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Estimation of Change Point in Process State on CUSUM $(\bar{x}, s)$ Control Chart

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Abstract. Control charts are used to distinguish between chance and assignable causes in the variability of quality characteristics. When a control chart signals that an assignable cause is present, process engineers must initiate a search for the assignable cause of the process disturbance. Identifying the time of a process change could lead to simplifying the search for the assignable cause and less process down time, as well as help to reduce the probability of incorrectly identifying the assignable cause. The change point estimation by likelihood theory and the built-in change point estimation in a control chart have been discussed until now. In this article, we discuss two kinds of process change point estimation when the CUSUM $(\bar{x}, s)$ control chart for monitoring process mean and variance simultaneously is operated. Throughout some numerical experiments about the performance of the change point estimation, the change point estimation techniques in the CUSUM $(\bar{x}, s)$ control chart are considered.

Keywords: Statistical Process Control, $(\bar{x}, s)$ Control Chart, CUSUM Charting Technique, Maximum Likelihood Estimator, Change Point Estimation

1. INTRODUCTION

Control charts play an important role in statistical process control tools (Montgomery, 2005). Control charts are used to distinguish between chance and assignable causes in the variability of quality characteristics. When a control chart signals that an assignable cause is present, process engineers must initiate a search for the assignable cause of the process disturbance. The control chart frequently takes long time to issue the signal of the change in the process after the change in the process state actually occurred. Therefore, the process engineers should identify the moment when the process state has changed into an out-of-control state at first. Consequently, this can simplify the search for the assignable cause.

Various types of control charts have been developed until now. In particular, cumulative sum (CUSUM) and exponential weighted moving average (EWMA) are well known charting techniques (Hawkins and Olwell, 1998; Woodall and Montgomery, 1999; Montgomery, 2005). The CUSUM control chart and the EWMA control chart are sensitive to a slight change of the process because not only the information about the process obtained by the last sampling observation but also a lot of information provided by a sequence of previous observations is efficiently utilized.

The CUSUM and EWMA control charts have built-in change point estimators that can be used to provide an estimate of the process change point from the behavior of the past plots on the control chart (Pignatiello and Samuel, 2001). Nishina (1992) has reported that the CUSUM control chart is superior in identifying the moment of the process change although these control charts have the similar performance with respect to detecting the process change. Samuel, Pignatiello, and Calvin (1998) have suggested the use of a maximum likelihood estimator (MLE) of the process change point after a Shewhart control chart issued a signal. The estimation of change point using the

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likelihood theory is applied to various types of control charts such as CUSUM and EWMA control charts. Then, Pignatiello and Samuel (2001a) have compared the performance of the MLE and the build-in estimator for each control chart. As the result, they have shown that the MLE is almost superior to the built-in estimator in identifying the change point. The change point estimation with respect to the process mean has been discussed in the literature mentioned above. The change point estimation with respect to the process variance has been discussed using a MLE (Samuel and Pignatiello, 1998; Khoo and Lee, 2006).

When process quality characteristics is normally distributed, we use the $\bar{x}$ control chart together with the s or R control chart in order to monitor simultaneously the change in the process mean and variance. Similarly, suppose that we monitor the process state by using both the CUSUM $\bar{x}$ control chart and the CUSUM s control chart. However, it is no longer easy to develop the simultaneous usage of those CUSUM control charts in order that the prescribed first kind of error is satisfied. Kanagawa, Arizono and Ohta (1997) have developed the $(\bar{x}, s)$ control chart which enables process engineers to monitor the shift in the process mean and the change in the process variance simultaneously using one statistic composed of the estimators of mean and variance. The magnitude of difference between an actual and an ideal process states is measured using Kullback-Leibler (K-L) information in the $(\bar{x}, s)$ control chart. Hence, note that the $(\bar{x}, s)$ control chart is not equivalent to the simple union of $\bar{x}$ and s control charts. In the viewpoint of the simultaneous observation of mean and variance in the process, it would be relatively easy to apply the CUSUM charting technique into the $(\bar{x}, s)$ control chart in comparison with the development of the simultaneous usage of both CUSUM $\bar{x}$ and s control charts. Takemoto, Watakabe and Arizono (2003) have developed the CUSUM $(\bar{x}, s)$ control chart using a mask form. Then, Takemoto and Arizono (2007) have proposed the CUSUM $(\bar{x}, s)$ control chart using a tabular form. As mentioned above, CUSUM control charts already have a technique of estimating the change point in the charting procedure. However, the built-in change point estimation of the respective CUSUM $(\bar{x}, s)$ control charts has not been yet discussed.

In this article, we formulate the change point estimation by the MLE for both change in the process mean and variance at first. Then, the CUSUM $(\bar{x}, s)$ control chart is employed for monitoring the process mean and variance simultaneously and we discuss the built-in change point estimation in the two kinds of CUSUM $(\bar{x}, s)$ control charts. Further, we compare the change point estimation by the likelihood theory with the built-in change point estimation in the CUSUM $(\bar{x}, s)$ control chart. Throughout some numerical experiments about the performance of the change point estimation, the change point estimation techniques in the CUSUM $(\bar{x}, s)$ control chart are discussed.

2. PROCESS STEP CHANGE MODEL

We assume that the process is initially in control with observations coming from a normal distribution with a known mean μ_0 and a known variance σ_0^2. After an unknown point in time the process state changes from (μ_0, σ_0^2) to (μ_1, σ_1^2), where μ_1 and σ_1^2 are unknown, respectively. We also assume that once this step change in the process state occurs, the process remains at an out-of-control state (μ_1, σ_1^2) until the assignable cause has been identified and removed.

Then, denote the size of samples by n. And then, t expresses the sampling turn. We assume that the control chart signals at Tth sampling turn and that this signal is not a false alarm. Then, let τ be the last sampling turn in the in-control state, where τ is unknown. Further, x_{ij} denotes the quality characteristic of jth sample in the ith sampling turn.

We formulate the change point estimation by the likelihood theory. Because the process state is in-control until τ and out-of-control after $\tau+1$, the log-likelihood $L(\tau)$ is obtained as follows:

$$L(\tau) = \log \prod_{j=1}^{n} \left\{ \prod_{i=1}^{\tau} f(x_{ij};\ \mu_0,\ \sigma_0^2) \prod_{i=\tau+1}^{T} f(x_{ij};\ \mu_1,\ \sigma_1^2) \right\}$$

$$= -\frac{n\tau}{2}\log 2\pi\sigma_0^2 - \sum_{i=1}^{\tau}\sum_{j=1}^{n}\frac{(x_{ij}-\mu_0)^2}{2\sigma_0^2}$$

$$-\frac{n(T-\tau)}{2}\log 2\pi\sigma_1^2 - \sum_{i=\tau+1}^{T}\sum_{j=1}^{n}\frac{(x_{ij}-\mu_1)^2}{2\sigma_1^2}. \qquad (1)$$

From Eq.(1), we have the maximum likelihood estimators $\bar{x}$ and s^2 for μ_1 and σ_1^2 as follows:

$$\bar{x} = \frac{1}{n(T-\tau)}\sum_{i=\tau+1}^{T}\sum_{j=1}^{n} x_{ij}, \qquad (2)$$

$$s^2 = \frac{1}{n(T-\tau)}\sum_{i=\tau+1}^{T}\sum_{j=1}^{n}(x_{ij}-\bar{x})^2. \qquad (3)$$

By Eqs.(1), (2), and (3), the change point estimator $\hat{\tau}_{MLE}$ by maximum likelihood estimators is such that

$$\hat{\tau}_{MLE} = \underset{1\le\tau<T}{\arg\max}\left\{ -\frac{n(T-\tau)}{2} - \frac{n\tau}{2}\log\left(2\pi\sigma_0^2\right) - \sum_{i=1}^{\tau}\sum_{j=1}^{n}\frac{\left(x_{ij}-\mu_0\right)^2}{2\sigma_0^2} - \frac{n(T-\tau)}{2}\log\left(2\pi s^2\right)\right\}. \qquad (4)$$

3. CUSUM $(\bar{x}, s)$ CONTROL CHARTS

In this section, we show the outline of the Mask and Tabular CUSUM $(\bar{x}, s)$ control charts. The CUSUM

$(\bar{x}, s)$ control charts employ common statistic based on the K-L information (Kullback, 1959) for the purpose of monitoring the change in the process mean and variance. The common statistic is given as follows:

$$\lambda_i = \frac{n}{2}\left\{\log\frac{s_i^2}{\sigma^2} - 1 + \frac{\sigma^2}{s_i^2} + \frac{(\bar{x}_i - \mu)^2}{\sigma^2}\right\}, \quad (5)$$

where

$$\bar{x}_i = \frac{1}{n}\sum_{j=1}^{n} x_{ij}, \quad (6)$$

$$s_i^2 = \frac{1}{n}\sum_{j=1}^{n}(x_{ij} - \bar{x}_i)^2. \quad (7)$$

The derivation of Eq. (5) is described in Kanagawa, Arizono and Ohta (1997) and Watakabe and Arizono (1999).

On designing and evaluating CUSUM control charts, the average run length (ARL) is employed as a criterion instead of a probabilistic criterion because the probability of first and second kinds of errors are not fixed in the respective sampling turns (Goldsmith and Whitefield, 1961). The ARL means the average number of samplings before an out-of-control signal is indicated. ARL_0 is defined as the design criterion parameter instead of the probability of first kind of error. Therefore, the chart parameters are decided in order that the specified ARL_0 is satisfied.

3.1 Mask CUSUM $(\bar{x}, s)$ Control Chart

In this subsection, we describe the Mask CUSUM $(\bar{x}, s)$ control chart. Since Takemoto, Watakebe and Arizono (2003) have developed the Mask CUSUM $(\bar{x}, s)$ control chart, we show the outline of their result. In the Mask CUSUM $(\bar{x}, s)$ control chart, the plotted statistic is defined using the statistic in Eq. (5) as follows:

$$\Lambda_{MASK}[t] = \sum_{i=1}^{t}\lambda_i, \quad (8)$$

where t means the present sampling turn. Then, the process state is judged to be in-control or not by using a mask in Figure 1. θ_{MASK} and d_{MASK} mean the mask parameters and are defined as:

$$\begin{cases}\theta_{MASK} = \angle ABC \\ d_{MASK} = \overline{AB}\end{cases}, \quad (9)$$

where the points A, B, and C express the last plotted point, the point at a distance d_{MASK} ahead of the point A, and the intersection of the line crossing the segment AB at an acute angle θ_{MASK} and the horizontal axis. The K-L information means the magnitude of difference between the present process state and the in-control state. The process state is in-control until τ and out-of-control after $\tau+1$. From this point, the statistic $\lambda_i,\ i > \tau$ is relatively larger as compared to the statistic $\lambda_i,\ i \le \tau$. Then, the slope of the CUSUM statistic $\Lambda_{MASK}[t]$ after $\tau+1$ is conspicuous in comparison with the slope of the CUSUM statistic before $\tau+1$. Hence, the point $\Lambda_{MASK}[t]$ plotted before τ falls outside the mask due to the conspicuous change of the statistic $\Lambda_{MASK}[t]$ after $\tau+1$.

Generally as a rule for judgment in tth sampling turn, if a series of points from the zero point to the latest point

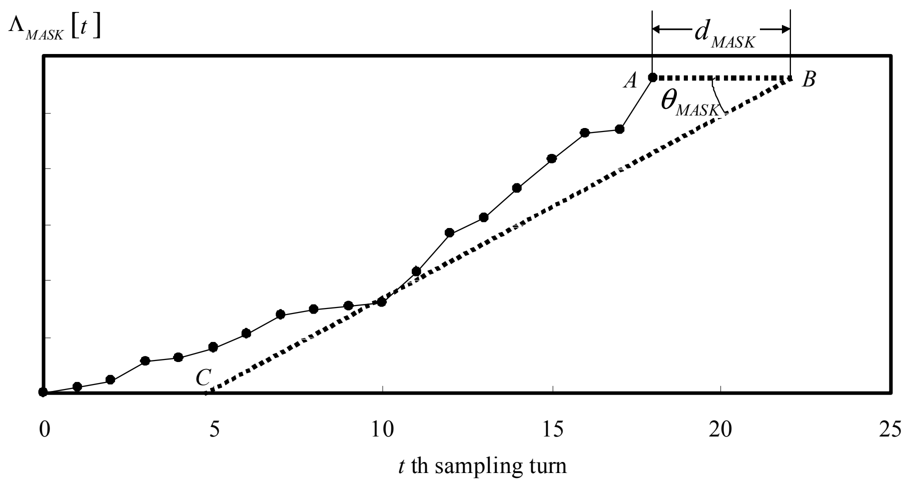

Figure 1. Mask CUSUM $(\bar{x}, s)$ control chart.

Table 1. Chart parameters in the Mask CUSUM $(\bar{x}, s)$ control chart for given $ARL_0 = 200$.

θ_{MASK}	50°	55°	60°	65°	70°	75°	80°
d_{MASK}	15.77	6.87	4.01	2.51	1.53	0.79	0.16

is inside the mask formed by chart parameters θ_{MASK} and d_{MASK}, in other words, is above the segment BC, then the process engineer judges that the present process is in-control and continues sampling. Note that the mask for the statistic $\Lambda_{MASK}[t]$ is renewed every sampling turn. Otherwise, the process engineer judges that the present process is not in-control and stops sampling.

As mentioned above, the CUSUM control chart employs ARL as the design and performance criteria. Therefore, ARL_0 is specified as the design criterion and the chart parameters θ_{MASK} and d_{MASK} are given in order that the specified ARL_0 is satisfied in the Mask CUSUM $(\bar{x},\ s)$ control chart. As an example without loss of generality, Table 1 illustrates the chart parameters θ_{MASK} and d_{MASK} for given $N(\mu_0,\ \sigma_0^2) = N(0.0,\ 1.0^2)$ and $ARL_0 = 200$.

Next, we discuss the built-in change point estimation in the Mask CUSUM $(\bar{x},\ s)$ control chart. In the CUSUM control chart using a mask form, the present plotted point is a standard point of detecting the departure of process from the in-control state, and then the process engineer judges the process state by going back to each previous point in turn. Assume that the control chart signaled and detected the process change at Tth sampling turn. Further, assume that the plotted point $\Lambda_{MASK}[i],\ t < i < T$ is above the segment BC and then $\Lambda_{MASK}[t]$ is below the segment BC. The engineer would understand that the change of the CUSUM statistic after $t + 1$ is such as to exceed the range of his expectation. As the result, the process state is judged to be not in-control. Simultaneously, the engineer estimates that t is the last sampling turn in the in-control state. In other word, the built-in change point estimation in the Mask CUSUM $(\bar{x},\ s)$ control chart is defined as follows:

$$\hat{\tau}_{MASK} = \max_{1 \le t < T} \{ t \mid \Lambda_M[t] \text{ is below the segement } BC \}. \quad (10)$$

3.2 Tabular CUSUM $(\bar{x},\ s)$ Control Chart

In this subsection, we describe the Tabular CUSUM $(\bar{x},\ s)$ control chart (Takemoto and Arizono, 2007). The plotted statistic is defined as:

$$\Lambda_{TAB}[t] = \max\{0,\ \lambda_t - r.v. + \Lambda_T[t-1]\}, \quad (11)$$

where "$r.v.$" means a reference value and it is a constant. The reference value is often chosen based on the expectation and variance of the original statistic. The expectation and variance of the statistic in Eq. (5) under the in-control condition are obtained as follows:

$$E[\lambda_i] = -\frac{n}{2}\varphi^{(0)}\left(\frac{n-1}{2}\right) + \frac{n}{2}\log\frac{n}{2}, \quad (12)$$

$$Var[\lambda_i] = \left(\frac{n}{2}\right)^2 \varphi^{(1)}\left(\frac{n-1}{2}\right) - \frac{n}{2}. \quad (13)$$

The derivation of Eq.(12) and Eq. (13) is described in Kanagawa, Arizono and Ohta (1997) and Watakabe and Arizono (1999). Using the expectation in Eq. (12) and variance in Eq. (13), the reference value is given as follows:

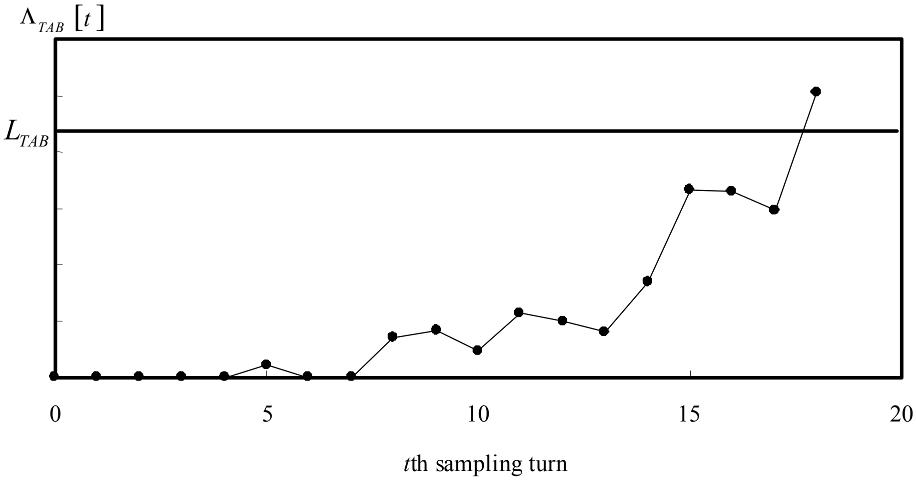

Figure 2. Tabular CUSUM $(\bar{x},\ s)$ control chart.

Table 2. chart parameters in the Tabular CUSUM $(\bar{x},\ s)$ control chart for given $ARL_0 = 200$.

ℓ_{TAB}	0.00	0.25	0.50	0.75	1.00	1.25	1.50
L_{TAB}	15.85	8.34	6.34	5.34	4.67	4.15	3.71

$$r.v. = E[\lambda_i] + \ell_{TAB}\sqrt{Var[\lambda_i]}, \quad (14)$$

where ℓ_{TAB} implies one of the chart parameters in the Tabular CUSUM $(\bar{x}, s)$ control chart.

Then, the process state is judged whether the in-control state or not by comparing the statistic in Eq. (11) with the control limit L_{TAB}, which implies one of the chart parameters in this chart and the width of control limit. As an outline of the Tabular CUSUM $(\bar{x}, s)$ control chart, Figure 2 is illustrated.

It is found from Eq. (11) that the accumulated information $\Lambda_{TAB}[t-1]$ is used on judgment in the tth sampling turn when the plotted statistic $\Lambda_{TAB}[t-1]$ is greater than the reference value. In contrast, the accumulated information $\Lambda_{TAB}[t-1]$ is canceled when $\Lambda_{TAB}[t-1]$ is smaller than the reference value. Incidentally, it is known that this chart doesn't essentially need charting and it is easy to operate.

The chart parameters ℓ_{TAB} and L_{TAB} are given in order to satisfy the specified ARL_0. As an example without loss of generality, Table 2 illustrates the chart parameters ℓ_{TAB} and L_{TAB} for given $N(\mu_0, \sigma_0^2) = N(0.0, 1.0^2)$ and $ARL_0 = 200$.

Next, we discuss the built-in change point estimation in the Tabular CUSUM $(\bar{x}, s)$ control chart. In the traditional CUSUM $\bar{x}$ control chart for process mean, the change point estimator is given by the maximum value of t at which the plotted CUSUM statistic is 0 (see Pignatiello and Samuel 2001). When a sequence of the plotted CUSUM statistics which are greater than 0 is obtained from the process, the original statistics $\bar{x}$ would almost exceed the reference value. Consequently, process engineers would guess that the process state has been just changed into an out-of-control state. Therefore, the change point of the process is supposed to be on the eve of a sequence of the plotted CUSUM statistics which is larger than 0. Similarly, a sequence of the plotted CUSUM statis-

Table 3. The estimated probabilities (%) of $\hat{\tau}_{MLE} = \tau$ and $\hat{\tau}_{MASK} = \tau$ for respective methods of estimating the process change point after a signal from a Mask CUSUM $(\bar{x}, s)$ control chart.

θ_{MASK}	μ_1	0.0	0.0	0.5	0.5	0.5	1.0	1.0	1.0
	σ_1	1.25	1.50	1.00	1.25	1.50	1.00	1.25	1.50
50.0	$E(T)$	138.30	112.42	119.90	114.34	108.28	105.63	105.27	104.34
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.38	33.71	28.18	31.63	43.06	59.25	56.84	57.89
	$\hat{P}(\hat{\tau}_{MASK} = \tau)$	1.91	5.15	3.83	4.84	8.04	12.03	12.48	14.63
55.0	$E(T)$	143.33	110.71	119.35	112.84	106.76	104.46	104.19	103.44
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.24	32.37	27.08	30.52	40.97	53.79	53.26	54.09
	$\hat{P}(\hat{\tau}_{MASK} = \tau)$	2.98	11.69	7.68	10.56	17.70	28.76	28.08	30.18
60.0	$E(T)$	151.32	110.71	122.98	113.60	106.59	104.10	103.90	103.19
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.58	32.12	26.42	31.00	40.64	53.86	52.61	52.76
	$\hat{P}(\hat{\tau}_{MASK} = \tau)$	2.23	11.85	5.41	9.35	18.11	32.93	30.71	34.28
65.0	$E(T)$	159.23	112.18	129.94	115.67	106.96	104.28	103.85	103.14
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.04	32.00	27.88	30.48	40.60	55.77	52.76	54.27
	$\hat{P}(\hat{\tau}_{MASK} = \tau)$	1.76	8.99	3.74	7.01	16.42	30.58	29.79	34.51
70.0	$E(T)$	167.88	113.70	139.29	118.63	107.67	104.93	104.30	103.29
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.84	32.98	28.14	31.78	41.58	57.87	54.78	55.12
	$\hat{P}(\hat{\tau}_{MASK} = \tau)$	1.48	7.46	2.65	5.56	13.88	24.07	25.23	32.00
75.0	$E(T)$	171.96	115.08	146.76	121.31	108.67	106.12	104.77	103.47
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	14.28	33.51	29.11	31.78	42.85	61.01	55.89	55.59
	$\hat{P}(\hat{\tau}_{MASK} = \tau)$	1.52	6.18	1.90	4.33	12.09	17.87	21.39	29.12
80.0	$E(T)$	174.32	116.52	151.70	122.99	109.40	107.38	105.37	103.78
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.31	33.13	28.11	31.25	42.43	61.24	56.53	58.16
	$\hat{P}(\hat{\tau}_{MASK} = \tau)$	1.49	6.08	1.89	4.57	10.24	13.49	18.75	26.28

tics $\Lambda_{TAB}[t]$ which are larger than 0 is considered to be a sign of the out-of-control condition. Therefore, the change point of the process is supposed to be on the eve of a sequence of the plotted CUSUM statistics $\Lambda_{TAB}[t]$. After all, we suggest that the estimator of the change point $\hat{\tau}_{TAB}$ is such that

$$\hat{\tau}_{TAB} = \max_{1 \le t < T} \{ t \mid \Lambda_{TAB}[t] = 0 \} . \quad (15)$$

4. NUMERICAL EXPERIMENTS

In this section, we compare the performance of the MLE change point estimator $\hat{\tau}_{MLE}$ from Eq. (4) with that of $\hat{\tau}_{MASK}$ from Eq. (10) when the Mask CUSUM $(\bar{x}, s)$ control chart signals a change. Then, we compare the performance of the MLE change point estimator $\hat{\tau}_{MLE}$ from Eq. (4) with that of $\hat{\tau}_{TAB}$ from Eq. (15) when the Tabular CUSUM $(\bar{x}, s)$ control chart signals a change.

4.1 Change Point Estimators Used with Mask CUSUM $(\bar{x}, s)$ Control Chart

We consider the Mask CUSUM $(\bar{x}, s)$ control charts with respective parameters $(\theta_{MASK}, d_{MASK})$. We use Monte Carlo simulation to investigate the performance of the change point estimators. The change point is simulated at $\tau = 100$. Observations are generated from the in-control state (μ_0, σ_0^2) for respective sampling turns 1, 2, ⋯, 100. Then, starting with sampling turn 101, observations are generated from an out-of-control state (μ_1, σ_1^2) until the Mask CUSUM $(\bar{x}, s)$ control chart produces a signal. At that point, both $\hat{\tau}_{MLE}$ and $\hat{\tau}_{MASK}$ are computed. This procedure is repeated a total of 10,000 times for every out-of-control state. The estimated probabilities of $\hat{\tau}_{MLE} = \tau$ and $\hat{\tau}_{MASK} = \tau$ obtained using each of the estimator form 10,000 simulation runs are computed and shown in Tables 3 and 4. Also, we show $E[T]$, the expected sampling turn at which the control chart signals a change in the process state that occurred at period τ. Then, $E[T]$ = ARL+τ, where ARL is differed by the magnitude of change. It is known that small θ_{MASK} has the superior power when the magnitude of change is relatively small while large θ_{MASK} has the superior power when the magnitude of change is relatively large (Takemoto, Watakabe, and Arizono, 2003).

Table 3 shows the following:

i) $\hat{P}(\hat{\tau}_{MLE} = \tau)$ is almost constant regardless of chart parameters for a given (μ_1, σ_1^2).

ii) $\hat{P}(\hat{\tau}_{MASK} = \tau)$ is concerned with the chart parameters. This is the reason why the built-in change point estimation depends on the chart parameters.

iii) $\hat{\tau}_{MLE}$ yields better results than $\hat{\tau}_{MASK}$ in term of accuracy.

Then, the observed frequency with which the change point estimators are within a given number of intervals of the actual change point is shown in Table 4. This provides an indication of the precision of each estimator. As the result, $\hat{\tau}_{MLE}$ yields better results than $\hat{\tau}_{MASK}$.

Overall, $\hat{\tau}_{MLE}$ yields better results than $\hat{\tau}_{MASK}$ in term of accuracy and precision. We have examined the performance of these estimators for some τ except $\tau = 100$. Then, we have confirmed the effectiveness of the MLE procedure in the change point estimation.

Table 4. Precision of $\hat{\tau}_{MLE}$ and $\hat{\tau}_{MASK}$ when used with a Mask CUSUM $(\bar{x}, s)$ control chart with $\theta_{MASK} = 60.0$.

μ_1	0.0	0.0	0.5	0.5	0.5	1.0	1.0	1.0
σ_1	1.25	1.50	1.00	1.25	1.50	1.00	1.25	1.50
$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.58	32.12	26.42	31.00	40.64	53.86	52.61	52.76
$\hat{P}(\lvert\hat{\tau}_{MLE} - \tau\rvert \le 1)$	27.51	54.52	47.91	53.13	63.61	73.07	73.20	74.32
$\hat{P}(\lvert\hat{\tau}_{MLE} - \tau\rvert \le 2)$	37.91	67.03	60.24	65.58	75.37	82.65	83.51	84.84
$\hat{P}(\lvert\hat{\tau}_{MLE} - \tau\rvert \le 3)$	45.99	75.17	68.51	73.75	82.59	88.56	89.37	90.52
$\hat{P}(\lvert\hat{\tau}_{MLE} - \tau\rvert \le 4)$	52.11	81.18	74.47	79.06	87.24	92.47	92.94	94.37
$\hat{P}(\lvert\hat{\tau}_{MLE} - \tau\rvert \le 5)$	57.40	85.26	79.21	83.54	90.59	94.88	95.21	96.41
$\hat{P}(\hat{\tau}_{MASK} = \tau)$	2.23	11.85	5.41	9.35	18.11	32.93	30.71	34.28
$\hat{P}(\lvert\hat{\tau}_{MASK} - \tau\rvert \le 1)$	5.25	24.55	12.54	20.19	35.95	58.62	55.80	59.68
$\hat{P}(\lvert\hat{\tau}_{MASK} - \tau\rvert \le 2)$	7.94	34.63	18.90	29.21	49.13	73.69	71.24	75.35
$\hat{P}(\lvert\hat{\tau}_{MASK} - \tau\rvert \le 3)$	10.27	43.17	24.20	36.91	59.67	83.17	81.32	85.00
$\hat{P}(\lvert\hat{\tau}_{MASK} - \tau\rvert \le 4)$	12.23	50.46	28.73	43.10	67.85	89.19	87.66	90.62
$\hat{P}(\lvert\hat{\tau}_{MASK} - \tau\rvert \le 5)$	21.33	58.52	44.25	55.12	71.29	80.66	81.41	85.05

4.2 Change Point Estimators Used with Tabular CUSUM $(\bar{x}, s)$ Control Chart

We consider Tabular CUSUM $(\bar{x}, s)$ control charts with respective parameters (ℓ_{TAB}, L_{TAB}). We use Monte Carlo simulation to compare the change point estimators $\hat{\tau}_{MLE}$ and $\hat{\tau}_{TAB}$. The change point is simulated at $\tau = 100$. Observations are generated from the in-control state (μ_0, σ_0^2) for respective sampling turns 1, 2, $\cdots$, 100. Then, starting with sampling turn 101, observations are generated from an out-of-control state (μ_1, σ_1^2) until the Tabular CUSUM $(\bar{x}, s)$ control chart produces a signal.

Table 5 shows $E[T]$, $\hat{P}(\hat{\tau}_{MLE} = \tau)$, and $\hat{P}(\hat{\tau}_{TAB} = \tau)$. $E[T]$ is differed by the magnitude of change. It is known that small ℓ_{TAB} has the superior power when the magnitude of change is relatively small while large ℓ_{TAB} has the superior power when the magnitude of change is relatively large (Takemoto and Arizono, 2007).

Table 5 shows the following:

i) $\hat{P}(\hat{\tau}_{MLE} = \tau)$ is almost constant regardless of chart parameters for a given (μ_1, σ_1^2).

ii) $\hat{P}(\hat{\tau}_{TAB} = \tau)$ is concerned with the chart parameters. This is the reason why the built-in change point estimation depends on the chart parameters.

iii) $\hat{\tau}_{MLE}$ yields better results than $\hat{\tau}_{TAB}$ in term of accuracy.

Then, the observed frequency with which the change point estimators are within a given number of intervals of the actual change point is shown in Table 6. Again, $\hat{\tau}_{MLE}$ yields better results than $\hat{\tau}_{TAB}$.

Overall, $\hat{\tau}_{MLE}$ yields better results than $\hat{\tau}_{TAB}$ in term of accuracy and precision. We have examined the performance of these estimators for some τ except $\tau = 100$. Then, we obtained the same results as Tables 5 and 6.

5. CONCLUSION

In this article, we have discussed the change point estimation in the CUSUM $(\bar{x}, s)$ control charts for monitoring process mean and variance simultaneously. At first, we have formulated the change point estimation by the MLE for both change in the process mean and variance. Then, when the CUSUM $(\bar{x}, s)$ control chart has

Table 5. The estimated probabilities (%) of $\hat{\tau}_{MLE} = \tau$ and $\hat{\tau}_{TAB} = \tau$ for respective methods of estimating the process change point after a signal from a Tabular CUSUM $(\bar{x}, s)$ control chart.

ℓ_{TAB}	μ_1	0.0	0.0	0.5	0.5	0.5	1.0	1.0	1.0
	σ_1	1.25	1.50	1.00	1.25	1.50	1.00	1.25	1.50
0.00	$E(T)$	139.63	111.72	119.33	114.01	107.84	105.25	104.93	104.10
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.24	33.17	27.44	31.25	41.82	57.98	56.23	57.84
	$\hat{P}(\hat{\tau}_{TAB} = \tau)$	2.11	5.56	4.37	5.28	6.54	9.23	9.04	8.95
0.25	$E(T)$	146.75	110.64	120.50	112.88	106.57	104.26	104.02	103.31
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	14.13	32.50	27.17	30.71	40.83	54.80	52.55	54.48
	$\hat{P}(\hat{\tau}_{TAB} = \tau)$	3.12	14.21	8.58	12.53	19.83	31.64	29.08	28.02
0.50	$E(T)$	153.28	111.25	124.85	114.11	106.67	104.15	103.84	103.14
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.37	31.96	27.66	30.32	41.51	55.33	51.88	53.02
	$\hat{P}(\hat{\tau}_{TAB} = \tau)$	2.51	14.01	6.86	11.69	23.15	39.66	35.63	34.91
0.75	$E(T)$	159.66	112.14	130.37	115.55	106.97	104.32	103.92	103.16
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.20	32.98	28.31	31.70	41.56	56.95	53.68	54.90
	$\hat{P}(\hat{\tau}_{TAB} = \tau)$	2.02	11.85	4.63	9.73	21.51	40.10	36.08	37.37
1.00	$E(T)$	164.62	113.20	135.61	117.10	107.31	104.57	104.07	103.19
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.62	32.93	28.43	30.86	40.85	56.91	53.53	54.30
	$\hat{P}(\hat{\tau}_{TAB} = \tau)$	1.91	9.92	3.64	8.05	18.03	36.71	34.21	36.27
1.25	$E(T)$	166.78	114.18	138.95	118.87	107.74	104.94	104.24	103.26
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.09	33.11	29.03	30.89	42.75	57.89	54.29	54.58
	$\hat{P}(\hat{\tau}_{TAB} = \tau)$	1.43	8.72	3.16	6.30	16.75	32.85	31.63	33.32
1.50	$E(T)$	169.00	114.40	142.73	119.75	108.14	105.29	104.46	103.35
	$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.15	33.02	29.05	31.26	40.73	58.67	54.75	55.67
	$\hat{P}(\hat{\tau}_{TAB} = \tau)$	1.60	8.03	2.69	5.98	14.64	27.93	28.61	31.13

Table 6. Precision of $\hat{\tau}_{MLE}$ and $\hat{\tau}_{TAB}$ when used with a Tabular CUSUM $(\bar{x},\ s)$ control chart with $\ell_{TAB} = 0.50$.

μ_1	0.0	0.0	0.5	0.5	0.5	1.0	1.0	1.0
σ_1	1.25	1.50	1.00	1.25	1.50	1.00	1.25	1.50
$\hat{P}(\hat{\tau}_{MLE} = \tau)$	13.37	31.96	27.66	30.32	41.51	55.33	51.88	53.02
$\hat{P}(\lvert \hat{\tau}_{MLE} - \tau \rvert \le 1)$	27.59	54.44	48.62	52.43	64.39	74.44	73.18	75.09
$\hat{P}(\lvert \hat{\tau}_{MLE} - \tau \rvert \le 2)$	36.84	67.65	61.37	65.31	75.97	83.35	83.42	85.58
$\hat{P}(\lvert \hat{\tau}_{MLE} - \tau \rvert \le 3)$	44.24	75.84	70.26	74.08	83.12	88.85	89.21	92.04
$\hat{P}(\lvert \hat{\tau}_{MLE} - \tau \rvert \le 4)$	50.13	81.49	76.18	79.76	87.79	92.70	93.09	95.20
$\hat{P}(\lvert \hat{\tau}_{MLE} - \tau \rvert \le 5)$	55.38	85.50	80.46	83.64	90.78	94.95	95.44	96.96
$\hat{P}(\hat{\tau}_{TAB} = \tau)$	2.28	2.51	14.01	6.86	11.69	23.15	39.66	35.63
$\hat{P}(\lvert \hat{\tau}_{TAB} - \tau \rvert \le 1)$	6.13	6.02	30.54	15.22	25.65	46.63	66.49	65.53
$\hat{P}(\lvert \hat{\tau}_{TAB} - \tau \rvert \le 2)$	9.09	8.72	41.88	21.52	35.41	60.50	78.99	78.37
$\hat{P}(\lvert \hat{\tau}_{TAB} - \tau \rvert \le 3)$	11.54	11.35	50.06	27.02	43.31	70.13	85.46	85.75
$\hat{P}(\lvert \hat{\tau}_{TAB} - \tau \rvert \le 4)$	13.74	13.75	57.04	31.54	49.25	76.99	90.05	90.32
$\hat{P}(\lvert \hat{\tau}_{TAB} - \tau \rvert \le 5)$	15.92	15.82	62.29	35.30	54.33	82.05	93.00	93.25

been employed for monitoring the process mean and variance simultaneously, we have discussed the built-in change point estimation for the two kinds of CUSUM $(\bar{x},\ s)$ control charts: Mask and Tabular. Further, we have compared the change point estimation by the MLE with the built-in change point estimation in the CUSUM $(\bar{x},\ s)$ control charts. Throughout some numerical experiments about the performance of the change point estimation, we have confirmed that the change point estimation by the MLE is almost superior to the built-in change point estimation. Especially, the probability of estimating correctly the change point as a time epoch and a time interval is shown through some numerical experiments. Note that it is very difficult to consider the confidence level and interval estimation in theoretical because the distribution of the estimated value of the change point is not revealed yet.

In the viewpoint of performance, it is shown that the MLE change point estimation is more useful than the built-in change point estimation in the previous literature including this article. Also, in the viewpoint of operation, the MLE change point estimation is convenient. The built-in change point estimation is concerned with the control charts and charting techniques such as Mask CUSUM, Tabular CUSUM, and EWMA, etc. In other word, the built-in change point has to be developed in correspondence to the control charts and charting techniques. While, the MLE change point estimation is applied into various charts regardless of the kinds of control charts and charting techniques.

The MLE change point estimation is applied to control charts based on the attribute property (Pignatiello and Samuel, 2001) as well as the variable property such as normal distribution and multivariable normal distribution (Nedumaran, Pignatiello, and Calvin, 2000). Pignatiello and Samuel (2001b) has shown the utility of MLE change point estimation in the *n* and *np* control chart for the fraction of nonconforming items. It is an interesting problem to reveal the utility in other control chars.

Knowing the time of a process change could lead to quicker identification of the assignable cause and less process down time. While, knowing not only the time of a process change but also the direction of a process change could lead to much quicker identification of the assignable cause. This will be our future research.

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