_1 = 0.1, C_{m_1} = 0.5, C_{p_1} = 0.1, c_2 = 0.2, C_{m_2} = 1, and C_{p_2} = 0.1, where the monetary unit is thousand dollars. The numerical computations are carried out by Wolfram Mathematica®.

As a leader, the infrastructure owner moves first. The first-order condition (16) corresponding to her action predicts that the subordinate facility owner will choose the inverse price λ^* = 61.1540. Substituting the inverse price to the inverse response function $r_1=r_1(\lambda)$ given by (15), her optimal revenue share is admitted at r_1^* = 0.3200. Finally, the infrastructure owner's maximal expected profit is $E\left[\pi_2\right]$ = 15.5047.

As a follower, the subordinate facility owner accepts the revenue share r_1^* = 0.3200 as provided. Hence her reaction determined by the first-order condition (15) gives the inverse price λ^* = 61.1540, which is exactly as predicted by the infrastructure owner. Transformed by the log-linear demand (1), her optimal service price is realized at $R^*=\left(\lambda_0/\lambda^*\right)^{1/a}$ = 1.0524. Finally, the subordinate facility owner's maximal expected profit is $E\left[\pi_1\right]$ = 5.4876.

The decision process of the two parties in this demonstrated decentralized service channel is summarized as follows. At first, knowing that the subordinate facility owner will follow her allocation on revenue shares, the infrastructure owner chooses the revenue shares r_1^* = 0.32 and r_2^* = 0.68 to realize her maximal expected profit $E\left[\pi_2\right]$ = 15.5047. Then, after observing the allocation of revenue shares, the subordinate facility owner takes the service price R^* = 1.0524 to obtain her maximal expected profit $E\left[\pi_1\right]$ = 5.4876. Therefore, the Nash equilibrium for the Stackelberg game is achieved as above.

5. CONCLUSION

This paper investigates a Stackelberg game in a decentralized service channel under revenue-sharing contracts. Significantly different from earlier studies, maintenance and depreciation costs of service facilities are considered in this paper. Mathematical models of the actions and reactions of the two parties in this system are developed and analyzed. Then analytical conditions are proposed to achieve a unique optimal Nash equilibrium. Finally, a numerical example is presented to illustrate the realization of the Nash equilibrium.

Although this study uses theoretical approach and no empirical data are acquired from the industries to illustrate the main results, our result still has its practical aspect that comes from the Weibull lifetime assumption because diverse types of facility lifetimes can be fitted by the Weibull distribution with various shape and scale parameters. However, the optimal equilibrium is somewhat complicated due to the intrinsic properties of a queuing process. An ad hoc approximation would be more straightforward to implement the solution process.

For future research, it is worth exploring coordination mechanisms for such decentralized service channel to improve its economic efficiency. It is also noteworthy that the Cournot-type game should be employed instead when the two players have equal power over the allocation of revenue shares. This study can also be extended in many other perspectives such as facilities with general-distributed lifetime, a number of subordinate facility owners, nonhomogeneous Poisson customer arrivals, heterogeneous customers, variable CM costs, and so forth.

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