A study of optimal periods in proportional reasoning

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Abstract: Proportional reasoning is one of the most widely used concepts in everyday life. It could be the most important basic concept in science and mathematics. In research where the subjects were animals, it has been found that learning effect rapidly decreased with any stimulation given after a optimalperiod. Therefore, it is necessary to research about optimal periods in order to instruct about proportional reasoning.

The purpose of this study was to investigate the optimal periods in proportional reasoning. The three programs for proportional reasoning instruction were developed by researchers. The titles of the programs were 'Block', 'Balance scale' and 'Water glass'. The subjects were 131 3rd to 6th grade students who were not expected to have any proportional reasoning skills yet. In order to find out the optimal periods in proportional reasoning, the programs were applied to these students. After 4-5 weeks of treatment, the researchers investigated whether their proportional reasoning skills were formed or not through the instrument. The results indicated that it would be most effective to teach proportional reasoning to 6th grade students. Teaching of proportional reasoning is essential not only for mathematics but also for science. The findings could be used to investigate the optimal periods of controlling variables, probability, combinational and correlational logic.

Key words: Proportional reasoning, optimal periods

Ⅰ. Introduction

For decades, the term 'critical period' has been used to describe an age window during which a particular type of experience is necessary for the development of a behavior or ability (Michel & Tyler, 2005). Development is an historical phenomenon in which previous events affect the manifestation of both current and subsequent events and current eventsthat affect subsequent events. Consequently, development must be defined by the illumination of the factors creating and governing the serial order and the processes of change and stability of that order over time (Michel & Moore, 1995; Michel & Tyler, 2005). To underline the distinction and to ensure that are referring to a window that is more variable in onset and offset than a classic critical period, the term optimal period was suggested (Cancedda *et al.*, 2004; Werker & Tees, 2005).

In Piaget's stages of development, proportional reasoning is considered to usher in the beginning of the formal operation stage (Inhelder & Piaget, 1958). As a result, the focus of many research studies revolves around adolescent students. Vernaud (1983) described a proportion as a multiplicative relationship between the measured quantities of two physically measurable attributes that he called measure spaces.

Although most people do not know the mathematical definition of proportion, they compare and contrast similarities and differences in various conditions as they experience proportional relationships in such conditions, and they use comparative reasoning, which enables people to understand qualitative relationships in the world (NCTM, 1994). Proportional reasoning also allows students to systematically organize their thoughts and ideas as they integrate them, and to expand their thoughts (Garofalo & Mtewa, 1990). It is likewise used in various forms and in various conditions. In fact, proportional reasoning is one of the most widely used concepts in day-to-day life and is the most important

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fundamental concept used in science and mathematics.

However, although proportional reasoning is utilized in various areas, many students consider the solution of problems related to the proportion concept difficult. For example, not only students who have reached the formal operational period, but also adults, feel constrained with respect to proportional reasoning (Capon & Kuhn, 1979; Jeon, Kwon, & Lawson, 1999; Karplus, Karplus & Wollman, 1974; Parllrand, 1979; Tourniaire & Pulos, 1985); in particular, most middle school students do not apply proportional reasoning (Chung, 1998; Kim, 1999; Kwon, 1997; Lee, 1998).

Studies on the development of proportional reasoning are classified into studies, the subjects of which are pre-school children, and studies, the subjectsof which are young boys and girls (Jeong, 2006). It has been reported that children aged below 5 are capable of accurate reasoning in conditions in which proportional reasoning is required (Acredolo et al., 1998; Lovett & Singer, 1991). Furthermore, recent studies assert that 5- to 6-month-old infants also have the core ability to undertake proportional reasoning (Huttenlocher, Duffy & Levine, 2002).

In a study, the subject of which were young boys and girls, Karplus, Adi & Lawson (1980) and Khoury (2002) proposed four proportional strategies using qualitative presumption, an additional strategy, an additional and a proportional strategy, and proportional reasoning.

On the other hand, as an active response measure to solve problems arising due to differences in cognitive standards, efforts to improve the effectiveness of science education by promoting students' cognitive development have been furthered based on the CASE (Cognition Acceleration through Science Education) program (Adey, Shayer & Yates, 1995). Studies applying CASE are being actively conducted not only in the UK (Adey, 1987; Adey & Shayer, 1994, 2002), but also in Hong Kong, China, Finland, Singapore, etc. (Choi *et al.*, 2002). These efforts are also being promoted in Korea (Nam, *et al.*, 2002; Choi, 2002).

In Korea, such studies report that the effects of promoting cognitive development by applying the CASE program are significant. In particular, in the study of Choi *et al.* (2002) the subjects of which were first year students in middle school, the acceleration effect of cognitive development was reported as significant, and the cognitive standards of the female students were found to have been more developed than that of the male students.

To accelerate cognitive development, proportional reasoning is guided and the thinking phases of proportion are examined. As Lenneberg (1967) proposed in his critical period hypothesis, a study was performed on second language learning. A recent study on the brain maintains that since there is a sharp growth in the cranium of 7-year-old children, it is the optimal time to teach them mathematics and reading; and that in 11-year-old children's brains, a sharp growth in logical and abstract thinking occurs. Regarding the development of the brain's nervous system, the question is asked as to whether or not there is an optimaltime to teach proportional reasoning. Therefore, this study aims to find the optimal time to teach proportional reasoning.

Ⅱ. Theoretical Background

In Korea, ratios and proportions are taught in the mathematics subject. A ratio is a comparison of relative sizes by setting a standard instead of by directly comparing two sizes. A proportion is an easier-to-understand form of ratio. Sixth-grade students learn ratios and proportions between June and July. The basic goal of teaching ratios and proportions is to make students understand 'regularity and function' (Ministry of Education & Human Resources Development, 2002). As they learn this regularity of mathematics, they are able to understand the order and the principles of their thoughts as they go through their lives in nature and in society.

Details on ratios and proportions, as taught to 6^{th} grade students, are aimed at helping the students understand the meaning of ratios, display the ratio of two numbers in symbols, and understand the comparative amount, the standard amount, and the value of the ratio. Furthermore, it allows students to understand fractions and proportions so as to display proportions, understand their relationship, find the relationship between the standard amount and the comparative amount, and solve proportional problems.

The primary goal of teaching ratios and proportions in 6th grade mathematics is to explain to students the meaning of the ratio, and to enable them to use the ratio as a symbol, and to display and read it. Other such goals are to enable the students to find the value of a ratio and to solve proportional problems. For the development of the class, after students learn the meaning of ratios, they are shown the values of ratios. Next, they are asked to display the ratio in fraction form and decimal fraction form, and are taught proportional expression and formulation of a rate graph. Before the students learn about ratios and proportions, they are taught fractions, decimal fractions, multiplication and division of fractions and decimal fractions, and comparison of the sizes of fractions and decimal fractions. Ratios and proportions are not included in education programs other than mathematics.

Ⅲ. Study Methodology

Subjects

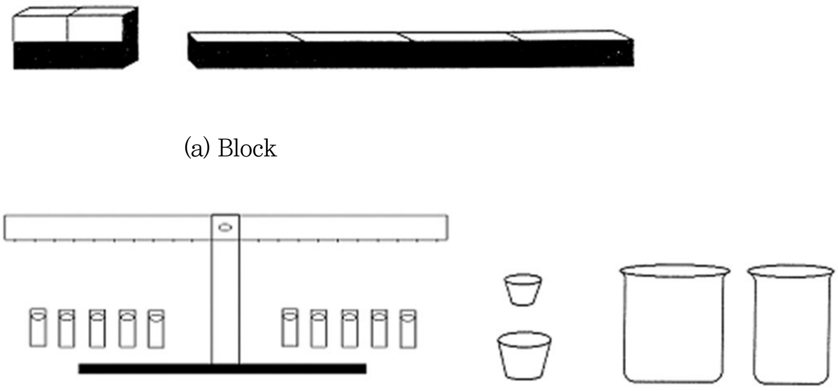
A proportional reasoning examination was conducted for students in the 3rd to the 7th grades. The subjects were selected from two schools in urban areas with populations of more than 1 million and from one school in another area with a population of less than 100,000. Proportional reasoning ability was found in only 10-25% of the examinees. This result corresponded with the results of the studies of Choi and Hur (1987) and of Hwang, Park & Kim (1989), which used Korean students as subjects. The subjects of this study were students with still unformed proportional reasoning abilities. The subjects were randomly selected and consisted of 33 students in the 3rd grade (age: 9.5 years ±0.3 years), 35 students in the 4th grade (age: 10.4 years ±0.3 years), 33 students in the 5th grade (age: 11.4 years ±0.5 years), and 30 students in the 6th grade (age: 12.4 years ±0.3 years).

Program Administration

The program for the formation of proportional reasoning was developed under the three conditions of a block, a balance scale, and a water glass. Each condition consisted of two sub-conditions. The block condition aimed to solve a problem using the proportion of a small-sized block and a large-sized block. The proportion of these blocks was 2:1. The balance scale asked where six weights of 10 g each should be placed when four weights of 10 g each were placed at a distance of 3 scales on the right to form a balance. In the water glass condition, the basic condition was to pour the amount of water corresponding to the notch mark 6 of the small water glass into a big water glass up to notch mark 4 (Fig. 1).

The program was administered by trained teachers. The teachers had six hours of advanced education and two preliminary trainings for 5th grade students. The program was administered in the order of block, balance scale, and water glass. In each condition, the first problems had to be solved using comparatively simple proportions; and for the second problem, a more complex condition was given.

For the administration of the program, the teachers directly guided groups of 2-3 students. The teachers made the students solve the problems themselves and checked their answers. Afterwards, they explained the process of solving each problem by proposing the multiplication strategy and the proportional strategy. At the end of each condition, an additional question was given to be solved. This process continued until all the students had solved the given problems. The time in which the students solved the problems in each condition was 54-90 minutes for the 3rd



(a) Block

(b) Balance scale (b) Water glass

Fig. 1 *3 conditions for the proportional program*

grade students (average: 68 min), 42–70 minutes for the 4th grade students (average: 47 min), 30–45 minutes for the 5th grade students (average: 36 min), and 10–25 minutes for the 6th grade students (average: 20 min).

The program, which was administered to the 3rd grade students and up, aimed to find the optimal grade at which to guide students on proportional reasoning. The 6th grade students completed the program.

Examination Tool

The examination tool for measuring the formation level of proportional reasoning ability was produced based on the examination paper for logical cognition. The pre-test and the post-test consisted of two questions each. The pre-test questions consisted of proportional reasoning problems using block and water glass, and the post-test questions consisted of proportional reasoning problems using gear and cylinder. All the questions in the pre-test and in the post-test were organized into questions that asked for answers and questions that asked for reasons. The validity of the examination tool was 0.97, and the reliability was measured as Cronbach α= 0.95.

Data Collection and Analysis

The pre-test was conducted by the students' teacher. The purpose and method of the examination were explained before it was conducted. The students were given sufficient time to solve the problems. The post-test was conducted 4–5 weeks after the administration of the program.

Regarding the strategy for determining the proportional reasoning abilities of the students, they were guided in each condition and asked to solve the problems, after which their answers were collected. For example, the teacher explained the basic condition of the block and gave the students the problem to solve. After the problem solving process was completed, the teacher explained the solution. It was repeated until the students understood the block condition, after which they were given the problem related to the block and were asked to solve it.

The formation of proportional reasoning ability was analyzed via two questions that measured such ability. Each questionsasked for the answer and the other, for the reason. The response to each question was scored 0, 1 and 2 points. One point was given when only one of the questions, whether it asked for the answer or for the reason, was answered; and two points were given when both the answer and the reason were correct. Therefore, proportional reasoning ability was deemed present when the student gave correct answers for the answer and the reason. If a student had only one correct answer or had all wrong answers, he was regarded as not having proportional reasoning ability. The students in the transitional period were counted as 'others.' For the processing of the statistics, SPSSWIN 12.0K was used.

Ⅳ. Results and Discussion

Strategy for the Proportional Reasoning Solution Process

The purpose of this study was to find the optimal grade at which to guide students in proportional reasoning. To achieve this, this study aimed to teach proportional reasoning to students whose proportional reasoning ability was still unformed, and to find the grade at which the formation level was the highest. The proportional reasoning program consisted of a block, a balance scale and a water glass. After the administration of each problem, the strategies that the students used to solve the given problems in each condition were observed.

The students who made intuitive prediction when they solved the proportional reasoning problem couldbe regarded as those whose proportional reasoning ability was still unformed. The students who used the multiplicative comparisonor mathematical proportional expression could be regarded as those who understood and used proportional reasoning. In intuitive prediction, the difference between the initial values in two physical condition was maintained or the approximate values were declared by making predictions. In the multiplicative comparison, the assumption that the small blocks made for 1 + 1 + 1 = 3 was used, so that the bigger blocks made for 2 + 2 + 2 = 6. The proportional strategy solved the problem using proportional reasoning.

In the block condition, 73% of the 3rd grade students who solved the block condition used the proportional strategy, and 86% of the 4th grade students, 94% of the 5th grade students and 100% of the 6th grade students also used the proportional strategy. Twenty-seven percent of the 3rd grade students and 6% of the 5th grade students were found not to have proportional reasoning ability. In the block condition, it was found that the higher the students'grade level was, the more they used mathematical proportional reasoning. This result was the same as that of the post-examination, which analyzed the proportional reasoning formation level four weeks after the exam.

In the balance scale condition, 85% of the 3rd grade students, and 94% of the 4th and 5th grade students, and all the 6th grade students used the multiplicative comparison strategy.

As with the block condition, in the balance scale condition, the higher-grade students used the proportional strategy more. As 97% of the 6th grade students used the proportional strategy, the 6th grade has been adjudged as the optimal time to teach students proportional reasoning.

The water glass condition showed the same result as that of the block and balance scale conditions. All the 6th grade students, 94% of the 4th- and 5th grade students, and 80% of the 3rd grade students used the multiplication strategy.

Table 1
Number of subjects according the problem solving strategy in the block condition (%)

Strategies / Grades	Intuitive prediction	Multiplicative comparison	Proportional reasoning
3rd	9(27.3)	19(57.6)	5(15.1)
4th	5(14.3)	11(31.4)	19(54.3)
5th	2(6.1)	7(21.2)	24(72.7)
6th	0(0.0)	2(6.7)	28(93.3)

Table 2

Number of subjects according the problem solving strategy in the balance scale condition (%)

Grades \ Strategies	Intuitive prediction	Multiplicative comparison	Proportional reasoning
3rd	5(15.1)	15(45.5)	13(39.4)
4th	3(8.6)	9(25.7)	23(65.7)
5th	2(6.1)	6(18.2)	25(75.8)
6th	0(0.0)	1(3.3)	29(96.7)

Table 3

Number of subjects according the problem solving strategy in the water glass condition (%)

Grades \ Strategies	Intuitive prediction	Multiplicative comparison	Proportional reasoning
3rd	7(21.2)	13(39.4)	13(39.4)
4th	2(5.7)	6(17.1)	27(77.2)
5th	2(6.1)	8(24.2)	23(69.7)
6th	0(0.0)	2(6.7)	28(93.3)

Level of Formation of Proportional Reasoning Ability

For the students whose proportional reasoning ability was found to have been still unformed, three types of proportional reasoning programs were conducted. Using a block, a balance scale, and a water glass. The level of formation of the proportional reasoning ability was measured four weeks after the program. The results are shown in Table 4.

The level of formation of the proportional reasoning ability was found to have been 12% for the 3rd grade students, 20% for the 4th grade students, 27% for the 5th grade students, and 67% for the 6th grade students. These results show that the level of formation of the proportional reasoning ability improved as the grade level became higher. In particular, 67% of the 6th grade students were found to have had proportional reasoning ability. Most of the 6th grade students in the transitional period gave correct answers but gave wrong answers for the reasons. Most of them checked the wrong answer of 'since the revolution of the gear makes a 2:3 proportion' among the choices for the question that asked for the reason.

On the other hand, while checking on the improvement of the subjects' proportional

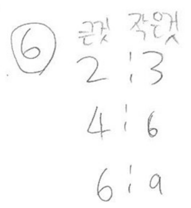
Table 4

Degree of formation of the proportional reasoning in post-test (%)

Grades	Not formed	Transition	Formed
3rd	8(24.2)	21(63.7)	4(12.1)
4th	6(17.1)	22(62.9)	7(20.0)
5th	2(6.1)	22(66.7)	9(27.2)
6th	1(3.3)	9(30.0)	20(66.7)

6. Water is poured into the narrow glass up to the 3rd mark. This water rises to the 2nd mark when poured into the wide glass. Water is now poured into the wide glass up to the 6th mark. How high would this water rise if it were poured into the empty narrow glass?

a) Student's example for multiplicative comparison



b) Student's example for proportional reasoning

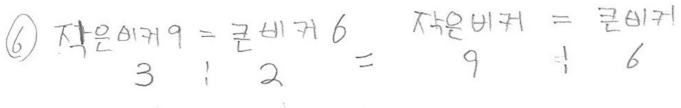


Fig. 2 *Student's response examples*

reasoning ability after the administration of the program, changes in the formation of proportional reasoning ability in the 3rd- to 6th grade students were observed. For this, the level of formation of the proportional reasoning ability among the students whose proportional reasoning ability was still unformed in the pre-test was measured in the post-examination. It was found that among the 3rd- to 6th grade students whose proportional reasoning ability was not yet formed and to whom the proportional reasoning program was administered, the proportional reasoning ability was found to have formed in 3% of the 6th grade students in the post-test. Therefore, the formation of the proportional reasoning ability in the students who participated in the study was considered the direct effect of the program.

As the level of formation of the proportional reasoning ability of the 6th grade students rapidly increased compared to that of the 5th grade students, the program was administered to 7th grade students whose proportional reasoning ability was still unformed, and the effect was observed. Twelve percent of 7th grade students were found to have been in the transitional period, and the proportional reasoning ability was found to have formed in 88% of the students.

Ⅴ. Conclusions and Implications

This study aimed to determine the optimal time at which to teach proportional reasoning to students with still unformed proportional reasoning abilities. The study found that it would be most effective to teach proportionalreasoning to 6th grade students. Teaching of proportional reasoning is essential not only for mathematics but also for science. Therefore, regarding the method of teaching logic in science or of teaching science concepts related to proportional reasoning, it is necessary to understand the level of formation of the proportional reasoning ability of the students.

Also, when teaching proportional reasoning, ratios and proportions are taught first, followed by fractions, decimal fractions, and proportional expressions. In science, however, it is considered helpful to form the proportional reasoning ability by explaining the multiplication strategy and the proportional strategy at the same time.

This study searched for the critical period at which to teach students proportional reasoning, which is the basic logic in science and mathematics and plays a role as the gate to the formal operational period. The following educational implications are being raised.

First, the 6th grade is considered the appropriate time to teach proportional reasoning. According to Korea's 7th national mathematics curriculum, ratios and proportions are currently taught at the 6th grade, so that this grade level is adjudged as the appropriate level at which to teach proportional reasoning. As shown in Tables 1 to 3, most of the 6th grade students in the study used the proportional strategy, and the formation of their proportional reasoning ability was found to have rapidly increased after the administration of the proportional reasoning program.

Second, although 6th grade students learn about ratios and proportions, the formation of the proportional reasoning ability among middle school and high school students appears low (Choi & Hur, 1987; Chung, Kwon & Kim, 1998; Jeon, Kwon, & Lawson, 1999). When teaching science in relation to proportional reasoning, it is necessary to understand the students' level of ability to understand ratios and proportions and their proportional reasoning ability.

Third, when there is a systematic method of teaching proportional reasoning, learning becomes more effective. For students with unformed proportional reasoning ability, their proportional reasoning ability was found to still unformed after a while. As proportional reasoning was taught to these students, however, the level of formation of their proportional reasoning ability appeared to be higher.

Fourth, the teaching of proportional reasoning in science should be developed in a direction different from the way it is taught in mathematics. In mathematics, ratios and proportions, fractions and decimal fractions, and proportional expressions are taught. The use of the multiplication strategy and the proportional strategy according to the method of development of proportional reasoning ability proposed in previous studies (Lawson, 1986; Hines & McMahon, 2005) is considered to be capable of promoting the formation of proportional reasoning ability.

In the new education curriculum, ratios and proportions are taught in the 5th grade. It is considered more appropriate, however, to teach it in the 6th grade. Also, neither the subfactors of logical cognition nor proportional reasoning are taught in science. Therefore, a study on this would have to be performed.

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